Řešené příklady z komplexní analýzy

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Autorem obrazu *Imaginární džungle* na titulní stránce je Jiří Bouchala (a majitelem Ondřej Bouchala).

Jestliže nejste příliš pyšní,
zkuste to jako já.

Jan Skácel

Předmluva

Tento text obsahuje řešení všech příkladů k procvičení uvedených v kapitole 10 skript Funkce komplexní proměnné (viz [1]).

Sazba a všechny obrázky jsou dílem mého syna Ondřeje. Ten také svými komentáři pomohl text na řadě míst vylepšit.

Není možné, že by se nám při korekturách podařilo odstranit všechny nedostatky a chyby. Budeme vděční za shovívavost a sdělení všech připomínek.[1]

Práce s tímto textem nás bavila. Přejeme čtenářům totéž.

V Orlové, 2020

Jiří Bouchala
(a Ondřej Bouchala)

[1]Všechny připomínky (výhrady, komentáře, doporučení, výhružky a dary) zasílejte (prosím) na moji e-mailovou adresu: `jiri.bouchala@vsb.cz`.

PŘÍKLAD 1.
Určete reálnou a imaginární část daného komplexního čísla

a) $z = (1+i)(3-2i)$;

b) $z = \frac{2-3i}{3+4i}$;

c) $z = \frac{1+i}{1-i}$;

d) $z = 2i - \overline{\frac{2-4i}{2}}$.

Řešení:

a) $z = (3+2)+i$; $\underline{\operatorname{Re} z = 5,\ \operatorname{Im} z = 1}$.

b) $z = \frac{2-3i}{3+4i} = \frac{(2-3i)(3-4i)}{9+16} = \frac{6-12-9i-8i}{25}$; $\underline{\operatorname{Re} z = -\frac{6}{25},\ \operatorname{Im} z = -\frac{17}{25}}$.

c) $z = \frac{1+i}{1-i} = \frac{(1+i)^2}{2} = \frac{1+2i-1}{2}$; $\underline{\operatorname{Re} z = 0,\ \operatorname{Im} z = 1}$.

d) $z = 2i - \overline{\frac{2-4i}{2}} = 2i - \frac{2+4i}{2} = -1$; $\underline{\operatorname{Re} z = -1,\ \operatorname{Im} z = 0}$.

PŘÍKLAD 2.
Zapište dané komplexní číslo v goniometrickém tvaru

a) $z = -1 + \sqrt{3}i$;

b) $z = i$;

c) $z = -8$;

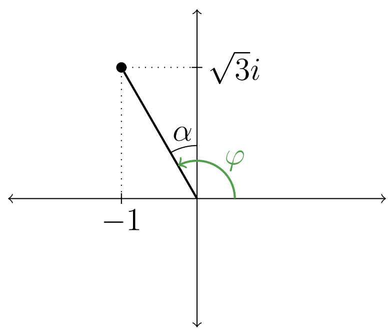
d) $z = -1 - \sqrt{3}i$;

e) $z = \frac{2+i}{3-2i}$;

f) $z = \frac{3-i}{2+i}$.

Řešení:

a)



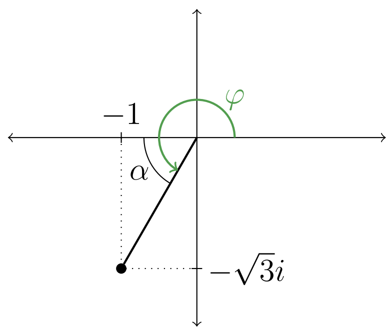
$$\cos\alpha = \frac{\sqrt{3}}{2},\ \alpha = \frac{\pi}{6},\ \varphi = \frac{\pi}{2} + \alpha = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2}{3}\pi;$$

$$\underline{z =} -1 + \sqrt{3}i = \sqrt{1+3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \underline{2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}.$$

b) $\underline{z =}\ i = \underline{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}$.

c) $\underline{z =} -8 = \underline{8(\cos\pi + i\sin\pi)}$.

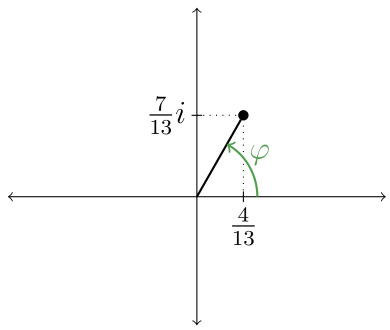
d)



$$\sin\alpha = \frac{\sqrt{3}}{2},\ \alpha = \frac{\pi}{3},\ \varphi = \pi + \alpha = \frac{4}{3}\pi;$$

$$\underline{z =} -1 - \sqrt{3}i = \underline{2\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right) = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)}.$$

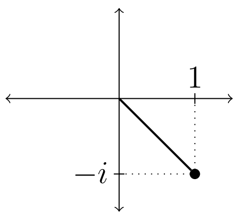
e) $z = \frac{2+i}{3-2i} = \frac{(2+i)(3+2i)}{9+4} = \frac{4}{13} + \frac{7}{13}i,$



$$|z| = \frac{1}{13}\sqrt{16+49} = \frac{\sqrt{65}}{13},\ \operatorname{tg}\varphi = \frac{\frac{7}{13}}{\frac{4}{13}} = \frac{7}{4},\ \varphi = \operatorname{arctg}\frac{7}{4};$$

$$\underline{z = \frac{\sqrt{65}}{13}\left(\cos\left(\operatorname{arctg}\frac{7}{4}\right) + i\sin\left(\operatorname{arctg}\frac{7}{4}\right)\right)}.$$

f) $z = \frac{3-i}{2+i} = \frac{(3-i)(2-i)}{5} = \frac{5-5i}{5} = 1 - i,$



$$\underline{z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)}.$$

Příklad 3.
Dokažte (matematickou indukcí) tzv. *Moivreovu větu*:

$$(\forall n \in \mathbb{N})\,(\forall \varphi \in \mathbb{R}) : \ \big(\cos\varphi + i\sin\varphi\big)^n = \cos(n\varphi) + i\sin(n\varphi).$$

Řešení:

1) Nejdříve ověříme, že tvrzení platí pro $n = 1$:

$$(\cos\varphi + i\sin\varphi)^1 = \cos(1\cdot\varphi) + i\sin(1\cdot\varphi)\,.$$

2) Nyní dokážeme implikaci $(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi) \overset{?}{\Rightarrow}$
$\overset{?}{\Rightarrow} (\cos\varphi + i\sin\varphi)^{n+1} = \cos((n+1)\varphi) + i\sin((n+1)\varphi)$:

$$\begin{aligned}(\cos\varphi + i\sin\varphi)^{n+1} &\overset{\text{i.p.}}{=} (\cos(n\varphi) + i\sin(n\varphi))\,(\cos\varphi + i\sin\varphi) = \\ &= (\cos(n\varphi)\cos\varphi - \sin(n\varphi)\sin\varphi) + i\,(\sin(n\varphi)\cos\varphi + \cos(n\varphi)\sin\varphi)\,,\end{aligned}$$

a nyní už jen stačí aplikovat známé „součtové vzorce“:

$$\begin{aligned}\cos(n\varphi)\cos\varphi - \sin(n\varphi)\sin\varphi &= \cos(n\varphi + \varphi) = \cos((n+1)\varphi),\\ \sin(n\varphi)\cos\varphi + \cos(n\varphi)\sin\varphi &= \sin(n\varphi + \varphi) = \sin((n+1)\varphi).\end{aligned}$$

Příklad 4.
Buď $\varphi \in \mathbb{R}$. Vyjádřete $\sin(4\varphi)$ a $\cos(4\varphi)$ pomocí $\sin\varphi$ a $\cos\varphi$.

