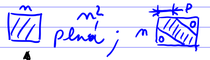
NUMERICKÉ ŘEŠENÍ SOUSTAV LINEÁRNÍCH ROVNIC

- **PŘÍMÉ METODY** (když se proces řešení vleze do RAM)

založené na ELIMINAČNÍCH PROCESECH



pásová
$p \cdot n$

pivotizovaná GEM

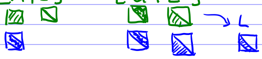
LU rozklad,

$A x_1 = b$

$\vdots$

$A \cdot x_k = b_k$

$[A|I] \longrightarrow [U|L^{-1}]$



Př! Pomocí LU rozkladu řeš soustavu $Ax = b = [32, 37, 21, 8]^T$:

$$\underbrace{\begin{bmatrix} 3 & 2 & 7 & 0 \\ 5 & 5 & 1 & 0 \\ 4 & 0 & 1 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix}}_{A} \Bigg| \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I} \begin{matrix} 3r_2 - 5r_1 \\ 3r_3 - 4r_1 \end{matrix} \rightarrow \underbrace{\begin{bmatrix} 3 & 2 & 7 & 0 \\ 0 & 5 & -32 & 0 \\ 0 & -8 & -25 & 9 \\ 0 & 2 & 1 & 0 \end{bmatrix}}_{T_1 \cdot A} \Bigg| \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ -4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_1} \begin{matrix} 5r_3 + 8r_2 \\ 5r_4 - 2r_2 \end{matrix}$$

$$T_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5/3 & 1/3 & 0 & 0 \\ 4/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

na počítači $1 \ddots 1$

$$\rightarrow \underbrace{\begin{bmatrix} 3 & 2 & 7 & 0 \\ 0 & 5 & -32 & 0 \\ 0 & 0 & -381 & 45 \\ 0 & 0 & \boxed{69} & 0 \end{bmatrix}}_{T_2 \cdot T_1 \cdot A} \Bigg| \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ -60 & 24 & 15 & 0 \\ 10 & -6 & 0 & 5 \end{bmatrix}}_{T_2 \cdot T_1} \; \frac{381}{3} r_4 + \frac{69}{3} r_3$$

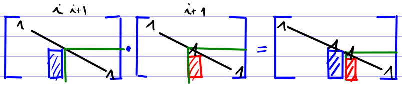
$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 0 & 5 \end{bmatrix} \qquad T_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -8/5 & 1/5 & 0 \\ 0 & 2/5 & 0 & 1/5 \end{bmatrix}$$

$$\rightarrow \underbrace{\begin{bmatrix} 3 & 2 & 7 & 0 \\ 0 & 5 & -32 & 0 \\ 0 & 0 & -381 & 45 \\ 0 & 0 & 0 & 1035 \end{bmatrix}}_{T_3 \cdot T_2 \cdot T_1 \cdot A \,=\, U} \Bigg| \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ -60 & 24 & 15 & 0 \\ -110 & -210 & 345 & 635 \end{bmatrix}}_{T_3 \cdot T_2 \cdot T_1 \,=\, L^{-1}}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 23 & 127 \end{bmatrix} \qquad T_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-25}{127} & 1/127 \end{bmatrix}$$

$= T_1^{-1} \cdot T_2^{-1} \cdot T_3^{-1}$

$A = L \cdot U$



Pivotace – záměna sloupců

$$\begin{bmatrix} \boxed{a_{11}} & \boxed{a_{12}} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} \boxed{x_1} \\ \boxed{x_2} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

LU rozklad a více pravých stran:

$$\underline{\underline{A}} \cdot \underline{x_i} = \underline{b_i} \qquad A(:, p) = L \cdot U$$

$$b_i = A x_i = A(:, p) \cdot \underbrace{x_i(p)}_{\text{ozn } z_i} = L \cdot \overbrace{U \cdot z_i}^{w_i}$$

w_i = back_subst(L, b_i);
z_i = back_subst(U, w_i);
$x_i = z_i(p)$;

n^2 složitost

• ITERAČNÍ METODY

- musím umět vynásobit maticí A, nebo části!

