MARS - 1. část projektu

1. [10 bodů] Řešte Cauchyovu úlohu:
$$\begin{cases} y'(t) = \frac{1}{y(t)-1}\sqrt{4-t} \\ y(0) = 2. \end{cases}$$

2. [10 bodů] Řešte diferenciální rovnici:
$$y'(t) = 2t(t^2 + y(t)).$$

Řešení

1.

$$\begin{aligned}
y'(t) &= \frac{1}{y(t)-1}\sqrt{4-t} && \text{tedy } t \in (-\infty, 4) \\
\left(\frac{y^2(t)}{2} - y(t)\right)' &= (y(t)-1)\,y'(t) = \sqrt{4-t} = \left(-\frac{(4-t)^{\frac{3}{2}}}{\frac{3}{2}}\right)' \\
\frac{y^2(t)}{2} - y(t) &= c - \frac{2\,(4-t)^{\frac{3}{2}}}{3} && \text{kde } c \in (-\frac{1}{2}, \infty) \\
(y(t)-1)^2 - 1 = y^2(t) - 2y(t) &= 2c - \frac{4\,(4-t)^{\frac{3}{2}}}{3} && \text{jelikož } \begin{matrix} y(0) = 2 \\ y(t) \neq 1 \end{matrix}\text{, zajímá nás } y(t) > 1 \\
|y(t)-1| = y(t) - 1 &= \sqrt{1 + 2c - \frac{4}{3}\,(4-t)^{\frac{3}{2}}} && \text{označme } 1 + 2c = \frac{4}{3}k,\ k \in (0, \infty) \\
y(t) &= 1 + \frac{2}{\sqrt{3}}\sqrt{k - (4-t)^{\frac{3}{2}}} && \text{na } (4 - k^{\frac{2}{3}}, 4) \\
\text{C.Ú:} \qquad 2 = y(0) &= 1 + \frac{2}{\sqrt{3}}\sqrt{k - 4^{\frac{3}{2}}} = 1 + \frac{2}{\sqrt{3}}\sqrt{k-8} \\
\frac{3}{4} &= k - 8 \quad \Rightarrow \quad k = 8 + \frac{3}{4} \\
\text{Tedy :} \qquad y(t) &= 1 + \frac{2}{\sqrt{3}}\sqrt{8 + \frac{3}{4} - (4-t)^{\frac{3}{2}}} \\
&= \underline{1 + \frac{1}{\sqrt{3}}\sqrt{35 - 4\,(4-t)^{\frac{3}{2}}}} && \text{na } (4 - (\frac{35}{4})^{\frac{2}{3}}, 4)
\end{aligned}$$

Link na wolframalpha.

2.

$$\begin{aligned}
y'(t) &= 2t\left(t^2 + y(t)\right) = 2ty(t) + 2t^3 \\
\text{Homogenní rovnice:} \qquad y_H'(t) &= 2ty_H(t) && y_H(t) = 0 \text{ je jedním z řešení} \\
(\ln(y_H(t)))' = \frac{1}{y_H(t)} y_H'(t) &= 2t = \left(t^2\right)' \\
\ln(y_H(t)) &= c + t^2 && c \in \mathbb{R} \\
|y_H(t)| &= e^{c+t^2} = Ke^{t^2} && \text{označili jsme } e^c = K \in \mathbb{R}^+ \\
y_H(t) &= ke^{t^2} && \text{kde } k \in \mathbb{R} \\
\text{Partikulární řešení:} \qquad y_P'(t) &= 2t\left(t^2 + y_P(t)\right) && \text{kde } y_P(t) = k(t)e^{t^2} \\
k'(t)e^{t^2} + k(t)2te^{t^2} &= 2t^3 + 2tk(t)e^{t^2} \\
k'(t) &= 2t^3 e^{-t^2} && \text{zajímá nás jen jedno řešení} \\
k(t) &= 2\int t^3 e^{-t^2}\,\mathrm{d}t = -e^{-t^2}(t^2+1) \\
y_P(t) &= -e^{-t^2}(t^2+1)e^{t^2} = -(t^2+1) \\
\text{Tedy} \qquad y(t) &= y_H(t) + y_P(t) = \underline{ke^{t^2} - (t^2+1)} && k \in \mathbb{R},\ t \in \mathbb{R}
\end{aligned}$$

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