1. Dokažte z definice, že $\lim \frac{n-2}{n+3} = 1$ a najděte takový index n^*, od kterého dále jsou všechny členy posloupnosti v intervalu $(1 - 10^{-16}, 1 + 10^{-16})$, tj. nerozlišitelné od 1 při běžně používané přesnosti na počítačích.

2. Dokažte, že

$$\left.\begin{array}{r}\lim a_n = a \in \mathbb{R} \\ \lim b_n = b \in \mathbb{R}\end{array}\right\} \Longrightarrow \lim (a_n + b_n) = a + b$$

3. Vypočítejte (existují–li, v opačném případě doložte proč neexistují) limity posloupností

   (a) $\lim \frac{2008n^2 - 2009n + 2010}{10000 - 0.0001n^2}$

   (b) $\lim (\sqrt{2n+3} - \sqrt{n+2})$

   (c) $\lim (\sqrt{n+3} - \sqrt{n+2})$

   (d) $\lim \frac{5^n + (-3)^n}{5^{n+2} + (-3)^{n+3}}$

   (e) $\lim \frac{2^n + (-2)^n}{2^{n+2}}$

   (f) $\lim \sqrt[n]{5n}$

   (g) $\lim \sqrt[5n]{n}$

   (h) $\lim \left(\frac{2n-1}{n}\right)^{2n+1}$

   (i) $\lim \left(\frac{n-1}{n}\right)^{2n+1}$

   (j) $\lim \frac{\sin(n!)}{\sqrt{n}}$

   (k) $\lim \frac{2n + \sin(n)}{3n - 1}$

Řešení

1. Z definice limity

$$\lim \frac{n-2}{n+3} = 1 \Longleftrightarrow \big(\forall\varepsilon > 0\big)\big(\exists n_0 \in \mathbb{N}\big)\big(\forall n \in \mathbb{N}, n > n_0\big) : \left|\frac{n-2}{n+3} - 1\right| < \varepsilon$$

Navíc $\big(\forall\varepsilon > 0\big)$ a $n \in \mathbb{N}$ je

$$\left(\left|\frac{n-2}{n+3} - 1\right| < \varepsilon\right) \Longleftrightarrow \left(\varepsilon > \left|\frac{n-2-(n+3)}{n+3}\right| = \frac{5}{n+3}\right) \Longleftrightarrow \left(n+3 > \frac{5}{\varepsilon}\right)$$
$$\Longleftrightarrow \left(n > \frac{5}{\varepsilon} - 3\right) \Longleftarrow \left(n > \frac{5}{\varepsilon}\right) \Longleftarrow \left(n > \left\lceil\frac{5}{\varepsilon}\right\rceil \overset{def.}{=} n_0\right)$$

kde $\lceil x \rceil$ označujeme zaokrouhlení čísla x nahoru. Ukázali jsme, že

$$\big(\forall\varepsilon > 0\big)\left(\exists n_0(\varepsilon) = \left\lceil\frac{5}{\varepsilon}\right\rceil \in \mathbb{N}\right)\big(\forall n \in \mathbb{N}, n > n_0\big) : \left|\frac{n-2}{n+3} - 1\right| < \varepsilon$$

a tedy z definice $\lim \frac{n-2}{n+3} = 1$. Navíc $a_n = \frac{n-2}{n+3}$ je v intervalu $(1-\varepsilon, 1+\varepsilon) = (1-10^{-16}, 1+10^{-16})$ pro všechna $n \in \mathbb{N}$ takové, že $n > n^* = \lceil \frac{5}{10^{-16}} \rceil = 5 \cdot 10^{16} + 1$.