Řešení:

$$\begin{aligned}\cos(4\varphi) + i\sin(4\varphi) &= (\cos\varphi + i\sin\varphi)^4 = \\ &= \big(\cos^2\varphi + 2i\sin\varphi\cos\varphi - \sin^2\varphi\big)^2 = \\ &= \cos^4\varphi - 4\sin^2\varphi\cos^2\varphi + \sin^4\varphi + \\ &\quad + 4i\sin\varphi\cos^3\varphi - 2\cos^2\varphi\sin^2\varphi - 4i\sin^3\varphi\cos\varphi = \\ &= \cos^4\varphi - 6\sin^2\varphi\cos^2\varphi + \sin^4\varphi + i\left(4\sin\varphi\cos^3\varphi - 4\sin^3\varphi\cos\varphi\right),\end{aligned}$$

a proto (stačí porovnat reálné a imaginární části)

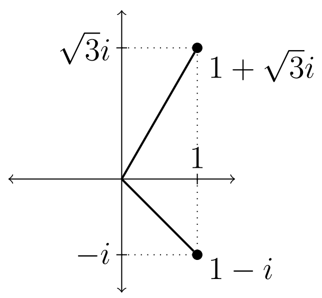
$$\underline{\cos(4\varphi) = \cos^4\varphi - 6\sin^2\varphi\cos^2\varphi + \sin^4\varphi},$$

$$\underline{\sin(4\varphi) = 4\sin\varphi\cos^3\varphi - 4\sin^3\varphi\cos\varphi}.$$

PŘÍKLAD **5**.

Určete $\operatorname{Re} z$ a $\operatorname{Im} z$, je-li $z = \left(\frac{1-i}{1+\sqrt{3}i}\right)^{24}$.

Řešení:



$$\frac{1-i}{1+\sqrt{3}i} = \frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)}{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)} = \frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(-\frac{7}{12}\pi\right)\right),$$

$$z = \frac{1}{2^{12}}\left(\cos\left(-\frac{24\cdot 7\pi}{12}\right) + i\sin\left(-\frac{24\cdot 7\pi}{12}\right)\right) = \frac{1}{2^{12}};$$

$$\underline{\operatorname{Re} z = \frac{1}{2^{12}},\ \operatorname{Im} z = 0.}$$

PŘÍKLAD **6**.

Určete $\operatorname{Arg} z$ a $\arg z$, je-li

a) $z = \left(\sqrt{3}+i\right)^{126}$;

b) $z = (1+i)^{137}$;

c) $z = -1-5i$.

Řešení:

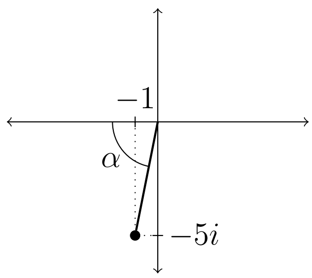
a) $z = (\sqrt{3}+i)^{126} = \left(2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})\right)^{126} = 2^{126}\left(\cos(21\pi) + i\sin(21\pi)\right) = -2^{126}$;

$$\underline{\operatorname{Arg} z = \{\pi + 2k\pi\colon\ k\in\mathbb{Z}\},\ \arg z = \pi.}$$

b) $z = 2^{\frac{137}{2}}\left(\cos\left(137\frac{\pi}{4}\right) + i\sin\left(137\frac{\pi}{4}\right)\right) = 2^{\frac{137}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$;

$$\underline{\operatorname{Arg} z = \left\{\frac{\pi}{4} + 2k\pi\colon\ k\in\mathbb{Z}\right\},\ \arg z = \frac{\pi}{4}.}$$

c)



$\operatorname{tg}\alpha = \frac{5}{1}$, $\alpha = \operatorname{arctg} 5$;

$$\underline{\operatorname{Arg} z = \{-\pi + \operatorname{arctg} 5 + 2k\pi\colon\ k \in \mathbb{Z}\},\ \arg z = -\pi + \operatorname{arctg} 5.}$$

PŘÍKLAD 7.
Znázorněte v Gaussově rovině množinu

a) $\{z \in \mathbb{C}\colon\ \operatorname{Re} z \leq 1\}$;

b) $\{z \in \mathbb{C}\colon\ \operatorname{Re}(z^2) = 2\}$;

c) $\{z \in \mathbb{C}\colon\ \operatorname{Im} \frac{1}{z} = \frac{1}{4}\}$;

d) $\{z \in \mathbb{C}\colon\ |\operatorname{Im} z| < 1\}$;

e) $\{z \in \mathbb{C}\colon\ |z| = \operatorname{Re} z + 1\}$;

f) $\{z \in \mathbb{C}\colon\ |z - 2| = |1 - 2\overline{z}|\}$;

g) $\{z \in \mathbb{C}\colon\ \left|\frac{z-2}{z-3}\right| = 1\}$;

h) $\{z \in \mathbb{C}\colon\ |1 + z| < |1 - z|\}$;

i) $\{z \in \mathbb{C}\colon\ |z + 1| = 2|z - 1|\}$;

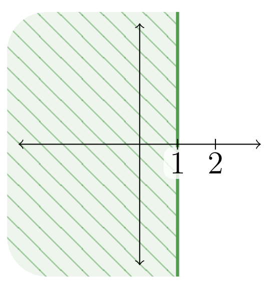
j) $\{z \in \mathbb{C}\colon\ 2 < |z + 2 - 3i| < 4\}$;

k) $\{z \in \mathbb{C}\colon\ \frac{\pi}{4} \leq \arg(z + 2i) \leq \frac{\pi}{2}\}$;

l) $\{z \in \mathbb{C}\colon\ |z| + \operatorname{Re} z \leq 1\ \wedge$
$\wedge -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{4}\}$.

