1. Vypočtěte derivaci funkce f, určete $\mathrm{D}f$ a $\mathrm{D}f'$, pro funkci

   (a) $f(x) = \sqrt{1-\sin(x^2)}$
   http://www.wolframalpha.com/input/?i=sqrt%281-sin%28x^2%29%29
   http://www.wolframalpha.com/input/?i=differentiate+sqrt%281-sin%28x^2%29%29+wrt+x

   (b) $f(x) = x^{\cos(x)}$
   http://www.wolframalpha.com/input/?i=x^cos%28x%29
   http://www.wolframalpha.com/input/?i=differentiate+x^cos%28x%29+wrt+x

   (c) $f(x) = 5^{\left(x^2\right)} - (5^x)^2$
   http://www.wolframalpha.com/input/?i=5^%28x^2%29-%285^x%29^2
   http://www.wolframalpha.com/input/?i=differentiate+5^%28x^2%29-%285^x%29^2+wrt+x

   (d) $f(x) = x \arcsin\left(\frac{x}{x+1}\right) + \operatorname{arctg}\left(\sqrt{x}\right) - \sqrt{x}$
   http://www.wolframalpha.com/input/?i=x*asin%28x%2F%28x%2B1%29%29%2Batan%28sqrt%28x%29%29-sqrt%28x%29
   differentiatex*asin(x/(x+1))+atan(sqrt(x))-sqrt(x)wrtx

   (e) $f(x) = \sqrt{\sqrt{1-x^2}-1}$

2. Vypočti $f(\frac{\pi}{4}) - 3f'(\frac{\pi}{4})$ funkce $f(x) = \frac{\cos^2(x)}{1+\sin^2(x)}$.

3. Vypočtěte druhou derivaci funkce $f(x) = e^{-x}\sin(x)$.

4. Napiš rovnici tečny ke grafu funkce $f(x) = \frac{8}{4+x^2}$ v bodě $[x_0,\, y_0] = [2,\, ?]$.

5. Pod jakým úhlem se protínají křivky $\varphi_1:\ x^2 + y^2 - 5 = 0$ a $\varphi_2:\ y^2 - 4x = 0$?

Řešení

1. Budeme používat nekorektní, ale zato přehlednější zápis výpočtu derivace

   (a) $f'(x) = \left(\left(1-\sin(x^2)\right)^{\frac{1}{2}}\right)' = \frac{1}{2}\left(1-\sin(x^2)\right)^{-\frac{1}{2}}\left(1-\sin(x^2)\right)' = \frac{\cos(x^2)\overbrace{\left(x^2\right)'}^{2x}}{2\sqrt{1-\sin(x^2)}} = \underline{\frac{x\cos(x^2)}{\sqrt{1-\sin(x^2)}}}$

   $Df = \mathbb{R};\quad 1-\sin(x^2) \neq 0 \Longrightarrow x^2 \notin \bigcup_{k\in\mathbb{Z}}\left\{\frac{\pi}{2}+2k\pi\right\} \Longrightarrow Df' = \mathbb{R} \setminus \bigcup_{k\in\mathbb{N}\cup\{0\}}\left\{-\sqrt{\frac{\pi}{2}+2k\pi}, \sqrt{\frac{\pi}{2}+2k\pi}\right\}$

   (b) $f'(x) = \left(e^{\cos(x)\ln(x)}\right)' = e^{\cos(x)\ln(x)}\left(\cos(x)\ln(x)\right)' = x^{\cos(x)}\left(-\sin(x)\ln(x)+\cos(x)\frac{1}{x}\right) =$
   $= \underline{x^{\cos(x)}\left(\frac{\cos(x)}{x}-\sin(x)\ln(x)\right)};\qquad Df = Df' = \mathbb{R}^+$

   (c) $f'(x) = 5^{\left(x^2\right)}\ln(5)2x - 2(5^x)5^x\ln(5) = \underline{2\ln(5)\left(x5^{\left(x^2\right)}-\left(5^x\right)^2\right)};\qquad Df = Df' = \mathbb{R}$