$\boxed{x_{k+1} = \underline{\Phi}(x_k)}$ Iterační metoda je

- **KONZISTENTNÍ** $\Leftrightarrow$ $\bar{x} = \underline{\Phi}(\bar{x})$

(se soustavou lin. rovnic $A\bar{x} = b$)

$(\forall b \in \mathbb{R}^n)$:

- **KONVERGENTNÍ** $\Leftrightarrow$ $\lim_{k\to\infty} x_k = \bar{x}$ nezávisí na x_0

Vondrák & Pospíšil V 3.15

"$\Rightarrow$" $\bar{x} = \phi(\bar{x}) \Longrightarrow A^{-1}b = MA^{-1}b + Nb$

$$\boxed{\begin{aligned}\bar{x} &= A^{-1}b\\ \phi(x) &= Mx + Nb\end{aligned}}$$

$(I - M)A^{-1}b = Nb \quad \Downarrow \quad \text{-- -- } \forall b \in \mathbb{R}^n$

$(I-M)A^{-1} = N$

$I - M = NA$

$\boxed{M = I - N\cdot A}$

"$\Leftarrow$" $\underbrace{M = I - N\cdot A}$, $A\bar{x} = b$, $\phi(x) = Mx + Nb$

chceme ukázat, že $\bar{x} = \phi(\bar{x})$ $\quad \bar{x} = A^{-1}b$

$$\boxed{\phi(\bar{x})} = M\cdot A^{-1}b + Nb = (I - NA)A^{-1}b + Nb = A^{-1}b \overbrace{- NAA^{-1}b + Nb}^{0} = A^{-1}b = \boxed{\bar{x}}$$

$$x^{k+1} = \phi(x^k) = Mx^k + Nb = (I - N\cdot A)x^k + Nb = x^k \underbrace{- N(Ax^k - b)}_{\text{oprava}}$$

Chyba lineární iterační metody: $e^k := \bar{x} - x^k$

$\bar{x} = M\bar{x} + Nb$ $\qquad e^k = M^k e^0$

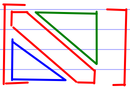
$$e^k = \bar{x} - x^k = \bar{x} - [Mx^{k-1} + Nb] = M(\underbrace{\bar{x} - x^{k-1}}_{e^{k-1}}) = \text{atd} = M^k \cdot e^0$$

$x^k = Mx^{k-1} + Nb$

$-e^{k+1} = -M^{k+1}e^0$

Lin. it. met. je konvergentní $\Leftrightarrow$ $\lim_{k} \|\overbrace{\phi(x^k)}^{x^{k+1}} - \bar{x}\| = 0$

$\|M\| < 1$



$A = \left[\ \right] = D + U + L$

$\boxed{1}$ Jacobi

$(D+L+U)\bar{x}$

$\|$

$A\bar{x} = b$

$D\bar{x} = b - (L+U)\bar{x}$

$\bar{x} = -D^{-1}(L+U)\bar{x} + D^{-1}b$

$x^{k+1} = -D^{-1}(L+U)x^k + D^{-1}b$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^k + b_i\right]$$

Konzistence: $\bar{\Phi}(\bar{x}) = -D^{-1}(L+U)\bar{x} + D^{-1}b \pm D^{-1}D\bar{x} = D^{-1}\Big[\overbrace{\underbrace{-(D+L+U)\bar{x}}_{A} + b}^{-b+b=0} + D\bar{x}\Big] = \bar{x}$

Konvergence $\Longleftrightarrow$ $\|M\| = \|D^{-1}(L+U)\| < 1$ což splňují diagonálně dominantní matice

$\forall i: \ |a_{ii}| > \sum_{j\neq i} a_{ij}$

2 Gauss-Seidel

x_j^{k+1} … $j<i$ už známe

z Jacobi: $x_i^{k+1} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij} \boxed{x_j^k} - \sum_{j=i+1}^{n} a_{ij} x_j^k + b_i\right]$

$$(D+L)x^{k+1} = b - Ux^k$$

je konzistentní (cv.). Je konvergentní $\Leftrightarrow \|(D+L)^{-1}U\| < 1$

3 SOR

Upravíme Gaussovu-Seidelovu metodu $x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^k \downarrow\right]$ $\pm a_{ii}x_i^k$

$$x_i^{k+1} = x_i^k + \omega \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - a_{ii}x_i^k - \sum_{j=i+1}^{n} a_{ij}x_j^k\right]$$

$$x_i^{k+1} = (1-\omega)x_i^k + \omega \underbrace{\frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k\right]}_{x_{GS\,i}^{k+1}} \qquad \omega \in (0,2)$$