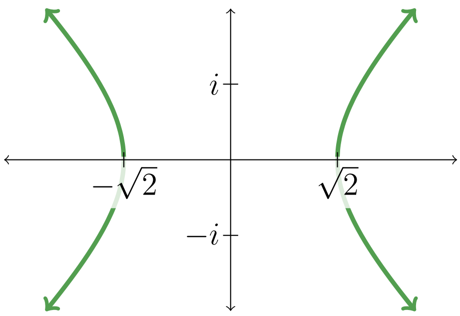
Řešení:

a) $\{z \in \mathbb{C}\colon\ \operatorname{Re} z \leq 1\}$:



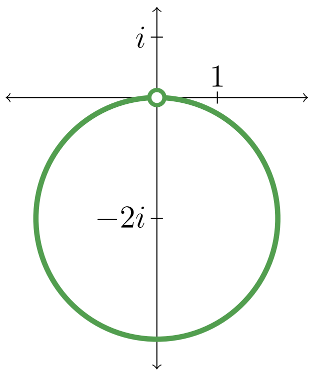
b)

$$\left\{z \in \mathbb{C}\colon\ \operatorname{Re}(z^2) = 2\right\} = \left\{x+iy\colon\ \operatorname{Re}(x^2+2ixy-y^2) = 2\right\} =$$
$$= \left\{x+iy\colon\ x^2-y^2=2\right\}:$$

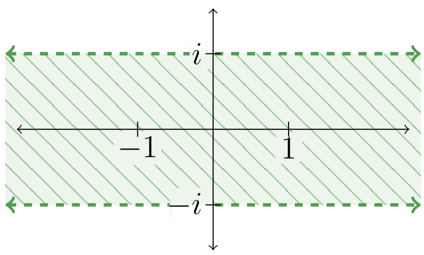


c)

$$\left\{z \in \mathbb{C}\colon\ \operatorname{Im}\frac{1}{z} = \frac{1}{4}\right\} = \left\{x+iy \in \mathbb{C}\colon\ \operatorname{Im}\frac{1}{x+iy} = \frac{1}{4}\right\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ \operatorname{Im}\frac{x-iy}{x^2+y^2} = \frac{1}{4}\right\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ -\frac{y}{x^2+y^2} = \frac{1}{4}\right\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ x^2+y^2 = -4y \wedge x^2+y^2 \neq 0\right\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ x^2+(y+2)^2 = 4\right\} \setminus \{0+0i\}:$$

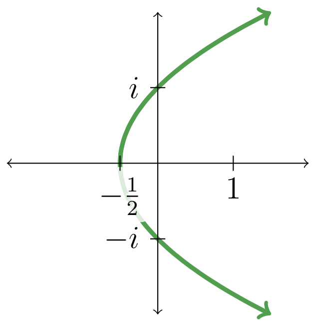


d) $\{z \in \mathbb{C}\colon\ |\operatorname{Im} z| < 1\}$:



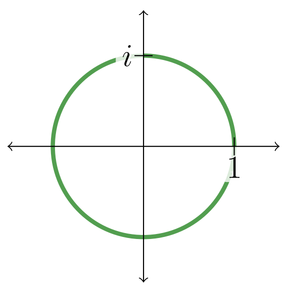
e)

$$\{z \in \mathbb{C}\colon\ |z| = \operatorname{Re} z + 1\} = \left\{x+iy\colon\ \sqrt{x^2+y^2} = x+1\right\} =$$
$$= \left\{x+iy\colon\ x^2+y^2 = x^2+2x+1\right\} =$$
$$= \left\{x+iy\colon\ y^2 = 2x+1\right\} =$$
$$= \left\{x+iy\colon\ x = \frac{y^2-1}{2}\right\}:$$

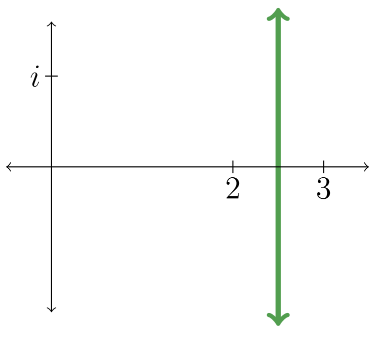


f)

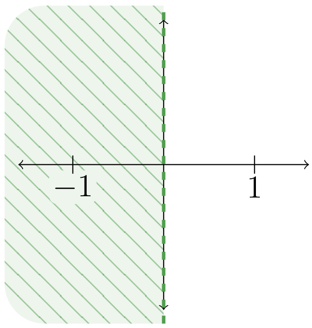
$$\{z \in \mathbb{C}\colon\ |z-2| = |1-2\overline{z}|\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ \sqrt{(x-2)^2+y^2} = |1-2(x-iy)|\right\} =$$
$$= \left\{x+iy\colon\ (x-2)^2+y^2 = (1-2x)^2+4y^2\right\} =$$
$$= \left\{x+iy\colon\ x^2-4x+4+y^2 = 1-4x+4x^2+4y^2\right\} =$$
$$= \left\{x+iy\colon\ 3x^2+3y^2 = 3\right\} =$$
$$= \left\{x+iy\colon\ x^2+y^2 = 1\right\}:$$



g) $\left\{z \in \mathbb{C}\colon\ \left|\frac{z-2}{z-3}\right| = 1\right\} = \{z \in \mathbb{C}\colon\ |z-2| = |z-3|\}$:

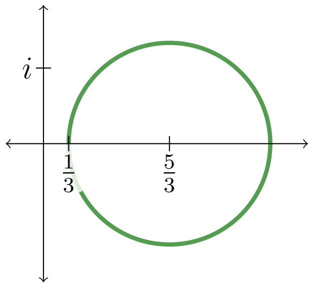


h) $\{z \in \mathbb{C}\colon\ |1+z| < |1-z|\} = \{z \in \mathbb{C}\colon\ |z-(-1)| < |z-1|\}$:

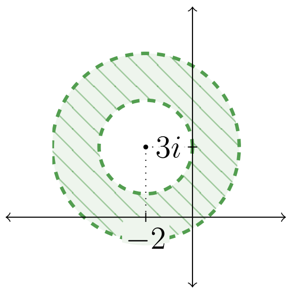


i)

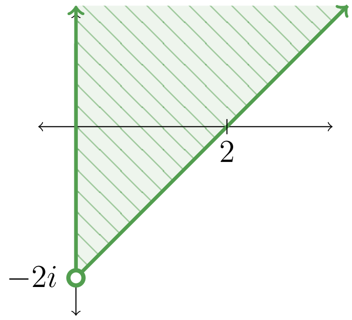
$$\begin{aligned}
\{z \in \mathbb{C}\colon\ |z+1| = 2|z-1|\} &= \\
&= \left\{x+iy \in \mathbb{C}\colon\ (x+1)^2 + y^2 = 4((x-1)^2+y^2)\right\} = \\
&= \left\{x+iy\colon\ x^2+2x+1+y^2 = 4x^2-8x+4+4y^2\right\} = \\
&= \left\{x+iy\colon\ 3x^2+3y^2-10x = -3\right\} = \\
&= \left\{x+iy\colon\ \left(x-\frac{5}{3}\right)^2 + y^2 = \frac{16}{9}\right\}:
\end{aligned}$$



j) $\{z \in \mathbb{C}\colon\ 2 < |z+2-3i| < 4\} = \{z \in \mathbb{C}\colon\ 2 < |z-(-2+3i)| < 4\}$:

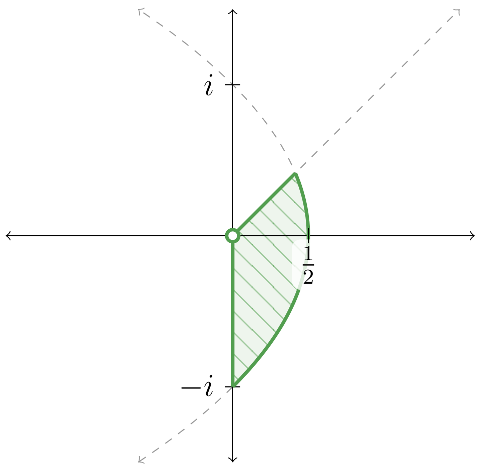


k) $\{z \in \mathbb{C}\colon \frac{\pi}{4} \leq \arg(z+2i) \leq \frac{\pi}{2}\}$:



l)

$$\begin{aligned}
&\left\{z \in \mathbb{C}\colon |z| + \operatorname{Re} z \leq 1 \wedge -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{4}\right\} = \\
&= \left\{x+iy \in \mathbb{C}\colon \sqrt{x^2+y^2} \leq 1-x \wedge -\frac{\pi}{2} \leq \arg(x+iy) \leq \frac{\pi}{4}\right\} = \\
&= \Big\{x+iy \in \mathbb{C}\colon x^2+y^2 \leq (1-x)^2 \wedge 1-x \geq 0 \wedge \\
&\quad \wedge -\frac{\pi}{2} \leq \arg(x+iy) \leq \frac{\pi}{4}\Big\} = \\
&= \left\{x+iy \in \mathbb{C}\colon y^2 \leq 1-2x \wedge -\frac{\pi}{2} \leq \arg(x+iy) \leq \frac{\pi}{4}\right\}:
\end{aligned}$$



Příklad 8.

Buď $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Dokažte následující implikace:

a) $\left.\begin{matrix}\varphi_1 \in \operatorname{Arg} z_1 \\ \varphi_2 \in \operatorname{Arg} z_2\end{matrix}\right\} \Rightarrow \varphi_1 + \varphi_2 \in \operatorname{Arg}(z_1 z_2)$;

b) $\left.\begin{matrix}\varphi_1 \in \operatorname{Arg} z_1 \\ \varphi_2 \in \operatorname{Arg} z_2\end{matrix}\right\} \Rightarrow \varphi_1 - \varphi_2 \in \operatorname{Arg}\left(\frac{z_1}{z_2}\right)$.

Řešení:

a)

$$\begin{aligned}
z_1 \cdot z_2 &= |z_1| \cdot |z_2| \cdot (\cos\varphi_1 + i\sin\varphi_1) \cdot (\cos\varphi_2 + i\sin\varphi_2) = \\
&= |z_1| \cdot |z_2| \cdot (\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)).
\end{aligned}$$

b)
$$\frac{z_1}{z_2}=\frac{|z_1|}{|z_2|}\cdot\frac{\cos\varphi_1+i\sin\varphi_1}{\cos\varphi_2+i\sin\varphi_2}=\left|\frac{z_1}{z_2}\right|(\cos\varphi_1+i\sin\varphi_1)\cdot(\cos\varphi_2-i\sin\varphi_2)=$$
$$=\left|\frac{z_1}{z_2}\right|\cdot\left(\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)\right).$$

PŘÍKLAD 9.

Rozhodněte, zda daná limita existuje, a pokud ano, vypočtěte ji

a) $\lim(3-4i)^n$;

b) $\lim\left((-1)^n+\frac{i}{n}\right)$;

c) $\lim\left(\frac{1+i}{\sqrt{2}}\right)^n$;

d) $\lim\left(\frac{1-\sqrt{3}i}{2}\right)^{6n}$.

Řešení:

a) $\underline{\lim(3-4i)^n=\infty}$, protože $|(3-4i)^n|=\left(\sqrt{9+16}\right)^n=5^n\to\infty$.

b) $\underline{\lim\underbrace{\left((-1)^n+\frac{i}{n}\right)}_{=:z_n}\text{ neexistuje}}$, protože
$$z_{2n}=(-1)^{2n}+\frac{i}{2n}=1+\frac{i}{2n}\to 1$$
a současně
$$z_{2n+1}=(-1)^{2n+1}+\frac{i}{2n+1}=-1+\frac{i}{2n+1}\to -1.$$

c) $\underline{\lim\underbrace{\left(\frac{1+i}{\sqrt{2}}\right)^n}_{=:z_n}\text{ neexistuje}}$, protože
$$z_n=\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^n=\cos\left(n\frac{\pi}{4}\right)+i\sin\left(n\frac{\pi}{4}\right),$$
a tedy
$$z_{8n}\to 1\wedge z_{8n+2}\to i.$$

d) $\underline{\lim\left(\frac{1-\sqrt{3}i}{2}\right)^{6n}=1}$, protože
$$\left(\frac{1-\sqrt{3}i}{2}\right)^{6n}=\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)^{6n}=$$
$$=\cos(-2\pi n)+i\sin(-2\pi n)\to 1.$$

PŘÍKLAD 10.
Buď (z_n) posloupnost komplexních čísel, $r \in \mathbb{R}^+$ a $\varphi \in \mathbb{R}$. Dokažte následující tvrzení:

a) $z_n \to 0 \;\Leftrightarrow\; \frac{1}{z_n} \to \infty$;

b) $\left.\begin{array}{r} |z_n| \to r \\ \arg z_n \to \varphi \end{array}\right\} \;\Rightarrow\; z_n \to r\big(\cos\varphi + i\sin\varphi\big)$;

a ukažte, že implikaci v tvrzení b) nelze obrátit.

Řešení:

a) Stačí si obě strany dokazované ekvivalence přepsat pomocí definice limity.

