1. Pomocí první substituční metody vypočtěte neurčitý integrál funkce f:

   (a) $f(x) = \mathrm{cotg}(x)$
   wolframalpha

   (b) $f(x) = \left(\sin(\frac{x}{2}) - \cos(\frac{x}{2})\right)^2$
   wolframalpha

   (c) $f(x) = x\sqrt{x^2 - 2011}$
   wolframalpha

   (d) $f(x) = \frac{1}{x^2+3}$
   wolframalpha

   (e) $f(x) = \frac{\ln(x)-3}{x\sqrt{\ln(x)}}$
   wolframalpha

   (f) $f(x) = \frac{-3x^{-1}-9x^2}{\sqrt[4]{\ln(x)+x^3}},\ Df = (1, \infty)$
   wolframalpha

2. Pomocí metody per-partes vypočtěte neurčitý integrál funkce f:

   (a) $f(x) = x^2 e^x$
   wolframalpha

   (b) $f(x) = (x^2 + x - 2)\ln(x)$
   wolframalpha

Řešení

1. (a) f je spojitá na $(0+k\pi, \pi+k\pi)$, $k \in \mathbb{Z}$. Na každém z těchto intervalů platí

$$\int \operatorname{cotg}(x)\,\mathrm{d}x = \int \frac{\cos(x)}{\sin(x)}\,\mathrm{d}x = \int \frac{1}{\sin(x)} \cdot \sin'(x)\,\mathrm{d}x \underbrace{=}_{\substack{t \;=\; \sin(x) \\ \mathrm{d}t \;=\; \cos(x)\mathrm{d}x}} \left(\int \frac{1}{t}\,\mathrm{d}t\right)_{|t=\sin(x)} =$$

$$= (\ln(|t|))_{|t=\sin(x)} = \underline{\ln(|\sin(x)|)}$$

(b)

$$\int \left(\sin(\frac{x}{2}) - \cos(\frac{x}{2})\right)^2 \mathrm{d}x = \int \sin^2(\frac{x}{2}) - 2\sin(\frac{x}{2})\cos(\frac{x}{2}) + \cos^2(\frac{x}{2})\,\mathrm{d}x = \int 1 - 2\sin(\frac{x}{2})\cos(\frac{x}{2})\,\mathrm{d}x =$$

$$= x - 2\int \sin(\frac{x}{2})\left(2\sin(\frac{x}{2})\right)'\,\mathrm{d}x \underbrace{=}_{\substack{t \;=\; \sin(\frac{x}{2}) \\ \mathrm{d}t \;=\; \frac{1}{2}\cos(\frac{x}{2})\mathrm{d}x}} x - 4\left(\int t\,\mathrm{d}t\right)_{|t=\sin(\frac{x}{2})} =$$

$$= x - 4\frac{\sin^2(\frac{x}{2})}{2} = \underline{x - 2\sin^2(\frac{x}{2})} \text{ na } \mathbb{R}$$

nebo jinak:

$$\int 2\sin(\frac{x}{2})\cos(\frac{x}{2})\,\mathrm{d}x = \int \sin(x)\,\mathrm{d}x = -\cos(x)$$

a tedy za výsledek můžeme brát také například $\underline{x + \cos(x)}$. Skutečně:

$$\cos(x) - \left(-2\sin^2(\frac{x}{2})\right) = \cos(x) + 2\frac{1 - \cos(2\frac{x}{2})}{2} = 2,$$

což je konstanta.

(c) f je spojitá na $(-\infty, -\sqrt{2011}\rangle$ a na $\langle\sqrt{2011}, \infty)$ a tedy

$$\int x\sqrt{x^2 - 2011}\,\mathrm{d}x = \int \sqrt{x^2 - 2011} \cdot \underbrace{\frac{1}{2}(x^2 - 2011)'}_{x}\,\mathrm{d}x \underbrace{=}_{\substack{t \;=\; x^2 - 2011 \\ \mathrm{d}t \;=\; 2x\mathrm{d}x}} \left(\frac{1}{2}\int t^{\frac{1}{2}}\,\mathrm{d}t\right)_{|t=x^2-2011} =$$

$$= \left(\frac{1}{3}t^{\frac{3}{2}}\right)_{|t=x^2-2011} = \underline{\frac{\left(x^2 - 2011\right)^{\frac{3}{2}}}{3}} \text{ na } (-\infty, -\sqrt{2011}) \text{ a na } (\sqrt{2011}, \infty)$$