$x_{GS\,i}^{k+1}$ … z Gaussovy-Seidelovy metody

$$= (1-\omega)x_i^k + \omega x_{GS\,i}^{k+1}$$

$$(D+\omega L)x^{k+1} = \omega b + [(1-\omega)D - \omega U]x^k$$

4 Richardsonova metoda

$\boxed{x^{k+1} = x^k + \omega r^k}$ kde $r^k = b - Ax^k$ … residuum, $\omega > 0$

- Konzistence: $\bar{x} + \omega \overbrace{(b - A\bar{x})}^{=0} = \bar{x}$
- Konvergence: … zde se podíváme na SPD matice. (symetrické pozitivně definitní)

spektrální rozklad $A = VDV^T$ (ortog. … diag.) $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_n$, $\langle v_1, \dots, v_n\rangle = \mathbb{R}^n$ ortonorm. báze

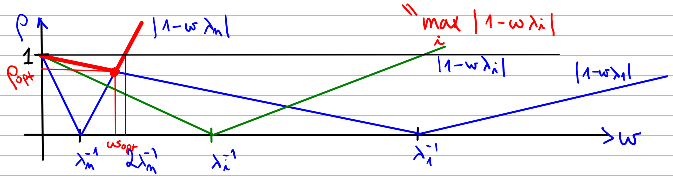
Pro reziduum $r^{k+1} = b - Ax^{k+1} = \overbrace{b - A[x^k}^{b-Ax^k = r^k} + \omega r^k] = \overbrace{(I - \omega A)r^k}^{= (I-\omega A)\sum_i \alpha_i^k v_i}$

$= \sum_i \alpha_i^k (I-\omega A)v_i$, $(I-\omega A)v_i = v_i - \omega\lambda_i v_i$

$$\sum_{i=1}^{n} \alpha_i^{k+1} v_i \quad \Downarrow \quad \underline{\alpha_i^{k+1} = (1-\omega\lambda_i)\alpha_i^k} \quad \sum_{i=1}^{n} \alpha_i^k v_i$$

$(\forall i): |1 - \omega\lambda_i| < 1 \quad \Leftrightarrow$ metoda konverguje

$$\|r^{k+1}\| \le \rho \|r^k\|, \qquad \rho = \max_i |1 - \omega\lambda_i|$$



Richardsonova metoda konverguje pro $\omega \in (0, 2\lambda_n^{-1})$, a ideální ω_{opt}: $1 - \omega_{opt}\lambda_1 = -(1 - \omega_{opt}\lambda_n) \Rightarrow \boxed{\omega_{opt} = \frac{2}{\lambda_1 + \lambda_n}}$

$$\boxed{\rho_{opt} = 1 - \frac{2}{\lambda_1 + \lambda_n}\lambda_1 = \frac{\lambda_1 + \lambda_n - 2\lambda_1}{\lambda_1 + \lambda_n} = \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1} = \frac{\frac{\lambda_n}{\lambda_1} - 1}{\frac{\lambda_n}{\lambda_1} + 1} = \frac{\varkappa(A) - 1}{\varkappa(A) + 1}}$$

$\varkappa(A)$ … číslo podmíněnosti matice A.

Počet iterací postačující k dosažení přesnosti ε:

$$\|r^k\| \le \rho_{opt}^k \|r^0\| \le \varepsilon \|r^0\| \Leftarrow k \ge \frac{\ln \varepsilon}{\ln \rho_{opt}} \quad \Big| \quad \rho_{opt}, \varepsilon \in (0,1)$$

$k \cdot \ln \rho_{opt} \le \ln \varepsilon$

Prakticky $k \ge \frac{\ln 10}{2} \varkappa(A) |\ln \varepsilon|$

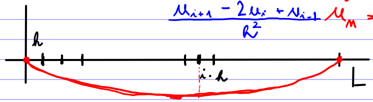
$\boxed{\text{CV}}$ Struna stacionární $\cancel{\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}} = \frac{\partial^2 u}{\partial x^2} + f(x,\cancel{t})$