- Levá strana:

$$z_n \to 0$$

$$\Updownarrow$$

$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0)\colon\ z_n \in U(0,\varepsilon)$$

$$\Updownarrow$$

$$\underline{(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0)\colon\ |z_n| < \varepsilon};$$

- pravá strana:

$$\frac{1}{z_n} \to \infty$$

$$\Updownarrow$$

$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0)\colon\ \frac{1}{z_n} \in U(\infty,\varepsilon)$$

$$\Updownarrow$$

$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0)\colon\ \left(\left|\frac{1}{z_n}\right| > \frac{1}{\varepsilon} \vee \frac{1}{z_n} = \infty\right)$$

$$\Updownarrow$$

$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0)\colon\ (\varepsilon > |z_n| \vee z_n = 0)$$

$$\Updownarrow$$

$$\underline{(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0)\colon\ |z_n| < \varepsilon}.$$

b) Z předpokladů plyne, že pro všechna dost velká n je

$$z_n = |z_n| \left(\cos\left(\arg z_n\right) + i \sin\left(\arg z_n\right)\right),$$

a tvrzení plyne přímo ze spojitosti funkcí kosinus a sinus a z věty o limitě součinu.

Jako protipříklad vyvracející platnost obrácené implikace dobře poslouží posloupnost

$$z_n := \cos\left(\pi + \frac{(-1)^n}{n}\right) + i \sin\left(\pi + \frac{(-1)^n}{n}\right)$$

a volba

$$r = 1,\ \varphi = \pi.$$

PŘÍKLAD 11.

Najděte všechna $z \in \mathbb{C}$, pro která platí

a) $z^3 = 1$;

b) $z^2 = i$;

c) $z^2 = 24i - 7$;

d) $\left(\frac{z-1}{z+1}\right)^2 = 2i$;

e) $z^4 = -1$;

f) $z^3 = i - 1$;

g) $z^5 = 1$;

h) $z^2 = -11 + 60i$;

i) $z^2 = 3 + 4i$.

Řešení:

a) $z = |z| \left(\cos\varphi + i \sin\varphi\right),\ 1 = \cos 0 + i \sin 0.$

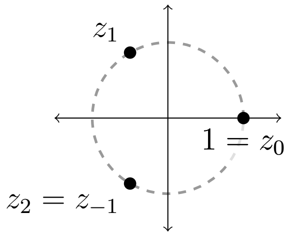
$$\begin{gathered} z^3 = |z|^3 \left(\cos(3\varphi) + i\sin(3\varphi)\right) = 1\left(\cos 0 + i \sin 0\right) \\ \Updownarrow \\ (|z|^3 = 1) \wedge (\exists k \in \mathbb{Z}\colon\ 3\varphi = 0 + 2k\pi) \\ \Updownarrow \\ (|z| = 1) \wedge \left(\exists k \in \mathbb{Z}\colon\ \varphi = k\tfrac{2\pi}{3}\right), \end{gathered}$$

a odtud

$$z = z_k = \cos\left(k\frac{2\pi}{3}\right) + i \sin\left(k\frac{2\pi}{3}\right) = \begin{cases} 1, & k \in \{3l\colon\ l \in \mathbb{Z}\}, \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2}, & k \in \{3l+1\colon\ l \in \mathbb{Z}\}, \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}, & k \in \{3l+2\colon\ l \in \mathbb{Z}\}, \end{cases}$$

tedy

$$z^3 = 1 \Leftrightarrow \underline{z \in \left\{1,\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}}.$$

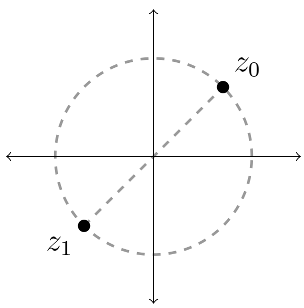


b) $z = |z|\left(\cos\varphi + i\sin\varphi\right)$, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$,

$$z^2 = |z|^2\left(\cos\left(2\varphi\right) + i\sin\left(2\varphi\right)\right) = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}$$
$$\Updownarrow$$
$$\left(|z|^2 = 1\right) \wedge \left(\exists k \in \mathbb{Z}\colon\ 2\varphi = \tfrac{\pi}{2} + 2k\pi\right),$$

a odtud

$$z = z_k = \cos\left(\frac{\pi}{4} + k\pi\right) + i\sin\left(\frac{\pi}{4} + k\pi\right) =$$
$$= \begin{cases} \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, & k \in \{2l : l \in \mathbb{Z}\}, \\ -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, & k \in \{2l+1\colon\ l \in \mathbb{Z}\}. \end{cases}$$



$$z^2 = i \Leftrightarrow \underline{z \in \left\{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right\}}.$$

c) Nechť $z = x + iy$. Pak

$$z^2 = x^2 + 2ixy - y^2 = 24i - 7 \Leftrightarrow \begin{pmatrix} x^2 - y^2 & = & -7 \\ 2xy & = & 24 \end{pmatrix} \Leftrightarrow$$
$$\Leftrightarrow \begin{pmatrix} x^2 - y^2 & = & -7 \\ y & = & \frac{12}{x} \end{pmatrix} \Leftrightarrow$$
$$\Leftrightarrow \begin{pmatrix} x^2 - \frac{144}{x^2} & = & -7 \\ y & = & \frac{12}{x} \end{pmatrix} \Leftrightarrow$$
$$\Leftrightarrow \begin{pmatrix} x^4 + 7x^2 - 144 & = & 0 \\ y & = & \frac{12}{x} \end{pmatrix},$$

což nastane právě tehdy, když $\underline{z = x + iy = 3 + 4i \text{ nebo } z = -3 - 4i}$.

d) Po substituci $\frac{z-1}{z+1} =: u = |u|\,(\cos\varphi + i\sin\varphi)$ řešíme nejdříve rovnici $u^2 = 2i$, tj.

$$|u|^2\,(\cos(2\varphi) + i\sin(2\varphi)) = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

s řešením

$$u = \pm\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \pm(1+i),$$

a pak už snadno $\underline{\frac{z-1}{z+1} = 1+i}$ právě tehdy, když $(z = x+iy)$

$$\begin{aligned} x + iy - 1 &= (1+i)(x+iy+1), \text{ neboli}\\ (x-1) + iy &= (x-y+1) + i(x+y+1), \text{ a proto}\\ (x-1 = x-y+1) &\wedge (y = x+y+1), \text{ tj.}\\ y = 2 &\wedge x = -1, \end{aligned}$$

a podobně $\underline{\frac{z-1}{z+1} = -1-i}$ právě tehdy, když

$$\begin{gathered} x + iy - 1 = -(1+i)(x+iy+1),\\ (x-1 = -x+y-1) \wedge (2y = -x-1), \text{ a proto}\\ y = -\frac{2}{5} \wedge x = -\frac{1}{5}. \end{gathered}$$