   (d) $\left(-1 \le \overbrace{\frac{x}{x+1}}^{1-\frac{1}{x+1}} \le 1\right) \Longleftrightarrow \left(0 \le \frac{1}{x+1} \le 2\right) \Longleftrightarrow \left(x+1 \in \langle\frac{1}{2},\infty)\right) \Longleftrightarrow \left(x \in \langle-\frac{1}{2},\infty)\right)$

   $\left(-1 \le \frac{x}{x+1} \le 1\right) \wedge \left(0 \le x\right) \Longrightarrow \left(0 \le x\right) \Longrightarrow Df = \langle 0,\infty)$

   $\left(\forall x \in Df\right):\ f'(x) = \arcsin\left(\frac{x}{x+1}\right) + x\frac{1}{\sqrt{1-\left(\frac{x}{x+1}\right)^2}}\frac{x+1-x}{(x+1)^2} + \frac{1}{1+\left(\sqrt{x}\right)^2}\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x}} =$

   $= \arcsin\left(\frac{x}{x+1}\right) + \frac{x}{(x+1)\sqrt{(x+1)^2-x^2}} + \frac{1}{2\sqrt{x}}\left(\underbrace{\frac{1}{1+x}-1}_{\frac{-x}{1+x}}\right) = \underline{\arcsin\left(\frac{x}{x+1}\right) + \frac{x}{x+1}\left(\frac{1}{\sqrt{2x+1}}-\frac{1}{2\sqrt{x}}\right)}$

   $Df' = \mathbb{R}^+ = (0,\infty)$

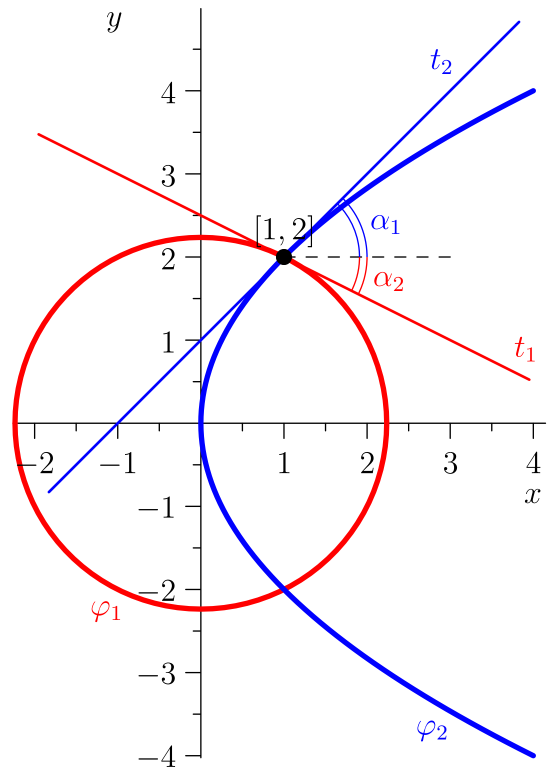
   (e) $\left(\sqrt{1-x^2}-1 \ge 0\right) \wedge \left(1-x^2 \ge 0\right) \Longrightarrow \left(|1-x^2| \ge 1\right) \wedge \left(x \in \langle-1,1\rangle\right) \Longrightarrow$
   $\Longrightarrow \left(-x^2 \ge 0\right) \wedge \left(x \in \langle-1,1\rangle\right) \Longrightarrow Df = \{0\} \Longrightarrow$Derivaci nemá smysl počítat, jelikož $Df' = \varnothing$