(d) f je spojitá na $\mathbb{R}$ a

$$\int \frac{1}{x^2 + 3}\,\mathrm{d}x = \int \frac{1}{3\left(\frac{x}{\sqrt{3}}\right)^2 + 3}\,\mathrm{d}x \underbrace{=}_{\substack{t \;=\; \frac{x}{\sqrt{3}} \\ \mathrm{d}t \;=\; \frac{1}{\sqrt{3}}\mathrm{d}x}} \left(\frac{\sqrt{3}}{3}\int \frac{1}{t^2 + 1}\,\mathrm{d}t\right)_{|t=\frac{x}{\sqrt{3}}} =$$

$$= \left(\frac{\sqrt{3}}{3}\operatorname{arctg}(t)\right)_{|t=\frac{x}{\sqrt{3}}} = \underline{\frac{1}{\sqrt{3}}\operatorname{arctg}(\frac{x}{\sqrt{3}})} \text{ na } \mathbb{R}$$

(e) f je spojitá na $(0,1)$ a na $(1,\infty)$ a dále

$$\int \frac{\ln(x) - 3}{x\sqrt{\ln(x)}}\,\mathrm{d}x \underbrace{=}_{\substack{t \;=\; \ln(x) \\ \mathrm{d}t \;=\; \frac{1}{x}\mathrm{d}x}} \left(\int \frac{t - 3}{t^{\frac{1}{2}}}\,\mathrm{d}t\right)_{|t=\ln(x)} = \left(\frac{2}{3}t^{\frac{3}{2}} - 6t^{\frac{1}{2}}\right)_{|t=\ln(x)} =$$

$$= \frac{2}{3}\ln^{\frac{3}{2}}(x) - 6\ln^{\frac{1}{2}}(x) = \underline{\frac{2}{3}\sqrt{\ln(x)}\,(\ln(x) - 9)} \text{ na } (0,1) \text{ a na } (1,\infty)$$

(f) f je spojitá na $(1,\infty)$ a dále

$$\int \frac{-3x^{-1}-9x^2}{\sqrt[4]{\ln(x)+x^3}}\,\mathrm{d}x \underbrace{=}_{\substack{t \;=\; \ln(x)+x^3 \\ \mathrm{d}t \;=\; \left(\frac{1}{x}+3x^2\right)\mathrm{d}x}} \left(-3\int t^{-\frac{1}{4}}\,\mathrm{d}t\right)_{|_{t=\ln(x)+x^3}} =$$

$$= \left(-3\frac{4}{3}t^{\frac{3}{4}}\right)_{|_{t=\ln(x)+x^3}} = \underline{-4\left(\ln(x)+x^3\right)^{\frac{3}{4}}}$$

2. (a) Funkce f je spojitá na $\mathbb{R}$ a

$$\int x^2e^x\,\mathrm{d}x \underbrace{=}_{\substack{u(x)=x^2 \implies u'(x)=2x \\ v'(x)=e^x \implies v(x)=e^x}} x^2e^x-2\int xe^x\,\mathrm{d}x \underbrace{=}_{\substack{u(x)=x \implies u'(x)=1 \\ v'(x)=e^x \implies v(x)=e^x}}$$

$$= x^2e^x-2\left(xe^x-\int e^x\,\mathrm{d}x\right) = \underline{e^x\left(x^2-2x+2\right)} \text{ na } \mathbb{R}$$

(b) Funkce f je spojitá na $Df=\mathbb{R}^+$ a

$$\int (x^2+x-2)\ln(x)\,\mathrm{d}x =$$

$$\underbrace{=}_{\substack{u(x)=\ln(x) \implies u'(x)=\frac{1}{x} \\ v'(x)=x^2+x-2 \implies v(x)=\frac{x^3}{3}+\frac{x^2}{2}-2x}} \left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)\ln(x)-\int \frac{\frac{x^3}{3}+\frac{x^2}{2}-2x}{x}\,\mathrm{d}x =$$

$$= \left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)\ln(x)+\int 2-\frac{x^2}{3}-\frac{x}{2}\,\mathrm{d}x = \left(\frac{x^3}{3}+\frac{x^2}{2}-2x\right)\ln(x)+2x-\frac{x^3}{9}-\frac{x^2}{4} =$$

$$= \underline{\frac{x}{36}\left[6\left(2x^2+3x-12\right)\ln(x)-4x^2-9x+72\right]} \text{ na } \mathbb{R}^+$$