$= 0$ $\qquad \frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$

$u_0 = 0$

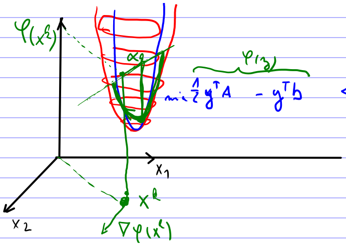
$u_n = 0$

$$\begin{bmatrix}1 & 0 & \cdots & & 0\\ -1 & 2 & -1 & 0 & \cdots\\ 0 & -1 & 2 & -1 & 0\\ & & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 2\\ 0 & \cdots & & & 0\end{bmatrix}\begin{bmatrix}u_0\\ \vdots\\ u_i\\ \vdots\\ u_n\end{bmatrix} = \begin{bmatrix}0\\ h^2 f\\ \vdots\\ h^2 f\\ 0\end{bmatrix}$$

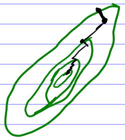


$L = n \cdot h$

$2u_i - u_{i+1} - u_{i-1} = h^2 f_i$



$\min \overbrace{\tfrac{1}{2}y^TA \;-\; y^Tb}^{\varphi(y)} \iff Ax = b$



$\boxed{5}$ Metoda největšího spádu

$Ax = b \iff x = \arg\min_{y\in\mathbb{R}^n} \overbrace{\tfrac{1}{2}y^TAy - b^Ty}^{\varphi(y)}, \qquad r(y) = -\nabla\varphi(y) = Ay - b$

$[\nabla\varphi(y)]_i = \frac{\partial\varphi}{\partial y_i}(y) = \frac{1}{2}\left([Ay]_i + [y^TA]_i - b_i\right) = [Ay - b]_i$

$x^{\ell+1} = x^\ell + \alpha^\ell r^\ell$

← Jaké je ideální (lokálně) α^ℓ?

$\alpha^\ell = \arg\min_\alpha \varphi(x^\ell + \alpha r^\ell) = \overbrace{\tfrac{1}{2}(x^\ell+\alpha r^\ell)^TA(x^\ell+\alpha r^\ell) - b^T(x^\ell+\alpha r^\ell)}$

$= \arg\min_\alpha \tfrac{1}{2}r^{\ell T}Ar^\ell\cdot\alpha^2 + \overbrace{(x^{\ell T}Ar^\ell - b^Tr^\ell)}^{r^{\ell T}(Ax^\ell - b)\;\|r^\ell\|}\alpha + \ldots$

$= \|r^\ell\| / r^{\ell T}Ar^\ell = \|r^\ell\| / \|r^\ell\|_A$

Pomalá konvergence: $\quad Av_i = \lambda_i v_i \qquad (I - \alpha^\ell A)\sum_i \beta_i^\ell v_i$

$r^{\ell+1} = b - Ax^{\ell+1} = \underbrace{(b - Ax^\ell)}_{r^\ell} - A\alpha^\ell r^\ell = \underbrace{(I - \alpha^\ell A)r^\ell}$

$\Vert$

$\sum_i \beta_i^{\ell+1} v_i$

$= \sum_i(\beta_i^\ell v_i - \alpha^\ell\lambda_i\beta_i^\ell v_i)$

$= \sum_i (1 - \alpha^\ell\lambda_i)\beta_i^\ell v_i$

$\rho_\ell = \max_{i\in\{1,\ldots,n\}} |1 - \alpha^\ell\lambda_i|$

$= \max_{i\in\{1,\ldots,n\}}\left|1 - \lambda_i\frac{\sum_j(\beta_j^\ell v_j)^T(\beta_j^\ell v_j)}{\sum_j(\quad)^TA(\quad)}\right| = \max_{i\in\{1,\ldots,n\}}\left|1 - \lambda_i\frac{\sum_j(\beta_j^\ell)^2}{\sum_j\lambda_j(\beta_j^\ell)^2}\right| \leq \left|1 - \frac{\lambda_1}{\lambda_n}\right|$

$(\beta_j^\ell)^2 v_j^T\underbrace{Av_j}_{=\lambda_j v_j} = (\quad)^2\lambda_j\underbrace{(v_j^Tv_j)}_{=1}$

$\boxed{6}$ METODA SPRUŽENÝCH GRADIENTŮ (CG)

A je SPD: $A = A^T$ sym., $\quad y^TAy \overset{y\neq 0}{>} 0$

conjugate gradient

$x^{k+1} = \arg\min_{y\in x^0+\mathcal{K}^k(A,r^0)} \varphi(y) = \arg\min_{y\in x^0+\mathcal{K}(A,r^0)} \|x - y\|_A$

$\|x - y\|_A = (x-y)^TA(x-y) = x^TAx - 2y^T\underbrace{Ax}_{b} + y^TAy = \underbrace{x^TAx}_{\text{konst.}} + 2\varphi(y)$

$Ax = b$

x... přesné řešení

$\mathcal{K}^k(A,r^0) = \langle r^0, Ar^0, A^2r^0, \ldots, A^kr^0\rangle$ .... Krylovovský prostor

Na rozdíl od metody největšího spádu se v každé iteraci CG realizuje globální minimum (na $\mathcal{K}^k(A,r^0)$).

