1. Vypočtěte integrály z racionálních funkcí

   (a) $f(x) = \frac{x+2}{(x^2-x+4)^2}$
   wolframalpha

2. Pomocí druhé substituční metody a uvedené subsituce vypočtěte neurčitý integrál funkce f:

   (a) $f(x) = \frac{1}{\sqrt{3+2x-x^2}}$ $\quad \frac{x-1}{2} = t$
   wolframalpha

   (b) $f(x) = \frac{\sqrt{1-x}}{x}$ $\quad \sqrt{1-x} = t$
   wolframalpha

   (c) $f(x) = \sqrt{\frac{1-x}{1+x}}$ $\quad \sqrt{\frac{1-x}{1+x}} = t$
   wolframalpha

3. Vypočtěte neurčitý integrál funkce f:

   (a) $f(x) = \frac{3}{x(\ln^3(2x)+1)}$
   wolframalpha

Řešení

1. (a)

$$\int \frac{x+2}{(x^2-x+4)^2}\mathrm{d}x = \int \frac{\frac{1}{2}(2x-1)+\frac{5}{2}}{(x^2-x+4)^2}\mathrm{d}x = \frac{1}{2}\int \frac{2x-1}{(x^2-x+4)^2}\mathrm{d}x + \frac{5}{2}\int \frac{1}{(x^2-x+4)^2}\mathrm{d}x =$$

$$\underbrace{=}_{\substack{t=x^2-x+4\\ \mathrm{d}t=(2x-1)\mathrm{d}x}} \frac{1}{2}\left(\int \frac{1}{t^2}\mathrm{d}t\right)_{|_{t=x^2-x+4}} + \frac{5}{2}\int \frac{1}{\left((x-\frac{1}{2})^2+\frac{15}{4}\right)^2}\mathrm{d}x =$$

$$= \frac{1}{2}\left(-t^{-1}\right)_{|_{t=x^2-x+4}} + \frac{5}{2}\int \frac{1}{\left(\frac{15}{4}\left[\frac{(x-\frac{1}{2})^2}{\frac{15}{4}}+1\right]\right)^2}\mathrm{d}x =$$

$$= \frac{-1}{2(x^2-x+4)} + \frac{5}{2}\cdot\frac{4^2}{15^2}\int \frac{1}{\left(\left(\frac{2x-1}{\sqrt{15}}\right)^2+1\right)^2}\mathrm{d}x =$$

$$\underbrace{=}_{\substack{u=\frac{2x-1}{\sqrt{15}}\\ \mathrm{d}u=\frac{2}{\sqrt{15}}\mathrm{d}x}} \frac{-1}{2(x^2-x+4)} + \frac{8}{9\cdot 5}\left(\int \frac{1}{\left(u^2+1\right)^2}\cdot\frac{\sqrt{15}}{2}\mathrm{d}u\right)_{|_{u=\frac{2x-1}{\sqrt{15}}}} =$$

$$= \frac{-1}{2(x^2-x+4)} + \frac{4}{3\sqrt{15}}\left(\int \frac{1}{\left(u^2+1\right)^2}\mathrm{d}u\right)_{|_{u=\frac{2x-1}{\sqrt{15}}}} =$$

$$= \frac{-1}{2(x^2-x+4)} + \frac{4}{3\sqrt{15}}\left(\frac{u}{2(1+u^2)} + \frac{1}{2}\mathrm{arctg}(u)\right)_{|_{u=\frac{2x-1}{\sqrt{15}}}} =$$

$$= \frac{-1}{2(x^2-x+4)} + \frac{4}{3\sqrt{15}}\left(\frac{\frac{2x-1}{\sqrt{15}}}{2(1+\frac{(2x-1)^2}{15})} + \frac{1}{2}\mathrm{arctg}\left(\frac{2x-1}{\sqrt{15}}\right)\right) =$$

$$= \frac{-1}{2(x^2-x+4)} + \frac{4x-2}{3}\underbrace{\frac{1}{15+(2x-1)^2}}_{4(x^2-x+4)} + \frac{2}{3\sqrt{15}}\mathrm{arctg}\left(\frac{2x-1}{\sqrt{15}}\right) =$$

$$= \frac{-3}{6(x^2-x+4)} + \frac{2x-1}{6(x^2-x+4)} + \frac{2}{3\sqrt{15}}\mathrm{arctg}\left(\frac{2x-1}{\sqrt{15}}\right)$$

$$= \underline{\frac{x-2}{3(x^2-x+4)} + \frac{2}{3\sqrt{15}}\mathrm{arctg}\left(\frac{2x-1}{\sqrt{15}}\right)} \text{ na } \mathbb{R},$$

kde jsme využili rekurentního vztahu z přednášek

$$\int \frac{1}{x^2+1}\mathrm{d}x = \mathrm{arctg}(x)\,; \qquad \left(\forall n\in\mathbb{N}\right):\ \int \frac{1}{\left(x^2+1\right)^{n+1}}\mathrm{d}x = \frac{x}{2n(1+x^2)^n} + \frac{2n-1}{2n}\int \frac{1}{\left(x^2+1\right)^n}\mathrm{d}x$$

