NUMERICKÁ INTEGRACE

Chceme spočítat $\boxed{I(f,a,b)=\int_a^b f(x)\,dx}$ (kde $a<b\in\mathbb{R}$, $f: \langle a,b\rangle\to\mathbb{R}$ spojitá)

přibližnou metodou $\boxed{I_n(f)=\sum_{i=0}^{n} w_i\, f(x_i)}$ (I_n, w_i, x_i závisí na a,b)

x_i … integrační body, w_i … váhy

$a\le x_0<x_1<\dots<x_n\le b$ takovou, že: $\boxed{(\forall p\in P_n):\ I(p)=I_n(p)}$

Jelikož I i I_n jsou lineární, stačí ověřit pro libovolnou bázi P_n. Známe-li integrační body, dopočteme váhy:

$p_0,\dots,p_n$ … báze P_n $\Rightarrow$ $\begin{bmatrix} p_0(x_0) & p_0(x_1) & \dots & p_0(x_n)\\ \vdots & & & \\ p_n(x_0) & p_n(x_1) & \dots & p_n(x_n)\end{bmatrix}\begin{bmatrix} w_0\\ \vdots\\ w_n\end{bmatrix}=\begin{bmatrix} I(p_0)\\ \vdots\\ I(p_n)\end{bmatrix}$

$\begin{bmatrix} 1 & 1 & \dots & 1\\ x_0 & x_1 & \dots & x_n\\ (x_0)^n & (x_1)^n & \dots & (x_n)^n\end{bmatrix}$

• dosazením $p_0=1,\ p_1=x,\ p_n=x^n$ dostaneme Vandermondovu matici

• Dokažte, že volbou $p_i(x)=L_i^n(x)=\prod_{\substack{j=0\\ j\ne i}}^{n}\frac{x-x_j}{x_i-x_j}$ (Lagrangeova báze) dostaneme $\boxed{w_i=\int_a^b L_i^n(x)\,dx}$.

$f_n(x)=\sum_{i=0}^{n} f(x_i)\,L_i^n(x)$

Integrál $I(f_n)$ z Lagrangeovy interpolace fn funkce f je pak roven $\boxed{I(f_n)=\int_a^b\sum_{i=0}^{n} f(x_i)\,L_i^n(x)\,dx=\sum_{i=0}^{n}\underbrace{\left[\int_a^b L_i^n(x)\,dx\right]}_{w_i} f(x_i)=I_n(f)}$

D: Lagrangeova báze $p_i(x_j)=\begin{cases}1 & \dots\ i=j\\ 0 & \dots\ i\ne j\end{cases}$ $\begin{bmatrix}1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ 0 & 0 & \dots & 1\end{bmatrix}\begin{bmatrix}w_0\\ w_1\\ \vdots\\ w_n\end{bmatrix}=\begin{bmatrix}\int L_0\\ \int L_1\\ \vdots\end{bmatrix}$, $I\cdot w=w=\dots$ $\square$

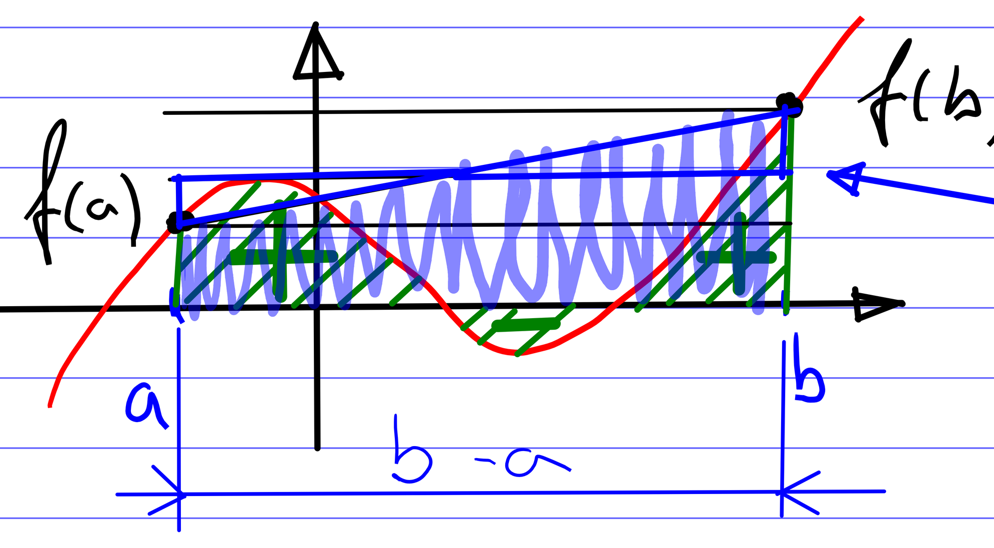
Zbývá určit body:

$\boxed{n=1}$ $a=x_0,\ x_1=b \implies h=b-a$

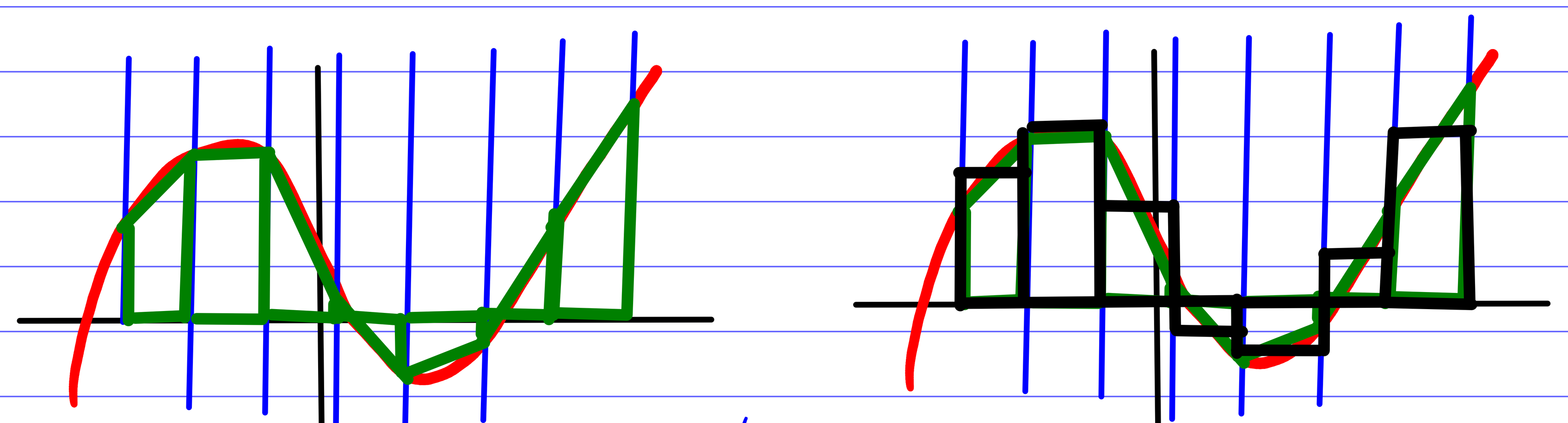
$\tilde w_0=\int_0^1\frac{t-1}{0-1}\,dt=-\left[\tfrac12(t-1)^2\right]_0^1=1/2$

$\tilde w_1=\int_0^1\frac{t-0}{1-0}\,dt=\left[\tfrac12 t^2\right]_0^1=1/2$

$I_1(f)=\frac{(b-a)}{2}\left[f(a)+f(b)\right]$



$\frac{f(a)+f(b)}{2}$



$I-I_1=\int_a^b f(x)\,dx-\int_a^b f_1(x)\,dx=\int_a^b\left(f(x)-f_1(x)\right)dx=$ (Lagrangeova interpolace)

$=\int_a^b(x-a)(x-b)\frac{f''(\xi)}{2!}\,dx$

$\boxed{f(x)-f_n(x)=(x-x_0)\cdots(x-x_n)\,\frac{f^{(n+1)}(\xi)}{(n+1)!}}$

$=-\frac12\int_a^b f''(\xi(x))\,\underbrace{(x-a)(b-x)}_{s'(x)>0}\,dx=-\frac12\int_{s(a)}^{s(b)} f''(\xi(x(s)))\,ds$

$s(x)=s$, $s'(x)\,dx=ds$

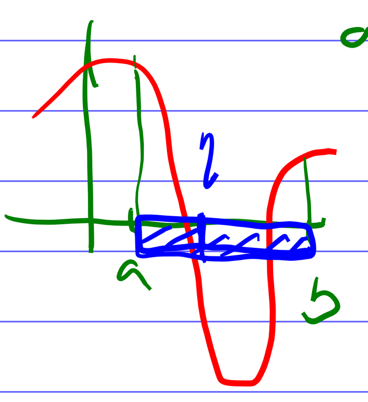
$=-\frac12 f''(\eta)\left[s(b)-s(a)\right]=\boxed{\frac{(b-a)^3}{12}f''(\eta)}$

$s(x)=\int(x-a)(b-x)\,dx=-\frac{x^3}{3}+\frac{(a+b)x^2}{2}-abx$ ($bx-x^2-ab+ax$)

$s'(x)>0$ na (a,b) $\Rightarrow$ s je na $\langle a,b\rangle$ rostoucí $\Rightarrow$ prostá

$\exists\, s^{-1}:\ s\to x$ …. lze provést $x(s)$

MA 1: $\int_a^b f(x)\,dx=(b-a)\cdot f(\eta)$, $\eta\in(a,b)$



$s(b)-s(a)=-\frac{b^3-a^3}{3}+\frac{(a+b)(b^2-a^2)}{2}-ab(b-a)$

$=(b-a)\left[-\frac{(b^2+ab+a^2)}{3}+\frac{(a+b)^2}{2}-ab\right]=\frac{-(b-a)^3}{6}$

$\boxed{n=0}$

$\boxed{n=2}$ $a=x_0,\ x_1=\frac{a+b}{2},\ x_2=b,\ h=(b-a)/2$ **SIMPSONOVO PRAVIDLO**

$\tilde w_0=\int_0^2\frac{x-\frac{a+b}{2}}{a-\frac{a+b}{2}}\cdot\frac{x-b}{a-b}\,dt=\dots=1/3$

$\tilde w_1=\int_0^2\frac{x-a}{\frac{a+b}{2}-a}\cdot\frac{x-b}{\frac{a+b}{2}-b}\,dt=\dots=4/3$

$\tilde w_2=\int_0^2\frac{x-a}{b-a}\cdot\frac{x-\frac{a+b}{2}}{b-\frac{a+b}{2}}\,dt=\dots=1/3$

$\boxed{I_2(x)=\frac{b-a}{6}\left[f(a)+4f\left(\tfrac{a+b}{2}\right)+f(b)\right]}$