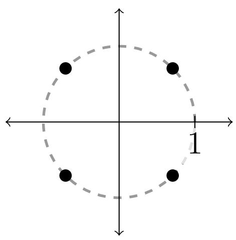
Shrnutí:

$$\underline{\left(\frac{z-1}{z+1}\right)^2 = 2i \Leftrightarrow \left(z = -1+2i \vee z = -\frac{1}{5} - \frac{2}{5}i\right).}$$

e) $|z|^4\,(\cos(4\varphi) + i\sin(4\varphi)) = \cos\pi + i\sin\pi$ právě tehdy, když

$$z = z_k = \cos\left(\frac{\pi}{4} + k\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + k\frac{\pi}{2}\right),\ k \in \mathbb{Z},\ \text{tzn.}$$

$$z^4 = -1 \Leftrightarrow \underline{z \in \left\{\frac{1+i}{\sqrt{2}},\ \frac{-1+i}{\sqrt{2}},\ \frac{-1-i}{\sqrt{2}},\ \frac{1-i}{\sqrt{2}}\right\}}.$$



f)
$$|z|^3\,(\cos(3\varphi) + i\sin(3\varphi)) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

právě tehdy, když

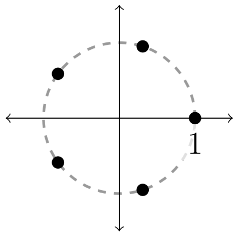
$$\left(|z| = \sqrt[3]{\sqrt{2}}\right) \wedge \left(3\varphi = \frac{3\pi}{4} + 2k\pi,\ k \in \mathbb{Z}\right).$$

Odtud již snadno plyne, že $z^3 = i - 1$ právě tehdy, pokud

$$\underline{z \in \left\{\sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right):\ k \in \{0,\ 1,\ 2\}\right\}}.$$

g)

$$\underline{z = \cos\left(\frac{2\pi}{5}k\right) + i\sin\left(\frac{2\pi}{5}k\right),\ \ k \in \{0,\ 1,\ 2,\ 3,\ 4\}.}$$



h)

$$z^2 = (x+iy)^2 \quad = \quad -11+60i$$

$$\Updownarrow$$

$$x^2 + 2ixy - y^2 \quad = \quad 11+60i$$

$$\Updownarrow$$

$$x^2 - y^2 = -11 \quad \wedge \quad 2xy = 60$$

$$\Updownarrow$$

$$x^2 - \tfrac{900}{x^2} = -11 \quad \wedge \quad y = \tfrac{30}{x}$$

$$\Updownarrow$$

$$y = \tfrac{30}{x} \wedge x^2 = \frac{-11 \pm \sqrt{121+3600}}{2} = \begin{cases} \dfrac{-11-\sqrt{3721}}{2} & \dots \text{ to nelze,} \\ \dfrac{-11+\sqrt{3721}}{2} = \dfrac{-11+61}{2} = 25, \end{cases}$$

a proto

$$z^2 = -11+60i \Leftrightarrow \underline{z = \pm(5+6i)}.$$

i) Buď $z = x+iy$. Pak

$$z^2 = (x+iy)^2 \quad = \quad 3+4i$$

$$\Updownarrow$$

$$x^2 - y^2 = 3 \quad \wedge \quad 2xy = 4$$

$$\Updownarrow$$

$$x^2 - \tfrac{4}{x^2} = 3 \quad \wedge \quad y = \tfrac{2}{x}$$

$$\Updownarrow$$

$$y = \tfrac{2}{x} \wedge x^2 = \frac{3 \pm \sqrt{9+16}}{2} = \begin{cases} \frac{3-5}{2} & \dots \text{ to nelze,} \\ 4, \end{cases}$$

a proto

$$z^2 = 3+4i \Leftrightarrow \underline{z = \pm(2+i)}.$$

PŘÍKLAD 12.
Určete a znázorněte množinu $M = \left\{\frac{1}{z}\colon\ z \in \Omega\right\}$, je-li

a) $\Omega = \{z \in \mathbb{C}\colon\ \arg z = \alpha\},\ \alpha \in (-\pi, \pi\rangle$;

b) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| = 1\}$;

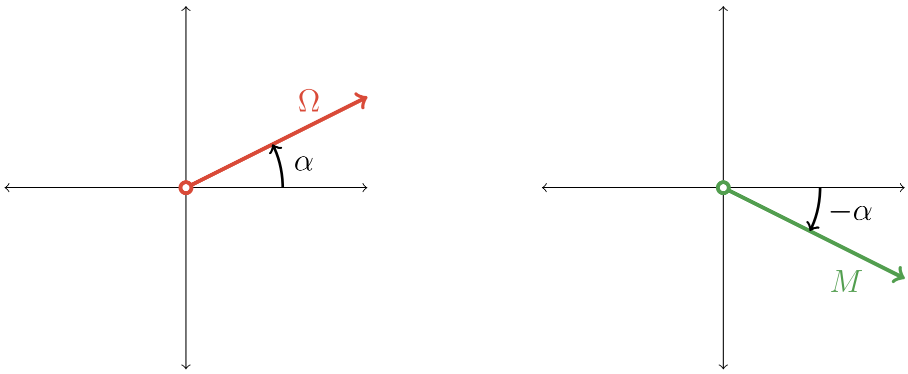
c) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z = \operatorname{Im} z\}$;

d) $\Omega = \{x + iy \in \mathbb{C}\colon\ x = 1\}$;

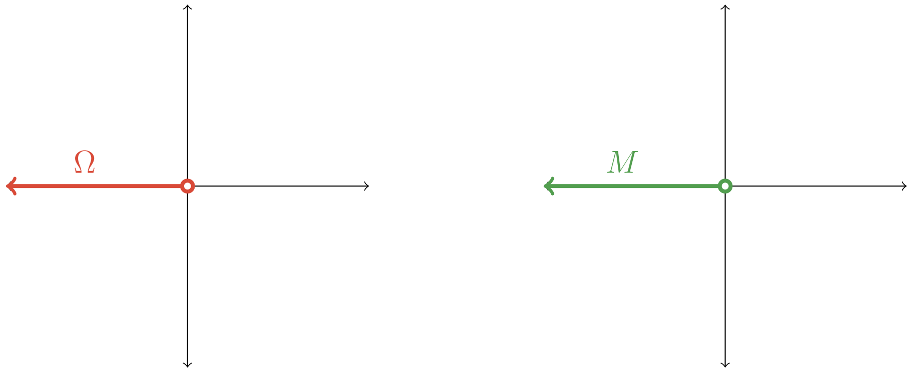
e) $\Omega = \{x + iy \in \mathbb{C}\colon\ y = 0\}$.

