1

$B(\underline{x},\underline{y}) = B\left(\begin{bmatrix}x_1\\x_2\end{bmatrix},\begin{bmatrix}y_1\\y_2\end{bmatrix}\right) = 4x_1y_2 - 2x_2y_2$ $\qquad (2x_1y_2+2x_2y_1-2x_2y_2)$

a) $B_S(\underline{x},\underline{y}) = \frac{1}{2}\left[B(\underline{x},\underline{y}) + B(\underline{y},\underline{x})\right] = \frac{1}{2}\left[4x_1y_2 - 2x_2y_2 + 4y_1x_2 - 2y_2x_2\right] =$

$B_A(\underline{x},\underline{y}) = \frac{1}{2}\left[B(\underline{x},\underline{y}) - B(\underline{y},\underline{x})\right] = \frac{1}{2}\left[4x_1y_2 - 2x_2y_2 + 4y_1x_2 - 2y_2x_2\right] = \underline{2x_1y_2 - 2x_2y_1}$

b) $F = (f_1, f_2) = \left(\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$

$$\left.\begin{aligned} B(f_1,f_1) &= 4\cdot1\cdot2 - 2\cdot2\cdot2 = 0\\ B(f_1,f_2) &= 4\cdot1\cdot1 - 2\cdot2\cdot1 = 0\\ B(f_2,f_1) &= 4\cdot1\cdot2 - 2\cdot1\cdot2 = 4\\ B(f_2,f_2) &= 4\cdot1\cdot1 - 2\cdot1\cdot1 = 2\end{aligned}\right\} [B]_F = \begin{bmatrix}B(f_1,f_1) & B(f_1,f_2)\\ B(f_2,f_1) & B(f_2,f_2)\end{bmatrix} = \begin{bmatrix}0&0\\4&2\end{bmatrix}$$

2. způsob: $B\left(\begin{bmatrix}x_1\\x_2\end{bmatrix},\begin{bmatrix}y_1\\y_2\end{bmatrix}\right) = [x_1\ x_2]\begin{bmatrix}0&4\\0&-2\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} \Rightarrow [B]_\varphi = \begin{bmatrix}0&4\\0&-2\end{bmatrix},\ \varphi = \Big(\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{e_2}\Big)$

$[\underline{v}]_\varphi = [P]_{\varphi\leftarrow F}\,[\underline{v}]_F$, dosazením $\underline{e}_1, \underline{e}_2$ za $\underline{v}$ dostaneme $[P]_{\varphi\leftarrow F} = [[f_1]_\varphi\ [f_2]_\varphi] = \begin{bmatrix}1&1\\2&1\end{bmatrix}$

$[B]_F = [P]^T_{\varphi\leftarrow F}[B]_\varphi[P]_{\varphi\leftarrow F} = \begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}0&4\\0&-2\end{bmatrix}\begin{bmatrix}1&1\\2&1\end{bmatrix} = \begin{bmatrix}0&0\\0&2\end{bmatrix}\begin{bmatrix}1&1\\2&1\end{bmatrix} = \begin{bmatrix}0&0\\4&2\end{bmatrix}$

c) $B(\underline{u},\underline{v}) = [\underline{u}]_F^T[B]_F[\underline{v}]_F = [3\ -4]\begin{bmatrix}0&0\\4&2\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix} = [3\ -4]\begin{bmatrix}0\\10\end{bmatrix} = -40$

2

$$\underbrace{\begin{bmatrix}2&-1&1\\-1&5&-2\\1&-2&1\end{bmatrix}}_{Q}\left|\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{I}\right. \begin{matrix}\\2r_2+r_1\\2r_2-r_1\end{matrix} \rightarrow \overset{2s_2+s_1\quad 2s_3-s_1}{\underbrace{\begin{bmatrix}2&-1&1\\0&9&-3\\0&-3&1\end{bmatrix}}_{T_1Q}}\left|\underbrace{\begin{bmatrix}1&0&0\\1&2&0\\-1&0&2\end{bmatrix}}_{T_1}\right. \rightarrow \underbrace{\begin{bmatrix}2&0&0\\0&18&-6\\0&-6&2\end{bmatrix}}_{T_1QT_1^T}\left|\underbrace{\begin{bmatrix}1&0&0\\1&2&0\\-1&0&2\end{bmatrix}}_{T_1}\right. \begin{matrix}\\ \\3r_3+r_2\end{matrix} \rightarrow$$

