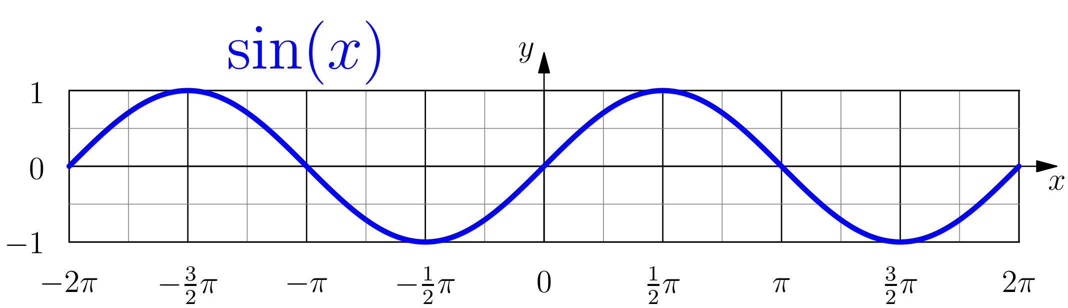
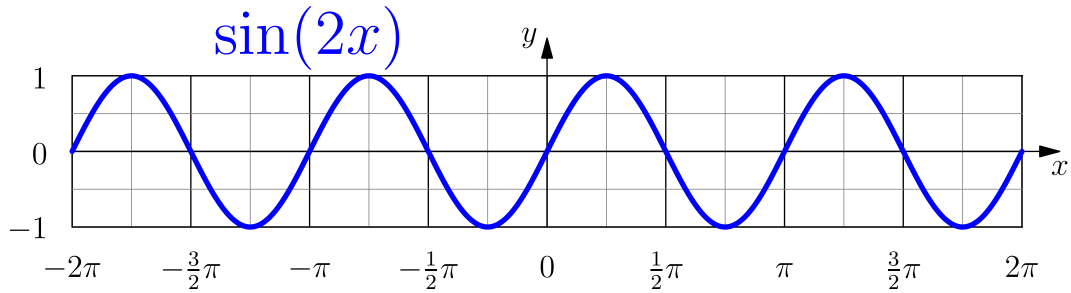
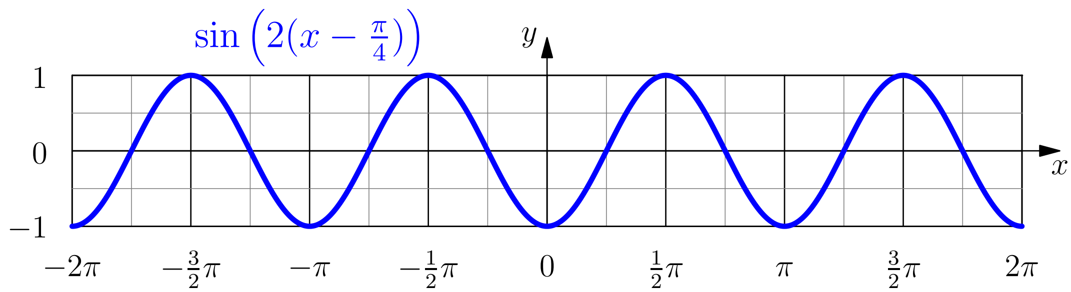
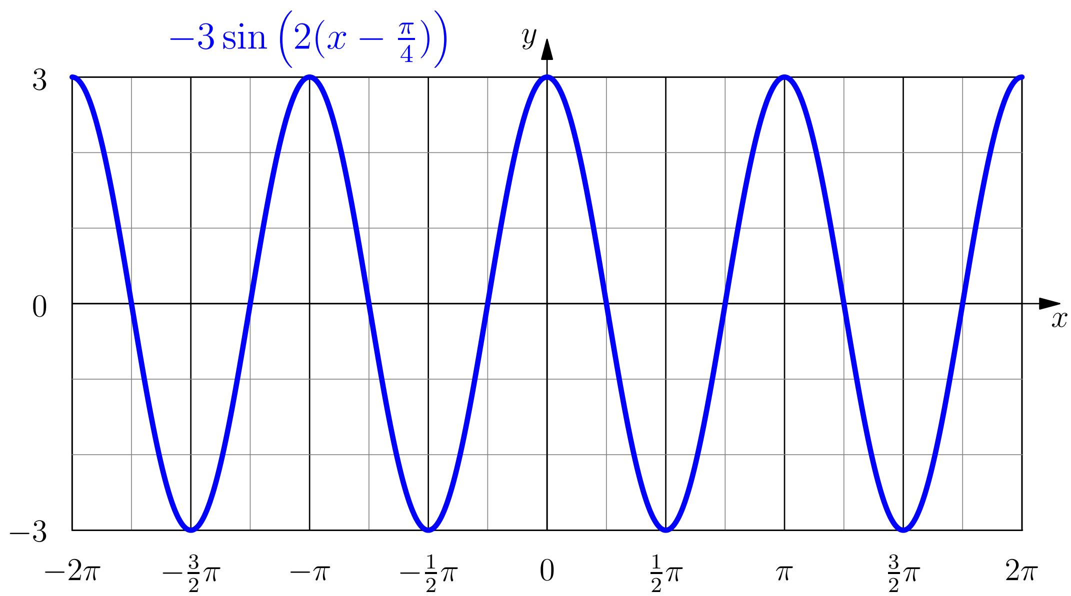
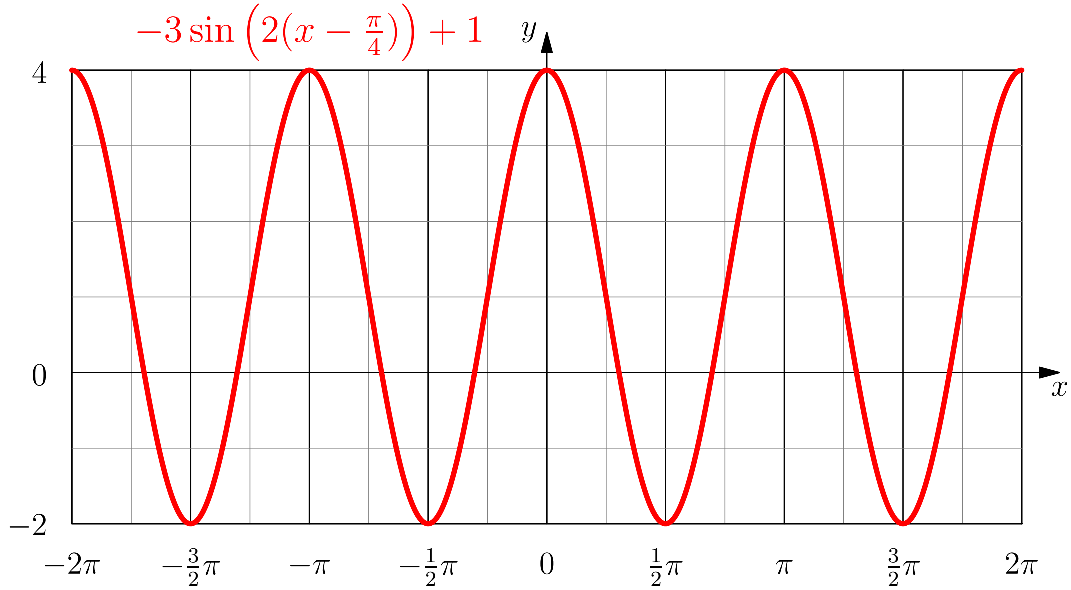
1. Určete Df funkce:
   (a) $f(x) \overset{def.}{=} \frac{3}{\log(2x-5)}$
   (b) $f(x) \overset{def.}{=} \frac{\sqrt{\sin(x)}}{\mathrm{tg}(2x)}$
   (c) $f(x) \overset{def.}{=} \sqrt{2\sin(x) - 1}$
2. Načrtni graf funkce f. Určete intervaly monotonie f. Zjistěte, zda je f sudá nebo lichá.
$$f(x) \overset{def.}{=} -3\sin(2x - \frac{\pi}{2}) + 1$$
3. Urči hodnoty zbývajících goniometrických funkcí, je-li:
$$\cos(x) = -\frac{8}{15} \quad \wedge \quad x \in (\frac{\pi}{2}; \pi) \ .$$
4. Je dáno $\sin(x) + \cos(x) = n$. Pomocí n vyjádřete výraz $\sin^3(x) + \cos^3(x)$.
5. Dokaž platnost identity:
$$\cos(x) + \sin(x)\mathrm{tg}(x) + \cos(x)\mathrm{cotg}(x) + \sin(x) = \frac{1}{\cos(x)} + \frac{1}{\sin(x)} \ .$$
6. Řeš soustavu:
$$\sin(x) > \cos(x) \quad \wedge \quad \mathrm{tg}(x) \leq \sqrt{3} \ .$$