2. (a) $3+2x-x^2 = -(x+1)(x-3) \geq 0 \Longrightarrow x \in \langle -1, 3\rangle$

$$\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x \underbrace{=}_{\substack{t=\frac{x-1}{2}\\ x=2t+1\\ \mathrm{d}x=2\mathrm{d}t}} \left(\int \frac{1}{\sqrt{3+2(2t+1)-(2t+1)^2}}\cdot 2\,\mathrm{d}t\right)_{|_{t=\frac{x-1}{2}}} = \left(\int \frac{1}{\sqrt{4-4t^2}}\cdot 2\,\mathrm{d}t\right)_{|_{t=\frac{x-1}{2}}} =$$

$$= \left(\int \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t\right)_{|_{t=\frac{x-1}{2}}} = (\arcsin(t))_{|_{t=\frac{x-1}{2}}} = \underline{\arcsin\left(\frac{x-1}{2}\right)} \text{ na } (-1,3)$$

(b) f je spojitá na $(-\infty, 0)$ a $(0, 1\rangle$ a

$$\int \frac{\sqrt{1-x}}{x}\,\mathrm{d}x \underbrace{=}_{\substack{t=\sqrt{1-x}\\ x=1-t^2\\ \mathrm{d}x=-2t\mathrm{d}t}} \left(\int \frac{t}{1-t^2}\cdot(-2t)\,\mathrm{d}t\right)_{|_{t=\sqrt{1-x}}} = \left(2\int \frac{-1+1-t^2}{1-t^2}\,\mathrm{d}t\right)_{|_{t=\sqrt{1-x}}} =$$

$$= \left(2\int\left(1+\frac{1}{t^2-1}\right)\,\mathrm{d}t\right)_{|_{t=\sqrt{1-x}}} = \left(2t+2\int\frac{1}{(t+1)(t-1)}\,\mathrm{d}t\right)_{|_{t=\sqrt{1-x}}} =$$

$$= \left(2t+\int\left(\frac{-1}{t+1}+\frac{1}{t-1}\right)\,\mathrm{d}t\right)_{|_{t=\sqrt{1-x}}} = (2t+-\ln(|t+1|)+\ln(|t-1|))_{|_{t=\sqrt{1-x}}} =$$

$$= \underline{2\sqrt{1-x}+\ln\left(\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right|\right)} \text{ na } (-\infty,0) \text{ a na } (0,1)$$

protože

$$\left(\frac{1}{(t+1)(t-1)} = \frac{A}{t+1}+\frac{B}{t-1}\right) \Longleftarrow \Big(1 = A(t-1)+B(t+1)\Big)$$

a dosazením $t=-1,1$ dostaneme $A=-\frac{1}{2}$, $B=\frac{1}{2}$

(c) $0 \leq \frac{1-x}{1+x} = \frac{2}{1+x}-1 \Longrightarrow 0 < 1+x \leq 2$, f je spojitá na $(-1, 1\rangle$ a

$$\int\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x \underbrace{=}_{\substack{t=\sqrt{\frac{1-x}{1+x}}\\ x=\frac{2}{t^2+1}+1\\ \mathrm{d}x=-\frac{4t}{(t^2+1)^2}\mathrm{d}t}} \left(\int t\cdot\left(-\frac{4t}{(t^2+1)^2}\right)\,\mathrm{d}t\right)_{|_{t=\sqrt{\frac{1-x}{1+x}}}} = -4\left(\int\frac{t^2+1-1}{(t^2+1)^2}\,\mathrm{d}t\right)_{|_{t=\sqrt{\frac{1-x}{1+x}}}} =$$

$$= -4\left(\int\left(\frac{1}{t^2+1}-\frac{1}{(t^2+1)^2}\right)\,\mathrm{d}t\right)_{|_{t=\sqrt{\frac{1-x}{1+x}}}} = \left(-4\mathrm{arctg}(t)+4\int\frac{1}{(t^2+1)^2}\,\mathrm{d}t\right)_{|_{t=\sqrt{\frac{1-x}{1+x}}}} =$$

