1. Vypočítejte (existují–li, v opačném případě doložte proč) limity funkcí

(a) $\lim\limits_{x\to-\infty} \frac{(5x-3)^{2000}(x+1)^{10}}{(3x-5)^{2010}}$

(b) $\lim\limits_{x\to-\infty} \frac{(5x-3)^{20}(x+1)^{10}}{(3x-5)^{2010}}$

(c) $\lim\limits_{x\to3} \frac{x-3}{x^2-2x-3}$

(d) $\lim\limits_{x\to-1} \frac{x-3}{x^2-2x-3}$

(e) $\lim\limits_{x\to-1} \frac{x+1}{|x+1|}$

(f) $\lim\limits_{x\to0} \frac{(1+px)^p-1}{x^3}$, v závislosti na parametru $p \in \mathbb{R}$.

(g) $\lim\limits_{x\to0} \frac{\sqrt{1-5x}-\sqrt{1+5x}}{x}$

(h) $\lim\limits_{x\to0} x^4 \sin\left(\frac{1}{x}\right)$

(i) $\lim\limits_{x\to0} \frac{1-\cos(x)}{x^2}$

(j) $\lim\limits_{x\to\frac{\pi}{4}} \frac{2\sin^2(x)-1}{\sin^2(x)+\sqrt{2}\sin(x)-\frac{3}{2}}$

(k) $\lim\limits_{x\to-\infty} \frac{1-x}{e^{2x}-1}$

(l) $\lim\limits_{x\to0} \frac{\pi-x}{e^{3x}-1}$

(m) $\lim\limits_{x\to0} \frac{\mathrm{tg}(x)\cdot\cos(3x)}{xe^{3x}(x-5)}$

(n) $\lim\limits_{x\to0} \frac{\sin(6x)}{\sin(5x)}$

(o) $\lim\limits_{x\to+\infty} \arcsin\left(\frac{x+3}{x^2+2x-15}\right)$

(p) $\lim\limits_{x\to0+} x^{\ln(x)}$

(q) $\lim\limits_{x\to\infty} x\left(\ln(x+1)-\ln(x)\right)$

(r) $\lim\limits_{x\to\infty} \left(1+\frac{7}{x}\right)^x$

Řešení

1. (a) $\lim\limits_{x\to-\infty} \frac{(5x-3)^{2000}(x+1)^{10}}{(3x-5)^{2010}} = \lim\limits_{x\to-\infty} \frac{x^{2010}\left(5-\frac{3}{x}\right)^{2000}\left(1+\frac{1}{x}\right)^{10}}{x^{2010}\left(3-\frac{5}{x}\right)^{2010}} = \lim\limits_{x\to-\infty} \frac{\left(5-\frac{3}{x}\right)^{2000}\left(1+\frac{1}{x}\right)^{10}}{\left(3-\frac{5}{x}\right)^{2010}} = \underline{\frac{5^{2000}}{3^{2010}}}$
wolframalpha

(b) $\lim\limits_{x\to-\infty} \frac{(5x-3)^{20}(x+1)^{10}}{(3x-5)^{2010}} = \lim\limits_{x\to-\infty} \frac{x^{30}\left(5-\frac{3}{x}\right)^{20}\left(1+\frac{1}{x}\right)^{10}}{x^{2010}\left(3-\frac{5}{x}\right)^{2010}} = \lim\limits_{x\to-\infty} \frac{\left(5-\frac{3}{x}\right)^{20}\left(1+\frac{1}{x}\right)^{10}}{x^{1980}\left(3-\frac{5}{x}\right)^{2010}} = \frac{5}{(-\infty)\cdot 3} = \underline{0}$
wolframalpha

(c) $\lim\limits_{x\to3} \frac{x-3}{x^2-2x-3} = \lim\limits_{x\to3} \frac{x-3}{(x-3)(x+1)} = \lim\limits_{x\to3} \frac{1}{x+1} = \underline{\frac{1}{4}}$
wolframalpha

(d) $\lim\limits_{x\to-1} \frac{x-3}{x^2-2x-3} = \lim\limits_{x\to-1} \frac{x-3}{(x-3)(x+1)} = \lim\limits_{x\to-1} \frac{1}{x+1}$ neexistuje, jelikož
$\lim\limits_{x\to-1-} \frac{1}{x+1} = \frac{1}{0-} = -\infty$ a $\lim\limits_{x\to-1+} \frac{1}{x+1} = \frac{1}{0+} = \infty$