Řešení

1. (a) $\left(\frac{3}{\log(2x-5)} \in \mathbb{R}\right) \Longrightarrow 2x + 5 > 0 \Longrightarrow \underline{Df = (-\frac{5}{2}; \infty)}$
   (b) $\left(\frac{\sqrt{\sin(x)}}{\mathrm{tg}(2x)} \in \mathbb{R}\right) \Longrightarrow (\sin(x) \geq 0) \wedge (\mathrm{tg}(2x) \neq 0) \wedge \left(2x \notin \bigcup_{k\in\mathbb{Z}} \left\{\frac{\pi}{2} + k\pi\right\}\right) \Longrightarrow$
   $\Longrightarrow x \in \bigcup_{k\in\mathbb{Z}} \langle 0 + 2k\pi; \pi + 2k\pi\rangle \wedge \left(2x \notin \bigcup_{k\in\mathbb{Z}} \{k\pi\}\right) \wedge \left(2x \notin \bigcup_{k\in\mathbb{Z}} \left\{\frac{\pi}{2} + k\pi\right\}\right) \Longrightarrow$
   $\Longrightarrow x \in \bigcup_{k\in\mathbb{Z}} \langle 0 + 2k\pi; \pi + 2k\pi\rangle \wedge \left(2x \notin \bigcup_{k\in\mathbb{Z}} \left\{k\frac{\pi}{2}\right\}\right) \Longrightarrow x \in \bigcup_{k\in\mathbb{Z}} \langle 0 + 2k\pi; \pi + 2k\pi\rangle \wedge \left(x \notin \bigcup_{k\in\mathbb{Z}} \left\{k\frac{\pi}{4}\right\}\right) \Longrightarrow$
   $\underline{Df = \bigcup_{k\in\mathbb{Z}} (0 + 2k\pi; \pi + 2k\pi) \setminus \left\{\frac{\pi}{4} + 2k\pi, \frac{\pi}{2} + 2k\pi, \frac{3\pi}{4} + 2k\pi\right\}}$
   (c) $\left(\sqrt{2\sin(x) - 1} \in \mathbb{R}\right) \Longrightarrow (2\sin(x) - 1 \geq 0) \Longrightarrow \left(\sin(x) \geq \frac{1}{2}\right) \Longrightarrow \underline{Df = \bigcup_{k\in\mathbb{Z}} \left\langle \frac{\pi}{6} + 2k\pi; \frac{5\pi}{6} + 2k\pi \right\rangle}$
2. •

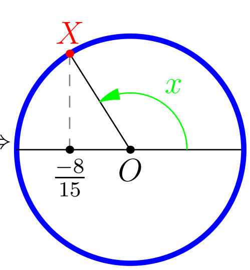


- 
$\sin(2x)$
y
1
0
-1
x
-2π
$-\frac{3}{2}\pi$
$-\pi$
$-\frac{1}{2}\pi$
0
$\frac{1}{2}\pi$
π
$\frac{3}{2}\pi$
2π
- 
$\sin\left(2(x-\frac{\pi}{4})\right)$
y
1
0
-1
x
-2π
$-\frac{3}{2}\pi$
$-\pi$
$-\frac{1}{2}\pi$
0
$\frac{1}{2}\pi$
π
$\frac{3}{2}\pi$
2π
- 
$-3\sin\left(2(x-\frac{\pi}{4})\right)$
y
3
0
-3
x
-2π
$-\frac{3}{2}\pi$
$-\pi$
$-\frac{1}{2}\pi$
0
$\frac{1}{2}\pi$
π
$\frac{3}{2}\pi$
2π