2. Z definice (konečné) limity posloupnosti plyne

$$(\lim a_n = a \in \mathbb{R}) \wedge (\lim b_n = b \in \mathbb{R}) \Longleftrightarrow$$
$$\Longleftrightarrow \text{současně platí} \left.\begin{array}{l} \big(\forall\varepsilon > 0\big)\big(\exists n_{0a} \in \mathbb{N}\big)\big(\forall n \in \mathbb{N},\, n > n_{0a}\big) : |a_n - a| < \varepsilon \\ \big(\forall\varepsilon > 0\big)\big(\exists n_{0b} \in \mathbb{N}\big)\big(\forall n \in \mathbb{N},\, n > n_{0b}\big) : |a_n - a| < \varepsilon \end{array}\right\} \Longrightarrow$$
$$\Longrightarrow \big(\forall\delta = 2\varepsilon > 0\big)\big(\exists n_0 = \max(n_{0a}, n_{0b}) \in \mathbb{N}\big)\big(\forall n \in \mathbb{N},\, n > n_0\big) : (|a_n - a| < \varepsilon) \wedge (|b_n - b| < \varepsilon) \Longrightarrow$$
$$\Longrightarrow \big(\forall\delta = 2\varepsilon > 0\big)\big(\exists n_0 = \max(n_{0a}, n_{0b}) \in \mathbb{N}\big)\big(\forall n \in \mathbb{N},\, n > n_0\big) :$$
$$|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \le |a_n - a| + |b_n - b| < \varepsilon + \varepsilon = \delta \Longleftrightarrow$$
$$\Longleftrightarrow \lim\,(a_n + b_n) = a + b$$

3. (a) $\lim \frac{2008n^2-2009n+2010}{10000-0.0001n^2} = \lim \frac{n^2\left(2008-\frac{2009}{n}+\frac{2010}{n^2}\right)}{n^2\left(\frac{10000}{n^2}-0.0001\right)} = \lim \frac{2008-\frac{2009}{n}+\frac{2010}{n^2}}{\frac{10000}{n^2}-0.0001} =$
$= \frac{\lim\, 2008 - \lim\, \frac{2009}{n} + \lim\, \frac{2010}{n^2}}{\lim\, \frac{10000}{n^2} - \lim\, 0.0001} = \frac{2008+0+0}{0-0.0001} = \underline{-20080000}$
wolframalpha

   (b) $\lim\,(\sqrt{2n+3} - \sqrt{n+2}) = (\lim\, \sqrt{n}) \cdot \left(\lim\,(\sqrt{2+\frac{3}{n}} - \sqrt{1+\frac{2}{n}})\right) = \infty \cdot \overbrace{(\sqrt{2}-1)}^{>0} = \infty$
wolframalpha

   (c) $\lim\,(\sqrt{n+3} - \sqrt{n+2}) = \lim\,(\sqrt{n+3} - \sqrt{n+2})\frac{\sqrt{n+3}+\sqrt{n+2}}{\sqrt{n+3}+\sqrt{n+2}} = \lim\, \frac{n+3-(n+2)}{\sqrt{n+3}+\sqrt{n+2}}$
$= \frac{\lim\, 1}{\lim\, \sqrt{n+3} + \lim\, \sqrt{n+2}} = \frac{1}{\infty} = \underline{0}$
wolframalpha

   (d) $\lim\, \frac{5^n+(-3)^n}{5^{n+2}+(-3)^{n+3}} = \lim\, \frac{5^n\left(1+(\frac{-3}{5})^n\right)}{5^n\left(5^2+(-3)^3(\frac{-3}{5})^n\right)} = \lim\, \frac{1+(\frac{-3}{5})^n}{5^2+(-3)^3(\frac{-3}{5})^n} = \frac{1+\lim\,(\frac{-3}{5})^n}{5^2+(-3)^3\lim\,(\frac{-3}{5})^n} = \underline{\frac{1}{25}}$
wolframalpha