(e) $\lim\limits_{x\to-1} \frac{x+1}{|x+1|}$ neexistuje, jelikož
$\left(\lim\limits_{x\to-1-} \frac{x+1}{|x+1|} = \lim\limits_{x\to-1-} \frac{x+1}{-(x+1)} = \lim\limits_{x\to-1-} (-1) = -1\right) \wedge \left(\lim\limits_{x\to-1+} \frac{x+1}{|x+1|} = \lim\limits_{x\to-1+} \frac{x+1}{x+1} = \lim\limits_{x\to-1+} 1 = 1\right)$

(f) $\lim\limits_{x\to0} \frac{(1+px)^p-1}{x^3} =$

(g) $\lim\limits_{x\to0} \frac{\sqrt{1-5x}-\sqrt{1+5x}}{x} = \lim\limits_{x\to0} \frac{\sqrt{1-5x}-\sqrt{1+5x}}{x} \cdot \frac{\sqrt{1-5x}+\sqrt{1+5x}}{\sqrt{1-5x}+\sqrt{1+5x}} = \lim\limits_{x\to0} \frac{\overbrace{1-5x-(1+5x)}^{-10x}}{x\left(\sqrt{1-5x}+\sqrt{1+5x}\right)} =$
$= \lim\limits_{x\to0} \frac{-10}{\sqrt{1-5x}+\sqrt{1+5x}} = \underline{-5}$

(h) $\lim\limits_{x\to0} x^4 \sin\left(\frac{1}{x}\right) = \underline{0}$, jelikož $0 \xleftarrow{x\to0} -x^4 \le x^4 \sin\left(\frac{1}{x}\right) \le x^4 \xrightarrow{x\to0} 0$

(i) $\lim\limits_{x\to0} \frac{1-\cos(x)}{x^2} = \lim\limits_{x\to0} \frac{1-\cos(x)}{x^2} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \lim\limits_{x\to0} \frac{\overbrace{1-\cos^2(x)}^{\sin^2(x)}}{x^2(1+\cos(x))} = \left(\lim\limits_{x\to0} \frac{\sin(x)}{x}\right)^2 \left(\lim\limits_{x\to0} \frac{1}{1+\cos(x)}\right) = \underline{\frac{1}{2}}$

(j) $\lim\limits_{x\to\frac{\pi}{4}} \frac{2\sin^2(x)-1}{\sin^2(x)+\sqrt{2}\sin(x)-\frac{3}{2}} = \lim\limits_{x\to\frac{\pi}{4}} \frac{2\left(\sin(x)-\frac{1}{\sqrt{2}}\right)\left(\sin(x)+\frac{1}{\sqrt{2}}\right)}{\left(\sin(x)-\frac{1}{\sqrt{2}}\right)\left(\sin(x)+\frac{3}{\sqrt{2}}\right)} = \lim\limits_{x\to\frac{\pi}{4}} \frac{2\left(\sin(x)+\frac{1}{\sqrt{2}}\right)}{\sin(x)+\frac{3}{\sqrt{2}}} = \frac{2\frac{2}{\sqrt{2}}}{\frac{4}{\sqrt{2}}} = \underline{1}$

(k) $\lim\limits_{x\to-\infty} \frac{1-x}{e^{2x}-1} = \frac{\infty}{-1} = \underline{-\infty}$

(l) $\lim\limits_{x\to0} \frac{\pi-x}{e^{3x}-1}$ neexistuje protože
$\left(\lim\limits_{x\to0-} \frac{\pi-x}{e^{3x}-1} = \frac{\pi}{0-} = -\infty\right) \wedge \left(\lim\limits_{x\to0+} \frac{\pi-x}{e^{3x}-1} = \frac{\pi}{0+} = \infty\right)$

(m) $\lim\limits_{x\to0} \frac{\operatorname{tg}(x)\cdot\cos(3x)}{xe^{3x}(x-5)} = \lim\limits_{x\to0} \frac{\sin(x)}{x} \frac{\cos(3x)}{\cos(x)e^{3x}(x-5)} = 1 \cdot \frac{1}{1\cdot1\cdot(-5)} = \underline{-\frac{1}{5}}$, protože
$\left(\lim\limits_{x\to0} 3x = 0\right) \wedge \left(e^y \text{ je spojitá v } 0\right) \wedge \left(\cos \text{ je spojitá v } 0\right) \Longrightarrow$
$\Longrightarrow \left(\lim\limits_{x\to0} e^{3x} = e^0 = 1\right) \wedge \left(\lim\limits_{x\to0} \cos(3x) = \cos(0) = 1\right)$