$$\overset{3s_3+s_2}{\underbrace{\begin{bmatrix}2&0&0\\0&18&-6\\0&0&0\end{bmatrix}}_{T_2T_1QT_1^T}}\left|\underbrace{\begin{bmatrix}1&0&0\\1&2&0\\-2&2&6\end{bmatrix}}_{T_2T_1}\right. \rightarrow \underbrace{\underbrace{\begin{bmatrix}2&0&0\\0&18&0\\0&0&0\end{bmatrix}}_{T_2T_1QT_1^TT_2^T}}_{D}\left|\underbrace{\underbrace{\begin{bmatrix}1&0&0\\1&2&0\\-2&2&6\end{bmatrix}}_{T_2T_1}}_{T}\right.$$ ... Q je pozitivně semidefinitní

Pro zájemce!

$$\underset{[Q]_F}{\overset{}{D}} = \begin{bmatrix}2&&\\&18&\\&&0\end{bmatrix} = T\cdot Q\cdot T^T = \underset{[P]^T_{\varphi\leftarrow F}}{\begin{bmatrix}1&0&0\\1&2&0\\-2&2&6\end{bmatrix}}\underset{[Q]_\varphi}{\begin{bmatrix}2&-1&1\\-1&5&-2\\1&-2&1\end{bmatrix}}\underset{[P]_{\varphi\leftarrow F}}{\begin{bmatrix}1&1&-2\\0&2&2\\0&0&6\end{bmatrix}}$$

Pozn.: $[P]^{-1}_{\varphi\leftarrow F} = \frac{1}{6}\begin{bmatrix}6&-3&3\\0&3&-1\\0&0&1\end{bmatrix}$

Existuje F:

$[P]_{\varphi\leftarrow F} = [[f_1]_\varphi, [f_2]_\varphi, [f_3]_\varphi] = \begin{bmatrix}1&1&-2\\0&2&2\\0&0&6\end{bmatrix} \Rightarrow F = (f_1, f_2, f_3) = \left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}-2\\2\\6\end{bmatrix}\right)$

$Q(\underline{x}) = [x_1\ x_2\ x_3]\begin{bmatrix}2&-1&1\\-1&5&-2\\1&-2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 2x_1^2 - 2x_1x_2 + 2x_1x_3 + 5x_2^2 - 4x_2x_3 + x_3^2$

$[x_1\ x_2\ x_3]\,\frac{1}{6}\begin{bmatrix}6&0&0\\-3&3&0\\3&-1&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&18&0\\0&0&0\end{bmatrix}\frac{1}{6}\begin{bmatrix}6&-3&3\\0&3&-1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \left[\left(x_1 - \frac{x_2}{2} + \frac{x_3}{2}\right), \left(\frac{x_2}{2} - \frac{x_3}{6}\right), \left(\frac{x_3}{6}\right)\right]\begin{bmatrix}2&&\\&18&\\&&0\end{bmatrix}\begin{bmatrix}x_1 - \frac{x_2}{2} + \frac{x_3}{2}\\ \frac{x_2}{2} - \frac{x_3}{6}\\ \frac{x_3}{6}\end{bmatrix}$

$= 2\left(x_1 - \frac{x_2}{2} + \frac{x_3}{2}\right)^2 + 18\left(\frac{x_2}{2} - \frac{x_3}{6}\right)^2 + 0\cdot\left(\frac{x_3}{6}\right)^2$