Mějme v $\mathbb{R}^n$ A-ortogonální bázi $p_0,\ldots,p_{n-1}$: $\quad p_i^TAp_j \overset{i\neq j}{=} 0, \quad \underset{i=j}{>} 0$

Řešení x soustavy $Ax = b$ zapíšeme v této bázi

$x = \sum_{i=0}^{n-1}\alpha_i p_i = \arg\min_{\substack{\gamma_i\in\mathbb{R}\\ i=0,\ldots,n-1}} \underbrace{\varphi\left(\sum_{i=0}^{n-1}\gamma_i p_i\right)} = \arg\min_{\substack{\gamma_i\in\mathbb{R}\\ i=0,\ldots,n-1}} \sum_{i=0}^{n-1}\left(\tfrac{1}{2}\gamma_i^2\underbrace{p_i^TAp_i}_{\text{čísla}} - \gamma_i\underbrace{b^Tp_i}_{\text{čísla}}\right)$

$\Vert$

$\arg\min_{y\in\mathbb{R}^n}\varphi(y)$

$\Vert$

$\frac{1}{2}\left(\sum_i\gamma_ip_i\right)^TA\left(\sum_j\gamma_jp_j\right) - b^T\left(\sum_i\gamma_ip_i\right)$

$\Vert$

$\underbrace{\frac{1}{2}\sum_i\sum_j\gamma_i\gamma_j\, p_i^TAp_j}$ $\quad \overset{i=j}{\to}\ 0, \quad \underset{j\neq i}{=} 0$

$\frac{1}{2}\sum_i\gamma_i^2\, p_i^TAp_i$

$$= \underset{\substack{\gamma_i \in \mathbb{R} \\ i=0,\dots,n-1}}{\arg\min} \sum_{i=0}^{n-1} \varphi(\gamma_i p_i) = \underset{\gamma_0}{\arg\min}\, \varphi(\gamma_0 p_0) + \underset{\gamma_1}{\arg\min}\, \varphi(\gamma_1 p_1) + \dots + \underset{n-1}{\arg\min}\, \varphi(\gamma_{n-1} p_{n-1})$$

oddělené [nezávislé] 1D minimalizace.

Jak budou vypadat jednotlivé α_i?

$ax^2+bx+c \dots \min v \ \frac{-b}{2a}$

$$\alpha_i = \arg\min \overbrace{\varphi(\gamma_i p_i)}^{\frac{1}{2}\gamma_i^2 p_i^T A p_i - \gamma_i b^T p_i} = \frac{b^T p_i}{p_i^T A p_i} \quad \square$$

Zatím jsme hledali x na celém $\mathbb{R}^n$

Nyní budeme méně ambiciózní a zkusme hledat minimum x^k na $\mathcal{Y}^k = \{x_0 + \gamma_0^k p_0 + \dots + \gamma_{k-1}^k p_{k-1}\} \subsetneq \mathbb{R}^n$.

$\varphi(z) = \frac{1}{2} z^T A z - b^T z$

$$\boxed{x^k = \underset{y^k \in \mathcal{Y}^k}{\arg\min}\, \varphi(y^k)} \iff \boxed{\alpha^k = \underset{\substack{\gamma_i^k \\ i=0,\dots,n-1}}{\arg\min}\, \varphi(x_0 + \gamma_0^k p_0 + \dots + \gamma_{k-1}^k p_{k-1})} =$$

$$= \underset{\gamma_i^k,\ i=0..k-1}{\arg\min} \sum_{i=0}^{k-1} \left[\frac{1}{2}(\gamma_i^k)^2 p_i^T A p_i - \gamma_i^k b^T p_i + \frac{1}{2} x_0^T A x_0 - b^T x_0 + \gamma_i^k x_0^T A p_i \right]$$