Řešení:

a) $\underline{\alpha \in (-\pi, \pi) \Rightarrow M = \{z \in \mathbb{C}\colon\ \arg z = -\alpha\}}$;

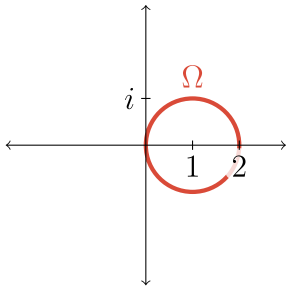


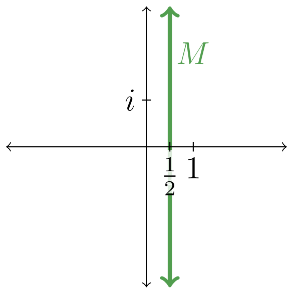
$\underline{\alpha = \pi \Rightarrow M = \Omega = \{z \in \mathbb{C}\colon\ \arg z = \pi\}}$.



b)

$$\underline{M =} \left\{u+iv\colon\ \frac{1}{u+iv} \in \Omega\right\} \cup \{\infty\} =$$
$$= \left\{u+iv\colon\ \left|\frac{1}{u+iv} - 1\right| = 1\right\} \cup \{\infty\} =$$
$$= \{u+iv\colon\ |1-u-iv| = |u+iv|\} \cup \{\infty\} =$$
$$= \left\{u+iv\colon\ (1-u)^2+v^2 = u^2+v^2\right\} \cup \{\infty\} =$$
$$= \{u+iv\colon\ 1-2u = 0\} \cup \{\infty\} =$$
$$= \underline{\left\{u+iv\colon\ u = \frac{1}{2}\right\} \cup \{\infty\}}.$$



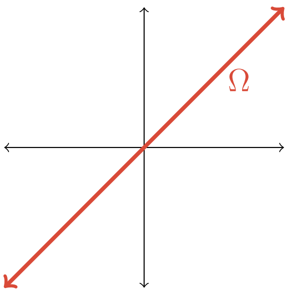


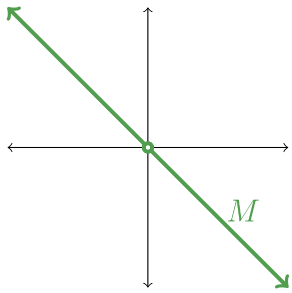
c)

$$M = \left\{u+iv\colon\ \frac{1}{u+iv} \in \Omega\right\} \cup \{\infty\} =$$
$$= \left\{u+iv\colon\ \frac{u-iv}{u^2+v^2} \in \Omega\right\} \cup \{\infty\},$$

a protože $\dfrac{u}{u^2+v^2} = \dfrac{-v}{u^2+v^2} \Leftrightarrow (u \neq 0 \wedge u = -v)$, je

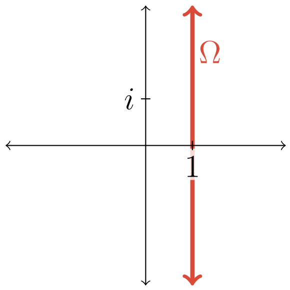
$$\underline{M = \{u+iv\colon\ u \neq 0 \wedge u = -v\} \cup \{\infty\}}.$$

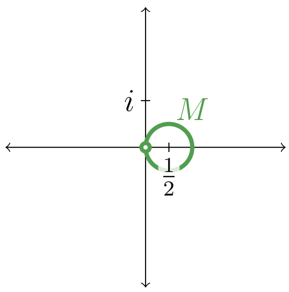




d)

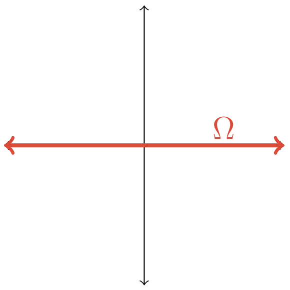
$$\underline{M =} \left\{u+iv\colon\ \frac{1}{u+iv} \in \Omega\right\} = $$
$$= \left\{u+iv\colon\ \frac{u}{u^2+v^2} = 1\right\} = $$
$$= \underline{\left\{u+iv\colon\ \left(u-\frac{1}{2}\right)^2 + v^2 = \frac{1}{4}\right\} \setminus \{0\}}.$$

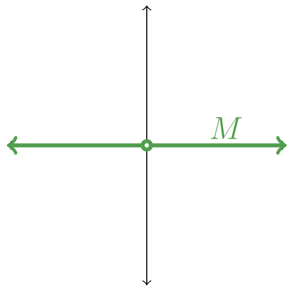




e)

$$\underline{M =} \left\{u+iv\colon\ \frac{1}{u+iv} \in \Omega\right\} \cup \{\infty\} = $$
$$= \left\{u+iv\colon\ \frac{-v}{u^2+v^2} = 0\right\} \cup \{\infty\} = $$
$$= \underline{\{u+iv\colon\ v = 0 \neq u\} \cup \{\infty\}}.$$





PŘÍKLAD 13.

Určete a znázorněte množinu $M = \{f(z)\colon\ z \in \Omega\}$, je-li

a) $\Omega = \{z \in \mathbb{C}\colon\ |\arg z| \leq \frac{\pi}{6}\},\ f(z) := z^2;$

b) $\Omega = \{z \in \mathbb{C}\colon\ |\operatorname{Im} z| < \frac{\pi}{2}\},\ f(z) := e^z;$

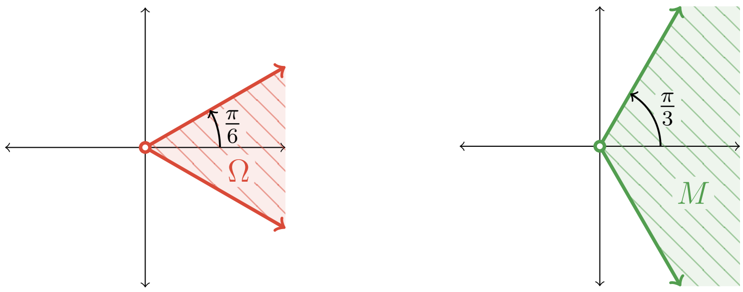
c) $\Omega = \{z \in \mathbb{C}\colon\ 0 < \operatorname{Re} z < \pi\ \wedge\ \operatorname{Im} z > 0\},\ f(z) := e^{iz};$

d) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Im} z = \frac{1}{2}\},\ f(z) := z^2.$

Řešení:

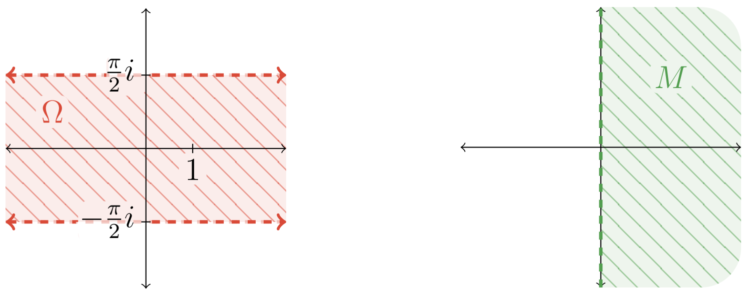
a)

$$\underline{M = \left\{z \in \mathbb{C}\colon\ |\arg z| \leq \frac{\pi}{6} \cdot 2 = \frac{\pi}{3}\right\}}.$$