3

① Rozvojem (podle 3 řádku)

$$\begin{vmatrix}2&1&2&0\\0&3&1&1\\-1&2&0&0\\0&-1&1&2\end{vmatrix} = (-1)\cdot(-1)^{3+1}\begin{vmatrix}1&2&0\\3&1&1\\-1&1&2\end{vmatrix} + 2(-1)^{3+2}\begin{vmatrix}2&2&0\\0&1&1\\0&1&2\end{vmatrix} = (-1)\overbrace{\begin{pmatrix}1\cdot1\cdot2 + 2\cdot1\cdot(-1) + 0\cdot3\cdot1\\ -0\cdot1\cdot(-1) - 1\cdot1\cdot1 - 2\cdot3\cdot2\end{pmatrix}}^{-13} + 2\overbrace{\begin{pmatrix}2\cdot1\cdot2 + 0\cdot1\cdot0 + 0\cdot2\cdot1\\ -0\cdot1\cdot0 - 1\cdot1\cdot2 - 2\cdot2\cdot0\end{pmatrix}}^{4}$$

$= \underline{9}$

② Eliminací

$$\begin{vmatrix}2&1&2&0\\0&3&1&1\\-1&2&0&0\\0&-1&1&2\end{vmatrix} \overset{r_1+2r_3}{=} \begin{vmatrix}0&5&2&0\\0&3&1&1\\-1&2&0&0\\0&-1&1&2\end{vmatrix} \begin{matrix}r_1+5r_4\\r_2+3r_4\\r_3+2r_4\end{matrix} = \begin{vmatrix}0&0&7&10\\0&0&4&7\\-1&0&2&4\\0&-1&1&2\end{vmatrix} = (-1)\cdot(-1)^{3+1}\begin{vmatrix}0&7&10\\0&4&7\\-1&1&2\end{vmatrix} =$$

$= (-1)\cdot(-1)\cdot(-1)^{3+1}\begin{vmatrix}7&10\\4&7\end{vmatrix} = 1\cdot(7\cdot7 - 4\cdot10) = \underline{9}$

4

$C = \begin{bmatrix}1&2&0\\2&-2&0\\0&0&3\end{bmatrix}$ $\qquad Cv = \lambda v \Rightarrow 0 = (C-\lambda I)v \Leftrightarrow \det(C-\lambda I) = 0$

$$0 = \begin{vmatrix}1-\lambda&2&0\\2&-2-\lambda&0\\0&0&3-\lambda\end{vmatrix} = (3-\lambda)\begin{vmatrix}1-\lambda&2\\2&-2-\lambda\end{vmatrix} = (3-\lambda)\big[\underbrace{(1-\lambda)(-2-\lambda) - 2\cdot2}_{\lambda^2+\lambda-6}\big] = (3-\lambda)(\lambda-2)(\lambda+3)$$

$\lambda_{1,2} = \frac{-1\pm\sqrt{1-4\cdot1\cdot(-6)}}{2} < \begin{matrix}2\\-3\end{matrix}$

$\lambda_1 = 3$
$\lambda_2 = 2$
$\lambda_3 = -3$

$\boxed{\lambda_1 = 3}$ $(C-3I)\underline{v}_1 = \underline{o}$ $\quad \left[\begin{array}{ccc|c}-2&2&0&0\\2&-5&0&0\\0&0&0&0\end{array}\right] \begin{matrix}-\frac{1}{2}r_1\\r_2+r_1\\ \rightarrow\end{matrix} \left[\begin{array}{ccc|c}1&-1&0&0\\0&-3&0&0\end{array}\right] \begin{matrix}r_1-\frac{1}{3}r_2\\-\frac{1}{3}r_2\end{matrix} \rightarrow \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\end{array}\right] \Rightarrow \hat{v}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}p$

$v_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\boxed{\lambda_2 = 2}$ $(C-2I)\underline{v}_2 = 0$ $\quad \left[\begin{array}{ccc|c}-1&2&0&0\\2&-4&0&0\\0&0&1&0\end{array}\right] \begin{matrix}-r_1\\r_2+2r_1\end{matrix} \rightarrow \left[\begin{array}{ccc|c}-1&2&0&0\\ \cancel{0}&\cancel{0}&\cancel{0}&\cancel{0}\\0&0&1&0\end{array}\right]$ $\hat{v}_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}p \Rightarrow v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}$

