MA2PM 2011, test 2 - demo

Zadání

1. Najděte lokální extrémy funkce
$$f(x,y)=\frac{xy-y}{1+x^2+y^2}\ .$$

2. Najděte globální extrémy funkce f na množině M, kde
$$f(x,y)=\cos(x)+\cos(y)+\cos(x+y)$$
$$M=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, x\in\langle-\frac{\pi}{2},\frac{\pi}{2}\rangle \wedge y\in\langle-\frac{\pi}{2},\frac{\pi}{2}\rangle\right\}\ .$$

3. Určete vázané lokální extrémy funkce f vzhledem k množině M, jestliže
$$f(x,y)=x+\frac{1}{2}y\,,\ M=\{(x,y)\in\mathbb{R}^2:\ x^2+y^2=1\}\,.$$

4. Vypočtěte $\iint_M \frac{x^2}{y^2}\,dx\,dy$ kde M je uzavřená množina ohraničená křivkami
$$xy=1\,,\ y=x\,,\ x=2\,.$$

5. Pomocí substituce $x=r\cos^3(t)$, $y=r\sin^3(t)$ vypočtěte $\iint_M xy\,dx\,dy$ kde $M=\{(x,y)\in\mathbb{R}^2:\ x\geq 0\wedge y\geq 0\wedge \sqrt[3]{x^2}+\sqrt[3]{y^2}\leq\sqrt[3]{a^2}\}$ a konstanta $a\in(0,\infty)$.

Řešení

1.

Jelikož $f\in C^\infty(\mathbb{R}^2,\mathbb{R})$, lokální extrémy mohou být pouze ve stacionárních bodech:

$$0=\frac{\partial f}{\partial x}(x,y)=\frac{y(1+x^2+y^2)-(xy-y)2x}{(1+x^2+y^2)^2}=$$
$$=\frac{y(1-x^2+y^2+2x)}{(1+x^2+y^2)^2}\Rightarrow 0=y(1-x^2+y^2+2x)$$
$$0=\frac{\partial f}{\partial y}(x,y)=\frac{(x-1)(1+x^2+y^2)-(xy-y)2y}{(1+x^2+y^2)^2}=$$
$$=\frac{(x-1)(1+x^2-y^2)}{(1+x^2+y^2)^2}\Rightarrow 0=(x-1)(1+x^2-y^2)$$

Tedy:

$$y = 0 \Rightarrow \begin{cases} x - 1 = 0 & \Rightarrow (1,0) \\ 1 + x^2 - 0^2 = 0 & \Rightarrow \emptyset \end{cases}$$

$$1 - x^2 + y^2 + 2x = 0 \Rightarrow \begin{cases} x - 1 = 0 & \Rightarrow \emptyset \\ 1 + x^2 - y^2 = 0 & \Rightarrow \text{diskuse dále} \ . \end{cases}$$

Zbývá prodiskutovat $(1 - x^2 + y^2 + 2x = 0) \wedge (1 + x^2 - y^2 = 0)$, tedy

$$(1 + 2x = x^2 - y^2) \wedge (x^2 - y^2 = -1) \Rightarrow (-1, \sqrt{2}), (-1, -\sqrt{2}) \ .$$

Spočtěme druhé parciální derivace

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\left(\frac{y(1 - x^2 + y^2 + 2x)}{(1 + x^2 + y^2)^2}\right) \\
&= y\frac{(-2x + 2)(1 + x^2 + y^2)^2 - (1 - x^2 + y^2 + 2x)2(1 + x^2 + y^2)2x}{(1 + x^2 + y^2)^4} = \\
&= 2y\frac{(1 - x)(1 + x^2 + y^2) - 2x(1 - x^2 + y^2 + 2x)}{(1 + x^2 + y^2)^3} \\
\frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{\partial}{\partial y}\left(\frac{(x - 1)(1 + x^2 - y^2)}{(1 + x^2 + y^2)^2}\right) = \\
&= (x - 1)\frac{-2y(1 + x^2 + y^2)^2 - (1 + x^2 - y^2)2(1 + x^2 + y^2)2y}{(1 + x^2 + y^2)^4} = \\
&= 2y(1 - x)\frac{3 + 3x^2 - y^2}{(1 + x^2 + y^2)^3} \\
\frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial}{\partial x}\left(\frac{(x - 1)(1 + x^2 - y^2)}{(1 + x^2 + y^2)^2}\right) = \\
&= \frac{1(1 + x^2 - y^2)}{(1 + x^2 + y^2)^2} + (x - 1)\frac{2x(1 + x^2 + y^2)^2 - (1 + x^2 - y^2)2(1 + x^2 + y^2)2x}{(1 + x^2 + y^2)^4} = \\
&= \frac{1 + x^2 - y^2}{(1 + x^2 + y^2)^2} + 2x(1 - x)\frac{1 + x^2 - 3y^2}{(1 + x^2 + y^2)^3}
\end{aligned}$$