- 

- Nechť $k \in \mathbb{Z}$;
  - funkce sin (tj. $x \mapsto \sin(x)$) je rostoucí na intervalech $\left\langle -\frac{\pi}{2}+2k\pi; \frac{\pi}{2}+2k\pi \right\rangle$ a klesající na intervalech $\left\langle \frac{\pi}{2}+2k\pi; \frac{3\pi}{2}+2k\pi \right\rangle$
  - funkce $x \mapsto \sin(2x)$ je $\begin{cases} \text{rostoucí na intervalech} & \left\langle -\frac{\pi}{4}+k\pi; \frac{\pi}{4}+k\pi \right\rangle \\ \text{klesající na intervalech} & \left\langle \frac{\pi}{4}+k\pi; \frac{3\pi}{4}+k\pi \right\rangle \end{cases}$
  - funkce $x \mapsto \sin\left(2(x-\frac{\pi}{4})\right)$ je $\begin{cases} \text{rostoucí na intervalech} & \left\langle k\pi; \frac{\pi}{2}+k\pi \right\rangle \\ \text{klesající na intervalech} & \left\langle \frac{\pi}{2}+k\pi; \pi+k\pi \right\rangle \end{cases}$
  - funkce $x \mapsto -3\sin\left(2(x-\frac{\pi}{4})\right)$ je $\begin{cases} \text{rostoucí na intervalech} & \left\langle \frac{\pi}{2}+k\pi; \pi+k\pi \right\rangle \\ \text{klesající na intervalech} & \left\langle k\pi; \frac{\pi}{2}+k\pi \right\rangle \end{cases}$
  - funkce f, tj. $x \mapsto -3\sin\left(2(x-\frac{\pi}{4})\right)+1$ je $\begin{cases} \text{rostoucí na intervalech} & \left\langle \frac{\pi}{2}+k\pi; \pi+k\pi \right\rangle \\ \text{klesající na intervalech} & \left\langle k\pi; \frac{\pi}{2}+k\pi \right\rangle \end{cases}$

- $\underline{f(-x)} = -3\sin\left(2(-x-\frac{\pi}{4})\right)+1 = -3\sin\left(-2x-\frac{\pi}{2}\right)+1 \overbrace{=}^{\sin(y)=-\sin(y+\pi)} 3\sin\left(-2x+\frac{\pi}{2}\right)+1 =$ $= 3\sin\left(-2(x-\frac{\pi}{4})\right)+1 \underbrace{=}_{\sin(y)=-\sin(-y)} -3\sin\left(2(x-\frac{\pi}{4})\right)+1 = \underline{f(x)}$, tedy f je sudá.

3. $\left(\cos(x) = -\frac{8}{15} \wedge x \in (\frac{\pi}{2};\pi)\right) \Longrightarrow$ 

$\Longrightarrow \begin{cases} \sin(x) = \sqrt{1-(-\frac{8}{15})^2} = \sqrt{\frac{225-64}{225}} = \frac{\sqrt{161}}{15} \\ \operatorname{tg}(x) = \frac{\sin(x)}{\cos(x)} = -\frac{\sqrt{161}}{8} \\ \operatorname{cotg}(x) = (\operatorname{tg}(x))^{-1} = -\frac{8}{\sqrt{161}} \end{cases}$

4. $n^2 = (\sin(x)+\cos(x))^2 = \overbrace{\sin^2(x)+\cos^2(x)}^{1} + 2\sin(x)\cos(x) \Longrightarrow \sin(x)\cos(x) = \frac{n^2-1}{2}$

$$\underline{\sin^3(x)+\cos^3(x)} = (\sin(x)+\cos(x))^3 - 3\underbrace{\left(\sin^2(x)\cos(x)+\sin(x)\cos^2(x)\right)}_{\underbrace{\sin(x)\cos(x)}_{\frac{n^2-1}{2}}\underbrace{(\sin(x)+\cos(x))}_{n}} = \underbrace{n^3 - 3n\frac{n^2-1}{2}}_{\frac{n}{2}(2n^2-3n^2+3)} = \underline{\frac{n}{2}\left(3-n^2\right)}$$

5. $\underline{\cos(x)+\sin(x)\text{tg}(x)+\cos(x)\text{cotg}(x)+\sin(x)} = \cos(x)+\sin(x)\frac{\sin(x)}{\cos(x)}+\cos(x)\frac{\cos(x)}{\sin(x)}+\sin(x) =$
$= \frac{\sin(x)\left[\cos^2(x)+\sin^2(x)\right]+\cos(x)\left[\cos^2(x)+\sin^2(x)\right]}{\sin(x)\cos(x)} = \underline{\frac{1}{\cos(x)}+\frac{1}{\sin(x)}}$

6. $(\sin(x)>\cos(x)) \wedge \left(\text{tg}(x) \leq \sqrt{3}\right) \iff \left(x \in \bigcup_{k\in\mathbb{Z}} \left(\frac{\pi}{4}+2k\pi; \frac{5\pi}{4}+2k\pi\right)\right) \wedge \left(x \in \bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{2}+k\pi; \frac{\pi}{3}+k\pi\right\rangle\right) \iff$
$\iff \underline{x \in \bigcup_{k\in\mathbb{Z}} \left(\frac{\pi}{4}+2k\pi; \frac{\pi}{3}+2k\pi\right\rangle \cup \left(\frac{\pi}{2}+2k\pi; \frac{5\pi}{4}+2k\pi\right)}$