2. $Df = \mathbb{R};\qquad f'(x) = \frac{2\cos(x)(-\sin(x))\left(1+\sin^2(x)\right)-\cos^2(x)2\sin(x)\cos(x)}{(1+\sin^2(x))^2} = -2\sin(x)\cos(x)\frac{1+\sin^2(x)+\cos^2(x)}{(1+\sin^2(x))^2} =$
   $= \frac{-2\sin(2x)}{(1+\sin^2(x))^2} \Longrightarrow \underline{f(\frac{\pi}{4})-3f'(\frac{\pi}{4})} = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{1+\left(\frac{1}{\sqrt{2}}\right)^2} - 3\frac{-2\cdot 1}{\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^2} = \frac{\frac{1}{2}}{\frac{3}{2}} + 6\frac{1}{\left(\frac{3}{2}\right)^2} = \frac{1}{3}+\frac{8}{3} = \underline{3}$

3. $f''(x) = \left(e^{-x}(-1)\sin(x)+e^{-x}\cos(x)\right)' = \left(e^{-x}\left(\cos(x)-\sin(x)\right)\right)' =$
   $= e^{-x}(-1)\left(\cos(x)-\sin(x)\right) + e^{-x}\left(-\sin(x)-\cos(x)\right) = \underline{-2e^{-x}\cos(x)}$

4. $[x_0,\, y_0] = [2,\, f(2)] = [2,1];\qquad f'(x) = 8(-1)(4+x^2)^{-2}2x \implies f'(2) = -16\cdot 8^{-2} = -\frac{1}{2};$
   tečna v $[x_0, f(x_0)]$: $y - f(x_0) = f'(x_0)(x-x_0) \implies y-1 = \left(-\frac{1}{2}\right)(x-2) \implies \underline{y = -\frac{1}{2}x+2}$

5. Průsečíky: $\left(x^2+y^2-5=0\right) \wedge \left(y^2-4x=0\right) \Longrightarrow \left(x^2+4x-5=(x-1)(x+5)=0\right) \Longrightarrow \left((x=1)\wedge(x=-5)\right)$ a jelikož pro $x=-5$ nelze najít reálné y aby bod $[x,y] \in \varphi_2$ (tj $y^2-4(-5)=0$), průsečíky křivek mohou mít x-ovou souřadnici pouze 1. Potom z $y^2-4\cdot 1=0$ plyne, že možné průsečíky jsou pouze $[1,-2]$, $[1,2]$. Všimněme si, že $\left(\left([x,y]\in\varphi_1\right) \Rightarrow \left([x,-y]\in\varphi_1\right)\right) \wedge \left(\left([x,y]\in\varphi_2\right) \Rightarrow \left([x,-y]\in\varphi_2\right)\right)$ tedy obě křivky jsou symetrické podle osy y a úhel, ve kterém se protínají, je u obou průsečíků stejný. Zabývejme se tedy pouze průsečíkem $[x,y]=[1,2]$.



V jeho okolí je $y > 0$ a protože

$\varphi_1 :\ \left(x^2 + y^2 - 5 = 0\right) \Longrightarrow \left(y^2 = 5 - x^2\right) \overset{y>0}{\Longrightarrow} \left(y = \sqrt{5 - x^2}\right)$,

$\varphi_2 :\ \left(y^2 - 4x = 0\right) \Longrightarrow \left(y^2 = 4x\right) \overset{y>0}{\Longrightarrow} \left(y = \sqrt{4x}\right)$,

lze navíc křivky φ_1, φ_2 na tomto okolí nahradit funkcemi $\varphi_1 \equiv f_1$, $\varphi_2 \equiv f_2$, kde $f_1(x) = \sqrt{5 - x^2}$ a $f_2(x) = \sqrt{4x} = 2\sqrt{x}$. Dále je $f_1'(1) = \left(\frac{-x}{\sqrt{5-x^2}}\right)_{|x=1} = -\frac{1}{2}$, $f_2'(x) = \left(\frac{1}{\sqrt{x}}\right)_{|x=1} = 1$, tedy úhly z obrázku jsou$\alpha_1 = \left|\operatorname{arctg}(-\frac{1}{2})\right| = \operatorname{arctg}(\frac{1}{2})$, $\alpha_2 = \operatorname{arctg}(1) = \frac{\pi}{4}$.

Zadané křivky se protínají pod úhlem $\frac{\pi}{4} + \operatorname{arctg}(\frac{1}{2})$.