1. (4b) Řešte Fourierovou metodou

$$\begin{aligned} u_{tt} &= u_{xx} && x \in (0,3),\ t \in \mathbb{R}^+ \\ u(0,t) &= u(1,t) = 0 && t \in \mathbb{R}^+ \\ u(x,0) &= 1 - \cos\left(\frac{\pi x}{3}\right),\ u_t(x,0) = 0 && x \in (0,3) \end{aligned}$$

Řešení: Víme (pokud si vztah nepamatujete, nezoufejtes a během cca 10 minut si jej odvoďte), že

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right) \\ \varphi(x) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) && \longrightarrow\ A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x \\ \psi(x) &= \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi x}{l}\right) && \longrightarrow\ B_n = \frac{2}{n\pi c}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x \end{aligned}$$

a v našem případě $c = 1$, $l = 3$. Vzpomeneme-li si na vztahy

$$\begin{aligned} \cos(\alpha)\sin(\beta) &= \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right) \\ \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \end{aligned}$$

je evidentní, že $(\forall n \in \mathbb{N}) : B_n = 0$ a

$$\begin{aligned} A_n &= \frac{2}{3}\int_0^3 \left[1 - \cos\left(\frac{\pi x}{3}\right)\right]\sin\left(\frac{n\pi x}{3}\right)\,\mathrm{d}x = \\ &= \frac{2}{3}\left(\int_0^3 \sin\left(\frac{n\pi x}{3}\right) - \frac{1}{2}\sin\left(\frac{(n+1)\pi x}{3}\right) + \frac{1}{2}\sin\left(\frac{(1-n)\pi x}{3}\right)\,\mathrm{d}x\right) = \\ &= \frac{2}{3}\left(-\frac{3}{\pi}\right)\left[\frac{1}{n}\cos\left(\frac{n\pi x}{3}\right) - \frac{1}{2(n+1)}\cos\left(\frac{(n+1)\pi x}{3}\right) + \frac{1}{2(1-n)}\cos\left(\frac{(1-n)\pi x}{3}\right)\right]_0^3 = \\ &= -\frac{1}{\pi}\left[\frac{2}{n}\cos(n\pi) - \frac{1}{n+1}\cos((n+1)\pi) + \frac{1}{1-n}\cos((1-n)\pi) - \left(\frac{2}{n} - \frac{1}{n+1} + \frac{1}{1-n}\right)\right] = \\ &= \frac{1}{\pi}\left[-\frac{2}{n}\cos(n\pi) + \frac{1}{n+1}\cos(n\pi) + \frac{1}{n-1}\cos(n\pi) + \frac{2(n^2-1) - n(n-1) - n(n+1)}{n(n^2-1)}\right] = \\ &= \frac{\left[-2(n^2-1) - n(n-1) - n(n+1)\right]\cos(n\pi) - 2}{\pi n(n^2-1)} = \\ &= \frac{(2-4n^2)\cos(n\pi) - 2}{\pi n(n^2-1)} = \begin{cases} \frac{4}{\pi n} & n = 2k-1,\ k \in \mathbb{N} \\ \frac{-4n}{\pi(n^2-1)} & n = 2k\quad,\ k \in \mathbb{N} \end{cases} \end{aligned}$$

Proto tedy

$$u(x,y) = \sum_{k=1}^{\infty}\left[\frac{4}{\pi(2k-1)}\cos\left(\frac{(2k-1)\pi t}{3}\right)\sin\left(\frac{(2k-1)\pi x}{3}\right) - \frac{8k}{\pi(4k^2-1)}\cos\left(\frac{2k\pi t}{3}\right)\sin\left(\frac{2k\pi x}{3}\right)\right]$$

2. (4b) S nápovědou $\cos(\alpha)\sin(\beta) = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$ řešte nehomogenní počátečně-okrajovou úlohu

$$\begin{aligned} u_t &= u_{xx} && x \in (0,1),\ t \in \mathbb{R}^+ \\ u(0,t) &= 1,\ u(1,t) = -1 && t \in \mathbb{R}^+ \\ u(x,0) &= \cos(\pi x) && x \in (0,1) \end{aligned}$$

Řešení: Funkci u hledejme ve tvaru $u(x,t) = v(x,t) + w(x,t)$ kde funkce v splňuje okrajovou podmínku a je co nejjednodušší, nejlépe aby $v_t = v_{xx} = 0$. Tomu vyhovuje například přímka $v(x,t) = 1 - 2x$. Pro w dostaneme úlohu

$$\begin{aligned} u_t - u_{xx} &= w_t - w_{xx} = 0 && x \in (0,1),\ t \in \mathbb{R}^+ \\ w(0,t) &= u(0,t) - v(0,t) = 0,\ w(1,t) = u(1,t) - v(1,t) = 0 && t \in \mathbb{R}^+ \\ w(x,0) &= u(x,0) - v(x,0) = \cos(\pi x) - 1 + 2x && x \in (0,1) \end{aligned}$$

pro jejíž řešení známe vzorec (lze opět v cca 10–ti minutách odvodit), $k = 1$, $l = 1$:

$$w(x,t) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right)$$

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \qquad \longrightarrow \quad A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x$$