(n) $\lim\limits_{x\to0} \frac{\sin(6x)}{\sin(5x)} = \lim\limits_{x\to0} \frac{\frac{\sin(6x)}{6x}6x}{\frac{\sin(5x)}{5x}5x} = \lim\limits_{x\to0} \frac{\frac{\sin(6x)}{6x}6}{\frac{\sin(5x)}{5x}5} = \underline{\frac{6}{5}}$, protože
$\left(\lim\limits_{x\to0} 6x = 0\right) \wedge \left(\lim\limits_{x\to0} 5x = 0\right) \wedge \left(\lim\limits_{y\to0} \frac{\sin(y)}{y} = 1 \ \ldots \text{ z přednášek}\right) \wedge \left(\forall x \in \mathbb{R} \setminus \{0\} : 6x \ne 0 \ne 5x\right) \Longrightarrow$
$\Longrightarrow \left(\lim\limits_{x\to0} \frac{\sin(6x)}{6x} = 1\right) \wedge \left(\lim\limits_{x\to0} \frac{\sin(5x)}{5x} = 1\right)$

(o) $\lim\limits_{x\to+\infty} \arcsin\left(\frac{x+3}{x^2+2x-15}\right) = 0$, protože
$\left(\lim\limits_{x\to+\infty} \frac{x+3}{x^2+2x-15} = \lim\limits_{x\to+\infty} \frac{x^2\left(\frac{1}{x}+\frac{3}{x^2}\right)}{x^2\left(1+\frac{2}{x}-\frac{15}{x^2}\right)} = \lim\limits_{x\to+\infty} \frac{\frac{1}{x}+\frac{3}{x^2}}{1+\frac{2}{x}-\frac{15}{x^2}} = 0\right) \wedge \left(\arcsin \text{ je spojitá v } 0\right) \Longrightarrow$
$\Longrightarrow \left(\lim\limits_{x\to+\infty} \arcsin\left(\frac{x+3}{x^2+2x-15}\right) = \arcsin(0) = 0\right)$

(p) $\lim\limits_{x\to 0+} x^{\ln(x)} = \lim\limits_{x\to 0+} e^{\ln^2(x)} = \infty$, protože
$\left(\lim\limits_{x\to 0+} \ln^2(x) = \infty\right) \wedge \left(\lim\limits_{y\to\infty} e^y = \infty\right) \wedge \left(\big(\exists P^+(0,\delta)\big)\big(\forall x \in P^+(0,\delta)\big) : \ln^2(x) \neq \infty \ \ldots \text{ pro jakékoli } \delta \in \mathbb{R}^+\right)$

(q) $\lim\limits_{x\to\infty} x\left(\ln(x+1) - \ln(x)\right) = \lim\limits_{x\to\infty} x \ln\left(\frac{x+1}{x}\right) = \lim\limits_{x\to\infty} x \ln\left(\frac{x+1}{x}\right) = \lim\limits_{x\to\infty} \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} = \underline{1}$, protože
$\left(\lim\limits_{x\to\infty} \frac{1}{x} = 0\right) \wedge \left(\lim\limits_{y\to 0} \frac{\ln(1+y)}{y} = 1 \ \ldots \text{ známý vzorec}\right) \wedge \left(\big(\exists P(\infty,\delta)\big)\big(\forall x \in P(\infty,\delta)\big) : \frac{1}{x} \neq 0 \ \ldots \text{ např. } \delta = 1\right)$

(r) $\lim\limits_{x\to\infty} \left(1 + \frac{7}{x}\right)^x = \lim\limits_{x\to\infty} e^{x \ln\left(1+\frac{7}{x}\right)} = \lim\limits_{x\to\infty} e^{\frac{\ln\left(1+\frac{7}{x}\right)}{\frac{7}{x}} \cdot 7} = \underline{e^7}$, protože
$\left(\lim\limits_{x\to\infty} \frac{7}{x} = 0\right) \wedge \left(\lim\limits_{y\to 0} \frac{\ln(1+y)}{y} = 1\right) \wedge \left(\big(\exists P(\infty,\delta)\big)\big(\forall x \in P(\infty,\delta)\big) : \frac{1}{x} \neq 0 \ \ldots \text{ např. } \delta = 1\right)$