$$= \underset{\gamma_0}{\arg\min} \underbrace{\frac{1}{2}(\gamma_0^k)^2 p_0 A p_0 - \gamma_0^k \overbrace{(b^T p_0 - x_0^T A p_0)}^{(\overbrace{b - A x_0}^{r_0})^T p_0}}_{\varphi_0(\gamma_0^k p_0)} + \dots + \underset{\gamma_{k-1}^k}{\arg\min} \underbrace{\frac{1}{2}(\gamma_{k-1}^k)^2 p_{k-1}^T A p_{k-1} - \gamma_{k-1}(\overbrace{b - A x_0}^{r_0})^T p_{k-1}}_{\varphi_0(\gamma_{k-1} p_{k-1})}$$

kde $\varphi_0(z) = \frac{1}{2} z^T A z - r_0^T z$

$$\alpha_i^k = \frac{r_0^T p_i}{p_i^T A p_i}$$

Předpokládejme, že známe $x^{k-1} = \underset{y^{k-1} \in \mathcal{Y}^{k-1}}{\arg\min}\, \varphi(y^{k-1})$ a hledáme $x^k = \arg\min \dots$.

$$\alpha_i^{\cancel{k}} = \boxed{\underset{\gamma_0^{\cancel{k}}}{\arg\min}\, \varphi_0(\gamma_0^{\cancel{k}} p_0)} + \dots + \boxed{\underset{\gamma_{k-2}^{\cancel{k}}}{\arg\min}\, \varphi_0(\gamma_{k-2}^{\cancel{k}} p_{k-2})} + \underset{\gamma_{k-1}^{\cancel{k}}}{\arg\min}\, \varphi_0(\gamma_{k-1}^{\cancel{k}} p_{k-1})$$

$$\alpha_0^{\cancel{k}} = \alpha_0^{\cancel{k-1}} \qquad \alpha_{k-2}^{\cancel{k}} = \alpha_{k-2}^{\cancel{k-1}}$$

Napočítané souřadnice (vypočtené jako 1D minimalizace) se nemění v závislosti na k!

$p_1^T A p_0 = 0 \Leftarrow p_1 \perp_A p_0$

x_0, $p_0 = r_0$, spočtu α_1, x_1, r_1, hledám $p_1 = r_1 + \beta_0^2 p_0$

x_i, $p_1, \dots p_i$, spočtu $\alpha_{i+1}, x_{i+1}, r_{i+1}$, hledám $\boxed{p_{i+1} = r_{i+1} + \beta_0^{i+1} p_0 + \dots + \beta_i^{i+1} p_i}$

$p_{i+1} \perp_A p_j \quad \forall j = 0, \dots, i$

$$0 = p_j^T A p_{i+1} = p_j^T A (r_{i+1} + \beta_0^{i+1} p_0 + \dots + \beta_i^{i+1} p_i) = p_j^T A r_{i+1} + \beta_0^{i+1} p_j^T A p_0 + \dots + \beta_i^{i+1} p_j^T A p_i$$

$$0 = p_j^T A r_{i+1} + \beta_j^{i+1} p_j^T A p_j$$

$$\beta_j^{i+1} = -\frac{p_j^T A r_{i+1}}{p_j^T A p_j}$$

(protože $p_j^T A p_k = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$)

Takto bychom museli : a) uchovávat v paměti $p_1, \dots, p_j$

b) v každém kroku spočítat $p_j^T A r_{i+1}$ $\forall j$

Pomocí následující strategie ale obě tyto nevýhody eliminujeme

▣ Hledejme x_i v $x_0 + \mathcal{K}_i$, kde $\mathcal{K}_i = \langle r_0, Ar_0, \dots, A^{k-1} r_0 \rangle$

- Platí: $\boxed{r_k \perp \mathcal{K}_k}$, tj $\forall d \in \mathcal{K}_k : r_k^T d = 0$

Důkaz (sporem): předpokládejme $r_k = b - Ax_k$, a $\exists d \in \mathcal{K}_k : r_k^T d \neq 0$