Pro $(1,0)$ je

$$\left(d^2 f\right)_{(1,0)}(h_1, h_2) = [h_1, h_2]\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = [h_1, h_2]\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

a pro $(x,y) \in \left\{(-1, \sqrt{2}), (-1, -\sqrt{2})\right\}$ je

$$\begin{aligned}
1 + x^2 + y^2 = 4, \qquad 1 + x^2 - y^2 = 0, \qquad 1 + x^2 - 3y^2 = -4 \\
1 - x = 2, \qquad 3 + 3x^2 - y^2 = 4 \ ,
\end{aligned}$$

tedy

$$\left(d^2f\right)_{(-1,\sqrt{2})}(h_1,h_2) = [h_1,h_2]\begin{bmatrix}\frac{\sqrt{2}}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{\sqrt{2}}{4}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix} = [h_1,h_2]\frac{1}{4}\begin{bmatrix}\sqrt{2} & 1\\ 1 & \sqrt{2}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix}$$

$$\left(d^2f\right)_{(-1,-\sqrt{2})}(h_1,h_2) = [h_1,h_2]\begin{bmatrix}-\frac{\sqrt{2}}{4} & \frac{1}{4}\\ \frac{1}{4} & -\frac{\sqrt{2}}{4}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix} = [h_1,h_2]\frac{1}{4}\begin{bmatrix}-\sqrt{2} & 1\\ 1 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix} .$$

Spočteme vlastní čísla matic:

$$\begin{vmatrix}\lambda_a & 1\\ 1 & \lambda_a\end{vmatrix} = \lambda_a^2 - 1 = 0$$

$$\begin{vmatrix}\sqrt{2}-\lambda_b & 1\\ 1 & \sqrt{2}-\lambda_b\end{vmatrix} = (\sqrt{2}-\lambda)^2 - 1 = \lambda_b^2 - 2\sqrt{2}\lambda_b + 1 = 0$$

$$\begin{vmatrix}-\sqrt{2}-\lambda_c & 1\\ 1 & -\sqrt{2}-\lambda_c\end{vmatrix} = (\sqrt{2}+\lambda)_b^2 - 1 = \lambda_c^2 + 2\sqrt{2}\lambda_c + 1 = 0 \ ,$$

$$\lambda_a = \begin{cases}1 & > 0\\ -1 & < 0\end{cases}$$

$$\lambda_b = \frac{2\sqrt{2} \pm \sqrt{8-4}}{2} = \begin{cases}\sqrt{2}+1 & > 0\\ \sqrt{2}-1 & > 0\end{cases}$$

$$\lambda_c = \frac{-2\sqrt{2} \pm \sqrt{8-4}}{2} = \begin{cases}-\sqrt{2}+1 & < 0\\ -\sqrt{2}-1 & < 0\end{cases}$$

tedy f má sedlový bod v $(-1,0)$, ostré lokální minimum v $(-1,\sqrt{2})$ a ostré lokální maximum v $(-1,\sqrt{2})$.

2.