$$= -4\mathrm{arctg}\left(\sqrt{\frac{1-x}{1+x}}\right)+4\left(\frac{t}{2(1+t^2)}+\frac{1}{2}\mathrm{arctg}(t)\right)_{|_{t=\sqrt{\frac{1-x}{1+x}}}} =$$

$$= -4\mathrm{arctg}\left(\sqrt{\frac{1-x}{1+x}}\right)+4\left(\frac{\sqrt{\frac{1-x}{1+x}}}{2(1+\frac{1-x}{1+x})}+\frac{1}{2}\mathrm{arctg}\left(\sqrt{\frac{1-x}{1+x}}\right)\right) =$$

$$= -2\mathrm{arctg}\left(\sqrt{\frac{1-x}{1+x}}\right)+2\frac{\sqrt{\frac{1-x}{1+x}}}{\frac{2}{1+x}} = \underline{(1+x)\sqrt{\frac{1-x}{1+x}}-2\mathrm{arctg}\left(\sqrt{\frac{1-x}{1+x}}\right)} \text{ na } (-1,1)$$

kde jsme opět využili rekurentního vztahu z přednášek

$$\int\frac{1}{x^2+1}\mathrm{d}x = \mathrm{arctg}(x)\,; \qquad \Big(\forall n\in\mathbb{N}\Big):\ \int\frac{1}{(x^2+1)^{n+1}}\mathrm{d}x = \frac{x}{2n(1+x^2)^n}+\frac{2n-1}{2n}\int\frac{1}{(x^2+1)^n}\mathrm{d}x$$

3. f je spojitá na $(0, \frac{e^{-1}}{2})$ a na $(\frac{e^{-1}}{2}, \infty)$. Na těchto intervalech platí

(a)

$$\int \frac{3}{x\left(\ln^3(2x)+1\right)}\,\mathrm{d}x \underbrace{=}_{\substack{t=\ln(2x)\\ \mathrm{d}t=\frac{1}{x}\mathrm{d}x}} \left(\int \frac{3}{(t^3+1)}\,\mathrm{d}t\right)_{|t=\ln(2x)} = \left(\int \frac{3}{(t+1)\left(t^2-t+1\right)}\,\mathrm{d}t\right)_{|t=\ln(2x)} =$$

$$= \left(\int \frac{1}{t+1} - \frac{t-2}{t^2-t+1}\,\mathrm{d}t\right)_{|t=\ln(2x)} = \left(\ln\left(|t+1|\right) + \int -\frac{1}{2}\frac{2t-1}{t^2-t+1} + \frac{\frac{3}{2}}{t^2-t+1}\,\mathrm{d}t\right)_{|t=\ln(2x)} =$$

$$\underbrace{=}_{\substack{u=t^2-t+1\\ \mathrm{d}u=(2t-1)\mathrm{d}t}} \ln\left(|\ln(2x)+1|\right) + \frac{1}{2}\left(-\left(\int \frac{1}{u}\mathrm{d}u\right)_{|u=t^2-t+1} + \int \frac{3}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}t\right)_{|t=\ln(2x)} =$$

$$= \ln\left(|\ln(2x)+1|\right) + \frac{1}{2}\left(-\Big(\ln(u)\Big)_{|u=t^2-t+1} + \int \frac{3}{\frac{3}{4}\left(\left(\frac{2}{\sqrt{3}}(t-\frac{1}{2})\right)^2+1\right)}\,\mathrm{d}t\right)_{|t=\ln(2x)} =$$

$$\underbrace{=}_{\substack{v=\frac{2}{\sqrt{3}}(t-\frac{1}{2})\\ \mathrm{d}v=\frac{2}{\sqrt{3}}\mathrm{d}t}} \ln\left(|\ln(2x)+1|\right) + \frac{1}{2}\left(-\ln(t^2-t+1) + \left(\int \frac{4\frac{\sqrt{3}}{2}}{(v^2+1)}\,\mathrm{d}v\right)_{|v=\frac{2}{\sqrt{3}}(t-\frac{1}{2})}\right)_{|t=\ln(2x)} =$$

$$= \ln\left(|\ln(2x)+1|\right) - \frac{1}{2}\ln\left(\ln^2(2x)-\ln(2x)+1\right) + \left(\sqrt{3}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right)\right)_{|t=\ln(2x)} =$$

$$= \underline{\ln\left(|\ln(2x)+1|\right) - \frac{1}{2}\ln\left(\ln^2(2x)-\ln(2x)+1\right) + \sqrt{3}\operatorname{arctg}\left(\frac{2\ln(2x)-1}{\sqrt{3}}\right)}$$