MARS - 1. část projektu

1. [10 bodů] Řešte Cauchyovu úlohu:

$$\begin{cases} y'(t) &= \frac{t\left(1-y^2(t)\right)}{y(t)(1-t^2)} \\ y(\sqrt{2}) &= 2. \end{cases}$$

Řešení:

$$\frac{y(t)}{1-y^2(t)} \cdot y'(t) = \frac{t}{1-t^2} \qquad \left(t \notin \{-1,1\} \wedge t_0 = \sqrt{2}\right) \Rightarrow t \in (1,\infty)$$

$$\frac{1}{1+y^2(t)} \cdot y'(t) = \frac{1}{1+t^2}$$

$$\int \frac{t}{1-t^2}\,\mathrm{d}t = -\frac{1}{2}\ln(|1-t^2|) = -\frac{1}{2}\ln(t^2-1)$$

$$\int \frac{y(t)}{1-y^2(t)} \cdot y'(t)\,\mathrm{d}t = \left(\int \frac{z}{1-z^2}\,\mathrm{d}z\right)\Big|_{z=y(t)} = -\frac{1}{2}\ln(|1-y^2(t)|)$$

$$-\frac{1}{2}\ln(|1-y^2(t)|) = c - \frac{1}{2}\ln(t^2-1)$$

$$\ln(|1-y^2(t)|) = \ln(t^2-1) + d \qquad d = -2c$$

$$|1-y^2(t)| = K \cdot (t^2-1) \qquad K = \mathrm{e}^d \in \mathbb{R}_+$$

$$1-y^2(t) = L \cdot (t^2-1) \qquad L \in \mathbb{R}$$

$$y(t) = |y(t)| = \sqrt{1-L\cdot(t^2-1)} \qquad \text{jelikož } y(\sqrt{2}) = 2 > 0$$

C.Ú: $\quad 2 = y(\sqrt{2}) = \sqrt{1-L\cdot(2-1)} = \sqrt{1-L}$

$$L = -3$$

Tedy : $\quad y(t) = \sqrt{1+3(t^2-1)} = \underline{\sqrt{3t^2-2}} \qquad$ na $(1,\infty)$

Link na wolframalpha.

2. [10 bodů] Metodou variace konstanty řešte diferenciální rovnici:

$$y'(t) - \frac{t^2}{1+t^3} \cdot y(t) = \sqrt[3]{1+t^3}$$

Řešení:

$$y'(t) - \frac{t^2}{1+t^3} \cdot y(t) = \sqrt[3]{1+t^3} \qquad t \in (-1, \infty)$$

Homogenní rovnice: $y'_H(t) - \frac{t^2}{1+t^3} y_H(t) = 0$ $\qquad y_H(t) = 0$ je jedním z řeše

$$(\ln|y_H(t)|)' = \frac{1}{y_H(t)} y'_H(t) = \frac{t^2}{1+t^3} = \left(\frac{1}{3}\ln|1+t^3|\right)'$$

$$\ln|y_H(t)| = c + \frac{1}{3}\ln|1+t^3| \qquad c \in \mathbb{R}$$

$$|y_H(t)| = e^{c+\frac{1}{3}\ln|1+t^3|} = K|\sqrt[3]{1+t^3}| \qquad K = e^c \in \mathbb{R}_+$$

$$y_H(t) = k\sqrt[3]{1+t^3} \qquad k \in \mathbb{R}$$

Partikulární řešení: $y'_P(t) - \frac{t^2}{1+t^3} \cdot y_P(t) = \sqrt[3]{1+t^3}$ $\qquad$ kde $y_P(t) = k(t)\sqrt[3]{1+t^3}$

$$k'(t) \cdot \sqrt[3]{1+t^3} + k(t)\frac{3t^2}{3\sqrt[3]{(1+t^3)^2}} - \frac{t^2}{1+t^3} \cdot k(t)\sqrt[3]{1+t^3} = \sqrt[3]{1+t^3}$$

$$k'(t) = 1 = (t)'$$

$$y_P(t) = t\sqrt[3]{1+t^3}$$

Tedy $\quad y(t) = y_H(t) + y_P(t) = \underline{(t+k)\sqrt[3]{1+t^3}} \qquad k \in \mathbb{R},\ t \in (-1, \infty)$

Link na wolframalpha.