Poznamenejme, že $f \in C^\infty(\mathbb{R}^2,\mathbb{R})$ a nabývá na uzavřené omezené množině M maxima a minima. Navíc

$$\partial M = N_1 \cup N_2 \cup N_3 \cup N_4$$

$$N_1 = \left\{(-\frac{\pi}{2},t)\middle|\ t \in \langle -\frac{\pi}{2},\frac{\pi}{2}\rangle\right\}$$

$$N_2 = \left\{(t,-\frac{\pi}{2})\middle|\ t \in \langle -\frac{\pi}{2},\frac{\pi}{2}\rangle\right\}$$

$$N_3 = \left\{(\frac{\pi}{2},t)\middle|\ t \in \langle -\frac{\pi}{2},\frac{\pi}{2}\rangle\right\}$$

$$N_4 = \left\{(t,\frac{\pi}{2})\middle|\ t \in \langle -\frac{\pi}{2},\frac{\pi}{2}\rangle\right\} \ .$$

- lokální extrémy uvnitř M:

$$0 = \frac{\partial f}{\partial x}(x,y) = -\sin(x) - \sin(x+y)$$
$$0 = \frac{\partial f}{\partial y}(x,y) = -\sin(y) - \sin(x+y) \ ,$$

Tedy pro stacionární body $(x,y) \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle^2$ platí $\sin(x) = \sin(y) = -\sin(x+y)$. Jelikož funkce sin je na $\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$ prostá, tak

$$(x,y) \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle^2 \wedge \sin(x) = \sin(y) \ \Rightarrow \ x = y$$
$$(x,y) \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle^2 \wedge x = y \wedge \sin(x) = -\sin(x+y) \ \Rightarrow \ (x,y) = (0,0) \ .$$

Pro úplnost dodejme, že

$$\frac{\partial^2 f}{\partial x^2}(0,0) = (-\cos(x) - \cos(x+y))_{|(x,y)=(0,0)} = -2$$
$$\frac{\partial^2 f}{\partial y^2}(0,0) = (-\cos(y) - \cos(x+y))_{|(x,y)=(0,0)} = -2$$
$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = (-\cos(x+y))_{|(x,y)=(0,0)} = -1$$
$$\left(d^2 f\right)_{(0,0)}(h_1,h_2) = [h_1,h_2] \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

a

$$\begin{vmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1 = \lambda^2 + 4\lambda + 1 = \left(\lambda - \underbrace{(-2+\sqrt{3})}_{<0}\right)\left(\lambda - \underbrace{(-2-\sqrt{3})}_{<0}\right) \ ,$$

tedy v $(0,0)$ má f ostré lokální maximum $f(0,0) = 3$.

- krajní body $N_1, \ldots, N_4$:

$$f(-\frac{\pi}{2}, -\frac{\pi}{2}) = -1$$
$$f(\frac{\pi}{2}, -\frac{\pi}{2}) = 1$$
$$f(-\frac{\pi}{2}, \frac{\pi}{2}) = 1$$
$$f(\frac{\pi}{2}, \frac{\pi}{2}) = -1 \ .$$

- uvnitř $N_1, \ldots, N_4$:

$$N_1 : \ g(t) := f(-\frac{\pi}{2}, t) = \cos(t) + \cos(t - \frac{\pi}{2}) \ \Rightarrow \ 0 = g'(t) = -\sin(t) - \sin(t - \frac{\pi}{2})$$
$$N_2 : \ g(t) := f(t, -\frac{\pi}{2}) = \cos(t) + \cos(t - \frac{\pi}{2}) \ \Rightarrow \ 0 = g'(t) = -\sin(t) - \sin(t - \frac{\pi}{2})$$
$$N_3 : \ g(t) := f(\frac{\pi}{2}, t) = \cos(t) + \cos(t + \frac{\pi}{2}) \ \Rightarrow \ 0 = g'(t) = -\sin(t) - \sin(t + \frac{\pi}{2})$$
$$N_4 : \ g(t) := f(t, \frac{\pi}{2}) = \cos(t) + \cos(t + \frac{\pi}{2}) \ \Rightarrow \ 0 = g'(t) = -\sin(t) - \sin(t + \frac{\pi}{2})$$