Ale $x_k = \arg\min\limits_{y_k \in x_0 + \mathcal{K}_k} \varphi(y^k) \Rightarrow \forall \varepsilon : \varphi(x_k) \le \varphi(\overbrace{x_k + \varepsilon d}^{\in x_0 + \mathcal{K}_k}) = \frac{1}{2}(x_k + \varepsilon d)^T A (x_k + \varepsilon d) - (x_k + \varepsilon d)^T b$

$$= \varphi(x_k) + \frac{\varepsilon^2}{2} \underbrace{d^T A d}_{>0} - \varepsilon d^T \underbrace{(b - Ax_k)}_{r_k}$$

$$0 \le \varepsilon \cdot \left[\frac{\varepsilon}{2} d^T A d - d^T r_k\right] = \underbrace{-\frac{1}{2 d^T A d}(d^T r_k)^2}_{< 0 \dots \text{spor}}$$

dosadíme-li $\varepsilon = \frac{d^T r_k}{d^T A d}$

Tedy $r_k \perp \mathcal{K}_k$ ▣ **Důsledek** Pokud $r_k \neq 0$, pak $r_k \in \mathcal{K}_{k+1}$

- Platí $\boxed{r_k \perp_A \mathcal{K}_{k-1}}$, tj $\forall d \in \mathcal{K}_{k-1} : r_k^T A d = 0$

Důkaz: $h = Ad \in \mathcal{K}_k \Rightarrow 0 = r_k^T h = r_k^T A d$ ▣

Důsledek: Připomeňme si ☼, kde jsme hledali

$$\underset{\overset{\cap}{\mathcal{K}_{i+1}}}{\overset{\perp_A \mathcal{K}_i}{p_{i+1}}} = \underset{\overset{\cap}{\mathcal{K}_{i+1}}}{\overset{\perp_A \mathcal{K}_{i-1}}{r_{i+1}}} + \overset{=0,\ \beta=0}{\beta_0} \underset{\overset{\cap}{\mathcal{K}_0}}{p_0} + \dots + \overset{=0}{\beta_{i-1}} \underset{\overset{\cap}{\mathcal{K}_{i-1}}}{p_{i-1}} + \beta_i \underset{\overset{\cap}{\mathcal{K}_i}}{p_i} = r_{i+1} + \beta_i p_i$$

Zbavili jsme se a) i b)

- **Efektivnější vzorce**

a) $r_{k+1} = b - Ax_{k+1} \overset{\clubsuit}{=} r_k - \alpha_k A p_k$

$$b - Ax_{k+1} = b - A(x_k + \alpha_k p_k) = b - Ax_k - \alpha_k A p_k = r_k - \alpha_k A p_k$$

b) $\alpha_k = \frac{r_0^T p_k}{p_k^T A p_k} = \frac{r_k^T p_k}{p_k^T A p_k} = \frac{r_k^T r_k}{p_k^T A p_k}$

$$r_k^T p_k \overset{\clubsuit}{=} (r_{k-1} - \alpha_{k-1} A p_{k-1})^T p_k = r_{k-1}^T p_{k-1} \overset{\clubsuit}{=} (r_{k-2} - \alpha_{k-2} A p_{k-2})^T p_k = r_{k-2}^T p_k \overset{\clubsuit}{=} \dots = r_0^T p_k$$

$$r_k^T p_k = r_k^T (r_k + \beta_{k-1} p_{i-1}) = r_k^T r_k + \beta_{k-1} \underbrace{r_k^T p_{i-1}}_{r_k \perp \mathcal{K}_k} = r_k^T r_k$$

b) $\beta_k = -\frac{p_k^T A r_{k+1}}{p_k^T A p_k} = \frac{r_{k+1}^T r_{k+1}}{r_k r_k}$

$$r_k^T r_{k-1} = (r_{k-1} - \alpha_{k-1} A p_{k-1})^T r_{k-1} = \|r_{k-1}\|^2 - \alpha_{k-1} p_{k-1}^T A r_{k-1}$$

$$p_k^T A p_k = p_k^T A (r_k + \beta_{k-1} p_{k-1}) = p_k^T A r_k = \frac{\|r_k\|^2}{\alpha_k}$$

$$-\frac{p_k^T A r_{k+1}}{p_k^T A p_k} = -\frac{\alpha_k \cdot r_{k+1}^T A p_k}{\|r_k\|^2} = \frac{\|r_{k+1}\|^2}{\|r_k\|^2}$$

$$r_{k+1}^T r_{k+1} = (r_k - \alpha_k A p_k) r_{k+1} = \underbrace{r_{k+1}^T r_k}_{0} - \alpha_k r_{k+1}^T A p_k$$