$\left\|\begin{bmatrix}2\\1\\0\end{bmatrix}\right\| = \sqrt{2^2+1^2} = \sqrt{5}$

$\boxed{\lambda_3 = -3}$ $(C+3I)\underline{v}_3 = 0$ $\quad \left[\begin{array}{ccc|c}\cancel{4}&\cancel{2}&\cancel{0}&\cancel{0}\\2&1&0&0\\0&0&6&0\end{array}\right] \Rightarrow \hat{v}_3 = \begin{bmatrix}1\\-2\\0\end{bmatrix} \rightarrow v_3 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\\0\end{bmatrix}$

$C\cdot[v_1\ v_2\ v_3] = [3v_1\ 2v_2\ -3v_3] = [v_1\ v_2\ v_3]\begin{bmatrix}3&&\\&2&\\&&-3\end{bmatrix} \Rightarrow C = [v_1\ v_2\ v_3]\begin{bmatrix}3&0&0\\0&2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}v_1^T\\v_2^T\\v_3^T\end{bmatrix}$

$$\begin{bmatrix}1&2&0\\2&-2&0\\0&0&3\end{bmatrix} = \begin{bmatrix}0&2/\sqrt{5}&1/\sqrt{5}\\0&1/\sqrt{5}&-2/\sqrt{5}\\1&0&0\end{bmatrix}\begin{bmatrix}3&0&0\\0&2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}0&0&1\\2/\sqrt{5}&1/\sqrt{5}&0\\1/\sqrt{5}&-2/\sqrt{5}&0\end{bmatrix}$$

5

$A = \Bigg(\underbrace{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}_{a_1}, \underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{a_2}, \underbrace{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}}_{a_3}\Bigg)$ $\qquad (\underline{u},\underline{v}) = \underline{u}^T\begin{bmatrix}2&&&\\&2&&\\&&2&\\&&&2\end{bmatrix}\underline{v}$

Hledáme $F = (f_1, f_2, f_3)$: $(f_i, f_j) = \begin{cases}1 & i=j\\0 & i\neq j\end{cases}$

$\boxed{\text{I}}$ $\hat{f}_1 = a_1$ $\qquad \|\hat{f}_1\| = \sqrt{[1\ 1\ 0\ 0]\,2I\begin{bmatrix}1\\1\\0\\0\end{bmatrix}} = \sqrt{4} = 2$

$f_1 = \frac{1}{\|\hat{f}_1\|}\hat{f}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$

$\boxed{\text{II}}$ $\hat{f}_2 = \underline{a_2} - \alpha_1 f_1$ $\qquad 0 = (\hat{f}_2, f_1) = (a_2, f_1) - \alpha_1\overbrace{(f_1, f_1)}^{1} =$

$= [1\ 0\ 0\ 0]\,2I\,\frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \alpha_1 \Rightarrow \alpha_1 = 1$

$\hat{f}_2 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$

$\|\hat{f}_2\| = \sqrt{\frac{1}{2}[1\ -1\ 0\ 0]\,2I\,\frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}} = 1 \Rightarrow f_2 = \frac{1}{\|\hat{f}_2\|}\hat{f}_2 = \frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$

$\boxed{\text{III}}$ $\hat{f}_3 = a_3 - \alpha_1 f_1 - \alpha_2 f_2$ $\qquad 0 = (\hat{f}_3, f_1) = (a_3, f_1) - \alpha_1\overbrace{(f_1, f_1)}^{1} - \alpha_2\overbrace{(f_2, f_1)}^{0} \Rightarrow \alpha_1 = \overset{2}{(a_3, f_1)}$

$= \begin{bmatrix}1\\1\\1\\0\end{bmatrix} - 2\cdot\frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - 0\cdot\frac{1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ $\qquad 0 = (\hat{f}_3, f_2) = (a_3, f_2) - \alpha_1\underbrace{(f_1, f_2)}_{0} - \alpha_2\underbrace{(f_2, f_2)}_{1} \Rightarrow \alpha_2 = \underset{0}{(a_3, f_2)}$

$\|\hat{f}_3\| = \sqrt{2} \Rightarrow f_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$