Jelikož pro $t \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$ je

$$\sin(t) = -\sin(t - \frac{\pi}{2}) \ \Rightarrow t = \frac{\pi}{4}$$
$$\sin(t) = -\sin(t + \frac{\pi}{2}) \ \Rightarrow t = -\frac{\pi}{4}$$

jsou $(-\frac{\pi}{2}, \frac{\pi}{4})$, $(\frac{\pi}{4}, -\frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{4})$, $(-\frac{\pi}{4}, \frac{\pi}{2})$, stacionární body na $N_1, \dots, N_4$ a

$$f(-\frac{\pi}{2}, \frac{\pi}{4}) = f(\frac{\pi}{4}, -\frac{\pi}{2}) = f(\frac{\pi}{2}, -\frac{\pi}{4}) = f(-\frac{\pi}{4}, \frac{\pi}{2}) = \sqrt{(2)}$$

- odpověď:

$$\min_{(x,y)\in M} f(x,y) = f(-\frac{\pi}{2}, -\frac{\pi}{2}) = f(\frac{\pi}{2}, \frac{\pi}{2}) = -1$$
$$\max_{(x,y)\in M} f(x,y) = f(0,0) = 3 \ .$$

3.

Množinu M lze parametrizovat

$$M = \left\{ (\cos(t), \sin(t)) \in \mathbb{R}^2 \middle| \ t \in \langle -\pi, \pi \rangle \right\} .$$

Tedy hledáme extrémy periodické funkce $g(t) = f(\cos(t), \sin(t)) = \cos(t) + \frac{\sin(t)}{2}$, $t \in \mathbb{R}$:

$$g(\pi) = g(-\pi) = -1 \, , \ 0 = g'(t) = -\sin(t) + \frac{\cos(t)}{2} \ \Rightarrow \ \sin(t) = \frac{\cos(t)}{2}$$

jelikož pro žádné $t \in \mathbb{R}$ neplatí $0 = \cos(t) = \sin(t)$, ve stacionárních bodech je nutně $\cos(t) \neq 0$. Pak

$$(\forall t \in \langle -\pi, \pi \rangle) : \ 0 = g'(t) \ \Rightarrow \ \operatorname{tg}(t) = \frac{1}{2} \ \Rightarrow \ t \in \left\{ \operatorname{arctg}(\frac{1}{2}), \operatorname{arctg}(\frac{1}{2}) - \pi \right\} .$$

Pro $t_1 = \operatorname{arctg}(\frac{1}{2})$ platí, že $\sin(t_1) > 0$ a $\cos(t_1) > 0$, tedy

$$\sin(t_1) = \frac{\cos(t_1)}{4} \ \Rightarrow \begin{cases} 1 - \cos^2(t_1) = \frac{\cos^2(t_1)}{4} \ \Rightarrow \ 1 = \frac{5}{4}\cos^2(t_1) & \Rightarrow \ \cos(t_1) = \frac{2}{\sqrt{5}} \\ \sin^2(t_1) = \frac{1 - \sin^2(t_1)}{4} \ \Rightarrow \ \frac{5}{4}\sin^2(t_1) = \frac{1}{4} & \Rightarrow \ \sin(t_1) = \frac{1}{\sqrt{5}} \end{cases}$$

a pro $t_2 = \operatorname{arctg}(\frac{1}{2}) - \pi$ je

$$\cos(t_2) = \cos(t_1 - \pi) = -\cos(t_1) \Rightarrow \ \cos(t_2) = -\frac{2}{\sqrt{5}}$$
$$\sin(t_2) = \sin(t_1 - \pi) = -\sin(t_1) \Rightarrow \ \sin(t_2) = -\frac{1}{\sqrt{5}} \, .$$

Dále $g''(t) = -\cos(t) + \frac{-\sin(t)}{2} = -g(t)$ a

$$g''(t_1) = -\frac{\sqrt{5}}{2} \, , \ g''(t_2) = \frac{\sqrt{5}}{2} \, .$$

Funkce g je hladká, periodická s periodou 2π, vyšetřovaný interval má délku 2π a $g'(\pi) = g'(-\pi) = -\frac{1}{2} \neq 0$. Tudíž v krajních bodech nemá g extrém. Funkce f má vázané lokální maximum $f(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) = \frac{\sqrt{5}}{2}$ a vázané lokální minimum $f(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}) = -\frac{\sqrt{5}}{2}$.