METODA SDRUŽENÝCH GRADIENTŮ

```
Vstup : $x_0, A, b, \varepsilon > 0$
$r_0 = b - A \cdot x_0$

$p_0 = r_0$
while $\|r_k\| \le \varepsilon \|b\|$ do
    $\alpha_k = (r_k^T r_k) / (p_k^T A p_k)$
    $x_{k+1} = x_k + \alpha_k p_k$
    $r_{k+1} = r_k - \alpha_k A p_k$

    $\beta_k = (r_{k+1}^T r_{k+1}) / (r_k^T r_k)$
    $p_{k+1} = r_{k+1} + \beta_k p_k$
    $k = k + 1$
end while
```

PŘEDPODMÍNĚNÁ METODA S.G.

```
Vstup : $x_0, A, b, P, \varepsilon > 0$
$r_0 = b - A \cdot x_0$
$z_0 = P^{-1} r_0$
$p_0 = z_0$
while $\|r_k\| \le \varepsilon \|b\|$ do
    $\alpha_k = (r_k^T z_k) / (p_k^T A p_k)$
    $x_{k+1} = x_k + \alpha_k p_k$
    $r_{k+1} = r_k - \alpha_k A p_k$
    $z_{k+1} = P^{-1} r_{k+1}$
    $\beta_k = (r_{k+1}^T z_{k+1}) / (r_k^T z_k)$
    $p_{k+1} = z_{k+1} + \beta_k p_k$
    $k = k + 1$
end while
```

$A^P = U^T D^P U$

Předpodmínění - místo $Ax = b$ zkusíme řešit

a) LEVÉ: $P^{-1}(Ax - b) = 0$

b) SYMETRICKÉ: $L^T A L \; L^{-1} x = L^T b$

c) PRAVÉ: $A P^{-1} P x = b$

$P^{-1} = LL^T$

$P^{-1} = LL^T$

Zkusme dosadit symetrické předpodmínění do algoritmu S.G.:

$$\underbrace{(L^TAL)}_{\bar A}\underbrace{L^{-1}x}_{\bar x} = \underbrace{L^Tb}_{\bar b}$$

$$\operatorname*{argmin}_{y\in x_0+\mathcal{K}_k} \frac{1}{2} y^TAy - y^Tb$$

$$\|$$

$$\operatorname*{argmin}_{\bar y\in L^{-1}x_0+L^{-1}\mathcal{K}_k} \frac{1}{2} \bar y^T\bar A\bar y - \bar y^T\bar b$$

$$z_k = P^{-T} r_k$$

$$\bar x = L^{-1}x$$

$$\bar r = \bar b - \bar A\bar x = L^T(b - ALL^{-1}x) = L^Tr$$

$$\boxed{\bar p = L^{-1}p}$$

0 krok: $\bar x_0 = L^{-1}x_0$

$\bar r_0 = L^Tr_0$

$\bar p_0 = \bar r_0 \;\dots\; L^{-1}p_0 = L^Tr_0 \;\dots\; p_0 = (LL^T)r_0 = \overbrace{P^{-1}r_0}^{z_0}$

i-tý krok: $\bar\alpha_k = (\bar r_k^T\bar r_k)/(\bar p_k^T\bar A\bar p_k) = (r_k^T\overbrace{LL^Tr_k}^{z_k})/p_k^T\underbrace{L^{-T}L^TALL^{-1}}_{A}p_k$

$= (r_k^Tz_k)/(p_k^TAp_k)$

$\bar x_{k+1} = \bar x_k + \bar\alpha_k\bar p_k \;\dots\; L^{-1}x_{k+1} = L^{-1}x_k + \bar\alpha_kL^{-1}p_k \;\dots\; x_{k+1} = x_k + \bar\alpha_kp_k$

$\bar r_{k+1} = \bar r_k - \bar\alpha_k\bar A\bar p_k \;\dots\; L^Tr_{k+1} = L^Tr_k - \bar\alpha_kL^TALL^{-1}p_k \;\dots\; r_{k+1} = r_k - \bar\alpha_kAp_k$

$\bar\beta_k = (\bar r_{k+1}^T\bar r_{k+1})/(\bar r_k^T\bar r_k) = (r_{k+1}^TLL^Tr_{k+1})/(r_k^TLL^Tr_k) = (r_{k+1}^Tz_{k+1})/(r_k^Tz_k)$

$\bar p_{k+1} = \bar r_{k+1} + \bar\beta_k\bar p_k \;\dots\; L^{-1}p_{k+1} = L^Tr_{k+1} + \bar\beta_kL^{-1}p_k \;\dots\; p_{k+1} = z_k + \bar\beta_kp_k$