Dosazením vypočítáme

$$\begin{aligned}
A_n &= 2\int_0^1 [\cos(\pi x) - 1 + 2x]\sin(n\pi x)\,\mathrm{d}x = \\
&= \int_0^1 2\cos(\pi x)\sin(n\pi x) - 2\sin(n\pi x) + 4x\sin(n\pi x)\,\mathrm{d}x = \\
&= -\frac{4}{n\pi}\left[x\cos(n\pi x)\right]_0^1 + \int_0^1 \sin((1+n)\pi x) - \sin((1-n)\pi x) - 2\sin(n\pi x) + \frac{4}{n\pi}\cos(n\pi x)\,\mathrm{d}x = \\
&= -\frac{4}{n\pi}\cos(n\pi) + \left[-\frac{\cos((1+n)\pi x)}{(1+n)\pi} + \frac{\cos((1-n)\pi x)}{(1-n)\pi} + \frac{2\cos(n\pi x)}{n\pi} + \frac{4\sin(n\pi x)}{(n\pi)^2}\right]_0^1 = \\
&= \left(-\frac{4}{n\pi}\right)\cos(n\pi) + \left(\frac{1}{(1+n)\pi} - \frac{1}{(1-n)\pi} + \frac{2}{n\pi}\right)\cos(n\pi) - \left(-\frac{1}{(1+n)\pi} + \frac{1}{(1-n)\pi} + \frac{2}{n\pi}\right) = \\
&= \frac{2(\cos(n\pi)+1)}{\pi n(n^2-1)} = \begin{cases} 0 & n = 2k-1,\ k \in \mathbb{N} \\ \frac{4}{\pi n(n^2-1)} & n = 2k \quad,\ k \in \mathbb{N} \end{cases}
\end{aligned}$$

a tedy

$$u(x,t) = 1 - 2x + 2\sum_{k=1}^{\infty} \frac{\mathrm{e}^{-(2k\pi)^2 t}\sin(2k\pi x)}{\pi k(4k^2-1)}$$

3. (4b) Separujte následující PDR na dvě ODR:

$$u_{tt} = (k(x)u_x)_x - mu_x + u$$

Řešení: Předpokládáme, že $u(x,t) = X(x)T(t)$. Dosazením do PDR dostaneme

$$\begin{aligned}
X(x)T''(t) &= (k(x)X'(x)T(t))_x - mX'(x)T(t) + X(x)T(t) = \\
&= k(x)'X'(x)T(t) + k(x)X''(x)T(t) - mX'(x)T(t) + X(x)T(t)
\end{aligned}$$

Po vydělení rovnice výrazem $X(x)T(t)$ dostaneme

$$\frac{T''}{T}(t) = \frac{kX'' + (k'-m)X' + 1}{X}(x) = -\lambda$$

z čehož vyseparujeme soustavu dvou ODR

$$\begin{aligned}
k(x)X''(x) + (k'(x) - m)X'(x) + (\lambda+1)X(x) &= 0 \\
T''(t) + \lambda T(t) &= 0
\end{aligned}$$

4. (3b) Sestavte tvar Laplaceovy rovnice $u_{xx} + u_{yy} = 0$ v polárních souřadnicích $x = r\cos(\varphi)$, $y = r\sin(\varphi)$.

Řešení: Definujme

$$\bar{u}(r,\varphi) = u(r\cos(\varphi), r\sin(\varphi))$$

Potom derivace složené funkce

$$\begin{aligned}
\bar{u}_r &= u_x\cos(\varphi) + u_y\sin(\varphi) \\
\bar{u}_\varphi &= u_x(-r\sin(\varphi)) + u_y r\cos(\varphi) \\
\bar{u}_{rr} &= u_x\cdot 0 + u_y\cdot 0 + u_{xx}\cos^2(\varphi) + u_{xy}2\cos(\varphi)\sin(\varphi) + u_{yy}\sin^2(\varphi) = \\
&= u_{xx}\cos^2(\varphi) + u_{xy}2\cos(\varphi)\sin(\varphi) + u_{yy}\sin^2(\varphi) \\
\bar{u}_{\varphi\varphi} &= u_x(-r\cos(\varphi)) + u_y(-r\sin(\varphi)) + u_{xx}r^2\sin^2(\varphi) - 2u_{xy}r^2\cos(\varphi)\sin(\varphi) + u_{yy}r^2\cos^2(\varphi)
\end{aligned}$$

a proto

$$
\begin{aligned}
&\bar{u}_{rr} + \frac{1}{r}\bar{u}_r + \frac{1}{r^2}\bar{u}_{\varphi\varphi} = \\
&\quad \left(u_{xx}\cos^2(\varphi) + u_{xy}2\cos(\varphi)\sin(\varphi) + u_{yy}\sin^2(\varphi)\right) + \\
&\quad + \frac{1}{r}\left(u_x\cos(\varphi) + u_y\sin(\varphi)\right) + \\
&\quad + \frac{1}{r^2}\left(u_x\left(-r\cos(\varphi)\right) + u_y\left(-r\sin(\varphi)\right) + u_{xx}r^2\sin^2(\varphi) - 2u_{xy}r^2\cos(\varphi)\sin(\varphi) + u_{yy}r^2\cos^2(\varphi)\right) = \\
&\quad = u_{xx} + u_{yy} = 0
\end{aligned}
$$