   (e) $\lim\, \frac{2^n+(-2)^n}{2^{n+2}} = \lim\, \frac{2^n(1+(-1)^n)}{4\cdot 2^n} = \lim\, \left(\frac{1}{4}(1+(-1)^n)\right) \ldots$
hledaná limita neexistuje, protože $1+(-1)^n = \begin{cases} 0 & \ldots\ n \text{ je liché} \\ 2 & \ldots\ n \text{ je sudé} \end{cases}$

   (f) $\lim\, \sqrt[n]{5n} = \lim\, \left(\sqrt[n]{5n}\right)^{\frac{5}{5}} = \lim\, \left(\sqrt[5n]{5n}\right)^5 = \left(\lim\, \sqrt[5n]{5n}\right)^5 = 1^5 = \underline{1}$
protože $(\sqrt[5n]{5n})$ je vybraná z $(\sqrt[n]{n})$ a $\lim\, \sqrt[n]{n} = 1$

Alternativní postup: $\left(1 \longleftarrow 1 = \sqrt[n]{1} \leq \sqrt[n]{5n} \overbrace{\leq}^{n>4} \sqrt[n]{n^2} = (\sqrt[n]{n})^2 \longrightarrow 1\right) \Longrightarrow \left(\sqrt[n]{5n} \longrightarrow 1\right)$
wolframalpha

(g) $\lim \sqrt[5n]{n} = \lim (\sqrt[n]{n})^{\frac{1}{5}} = (\lim \sqrt[n]{n})^{\frac{1}{5}} = 1^{\frac{1}{5}} = \underline{1}$
wolframalpha

(h) $\lim \left(\frac{2n-1}{n}\right)^{2n+1} = \infty$, protože $\infty \longleftarrow 1.5^n \leq 1.5^{2n+1} \overbrace{\leq}^{n>2} \left(2 - \frac{1}{n}\right)^{2n+1} = \left(\frac{2n-1}{n}\right)^{2n+1}$
wolframalpha

(i) $\lim \left(\frac{n-1}{n}\right)^{2n+1} = \lim \left(\left(1-\frac{1}{n}\right)^n\right)^2 \left(1-\frac{1}{n}\right) = \left(\lim \left(1-\frac{1}{n}\right)^n\right)^2 \cdot \lim \left(1-\frac{1}{n}\right) = \left(e^{-1}\right)^2 \cdot 1 = \underline{e^{-2}}$,
protože $\lim \left(1-\frac{1}{n}\right)^n = \lim \left(\frac{n-1}{n}\right)^n \overset{n\geq 1}{=} \lim \left(\frac{1}{\frac{n}{n-1}}\right)^n = \frac{1}{\lim \left(\frac{n-1+1}{n-1}\right)^n} =$
$= \frac{1}{\lim \left(1+\frac{1}{n-1}\right)^{n-1} \lim \left(1+\frac{1}{n-1}\right)} = \frac{1}{e \cdot 1} = \underline{e^{-1}}$
jelikož pro $n > 2$ je $b_n = \left(1 + \frac{1}{n-1}\right)^{n-1}$ a $\left(b_{n+1} = a_n = \left(1+\frac{1}{n}\right)^n\right) \wedge (\lim a_n = e)$,
je $\lim b_n = \lim b_{n+1} = \lim a_n = e$
wolframalpha

(j) $\lim \frac{\sin(n!)}{\sqrt{n}} = \underline{0}$, protože $0 \longleftarrow \frac{-1}{\sqrt{n}} \leq \frac{\sin(n!)}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \longrightarrow 0$
wolframalpha

(k) $\frac{2}{3} \longleftarrow \frac{2(1-\frac{1}{n})}{3(1-\frac{1}{n})} = \frac{2n-1}{3n-1} \leq \frac{2n+\sin(n)}{3n-1} \leq \frac{2n+1}{3n-1} = \frac{2(1+\frac{1}{n})}{3(1-\frac{1}{n})} \longrightarrow \frac{2}{3}$ a tedy $\frac{2n+\sin(n)}{3n-1} \longrightarrow \underline{\frac{2}{3}}$
wolframalpha