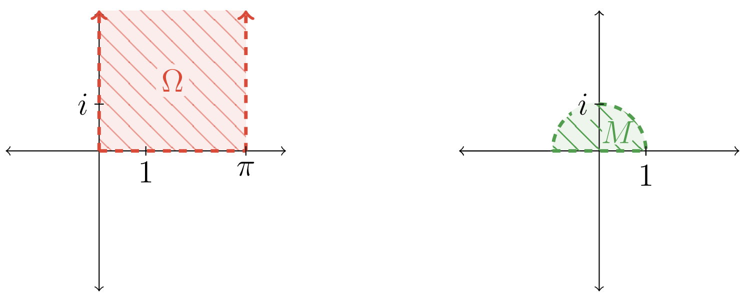
b)

$$\underline{M =} \left\{e^{x+iy}\colon\ |y| < \frac{\pi}{2}\right\} =$$
$$= \left\{e^{x}\left(\cos\left(y\right) + i\sin\left(y\right)\right)\colon\ |y| < \frac{\pi}{2}\right\} =$$
$$= \underline{\{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\}}.$$



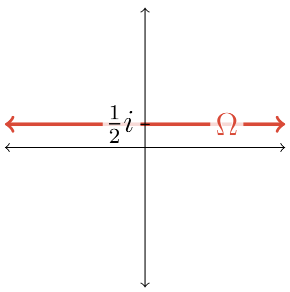
c)

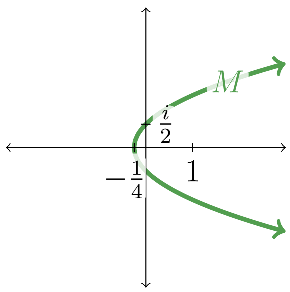
$$\underline{M =} \left\{e^{i(x+iy)} = e^{-y}\left(\cos x + i\sin x\right)\colon\ 0 < x < \pi\ \wedge\ y > 0\right\} =$$
$$= \underline{\{z \in \mathbb{C}\colon\ |z| < 1\ \wedge\ \operatorname{Im} z > 0\}}.$$



d)

$$\underline{M} = \left\{ \left(x + \frac{1}{2}i\right)^2 :\ x \in \mathbb{R} \right\} = \\ = \left\{ x^2 - \frac{1}{4} + xi :\ x \in \mathbb{R} \right\} = \\ = \underline{\left\{ y^2 - \frac{1}{4} + yi :\ y \in \mathbb{R} \right\}}.$$





PŘÍKLAD 14.
Vypočtěte

a) $\sin(2-3i)$;

b) $\cos i$;

c) $\cosh i$;

d) $\mathrm{Ln}(-5+3i)$ a $\ln(-5+3i)$;

e) $\mathrm{Ln}(-4-\sqrt{3}i)$ a $\ln(-4-\sqrt{3}i)$;

f) $\mathrm{Ln}(ie^2)$.

Řešení:

a)

$$\underline{\sin(2-3i)} = \frac{e^{i(2-3i)} - e^{-i(2-3i)}}{2i} = \\ = \frac{e^3(\cos(2) + i\sin(2)) - e^{-3}(\cos(-2) + i\sin(-2))}{2i} = \\ = \frac{e^3 - e^{-3}}{2i} \cdot \cos 2 + \frac{i(e^3 + e^{-3}) \cdot \sin 2}{2i} = \\ = \underline{\cosh 3 \cdot \sin 2 - (\sinh 3 \cdot \cos 2)i} \doteq \\ \doteq 9.15 + 4.17i.$$

b)

$$\underline{\cos i =} \frac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \underline{\cosh 1} \doteq 1.54.$$

c)

$$\underline{\cosh i =}\frac{e^i+e^{-i}}{2} = \frac{\cos 1 + i\sin 1 + \cos(-1) + i\sin(-1)}{2} = \underline{\cos 1} \doteq 0.54.$$

d)

$$-5+3i = \sqrt{34}\left(\cos\left(\frac{\pi}{2}+\operatorname{arctg}\frac{5}{3}\right) + i\sin\left(\frac{\pi}{2}+\operatorname{arctg}\frac{5}{3}\right)\right),$$

a proto

$$\underline{\operatorname{Ln}(-5+3i) = \ln\sqrt{34} + i\left(\frac{\pi}{2}+\operatorname{arctg}\frac{5}{3}\right) + 2k\pi i,\ k\in\mathbb{Z};}$$

$$\underline{\ln(-5+3i) = \ln\sqrt{34} + i\left(\frac{\pi}{2}+\operatorname{arctg}\frac{5}{3}\right)}.$$

e)

$$-4-\sqrt{3}i = \sqrt{19}\left(\cos\left(-\pi+\operatorname{arctg}\frac{\sqrt{3}}{4}\right) + i\sin\left(-\pi+\operatorname{arctg}\frac{\sqrt{3}}{4}\right)\right),$$

a proto

$$\underline{\operatorname{Ln}(-4-\sqrt{3}i) = \ln\sqrt{19} + i\left(-\pi+\operatorname{arctg}\frac{\sqrt{3}}{4}\right) + 2k\pi i,\ k\in\mathbb{Z};}$$

$$\underline{\ln(-4-\sqrt{3}i) = \ln\sqrt{19} + i\left(-\pi+\operatorname{arctg}\frac{\sqrt{3}}{4}\right)}.$$

f)

$$\underline{\operatorname{Ln}(ie^2)} = \ln(e^2) + i\frac{\pi}{2} + 2k\pi i = \underline{2 + \frac{\pi}{2}i + 2k\pi i,\ k\in\mathbb{Z}.}$$

Příklad 15.

Najděte všechna $z\in\mathbb{C}$, pro která platí

a) $\sin z = 3$;

b) $\cos z = \frac{\sqrt{3}}{2}$;

c) $\sin z + \cos z = 2$;

d) $\sin z - \cos z = 3$;

e) $z^2 + 2z + 9 + 6i = 0$.

Řešení:

a)

$$\sin z = 3$$

$$\Updownarrow$$

$$e^{iz} - e^{-iz} = 6i$$

$$\Updownarrow$$

$$e^{2iz} - 6ie^{iz} - 1 = 0$$

a odtud pro $z = x + iy$ dostaneme

$$e^{iz} = e^{i(x+iy)} = \frac{6i \pm \sqrt{-36+4}}{2} = (3 \pm \sqrt{8})i$$

$$\Updownarrow$$

$$e^{-y}(\cos x + i \sin x) = (3 \pm \sqrt{8})i$$

$$\Updownarrow$$

$$e^{-y} = 3 \pm \sqrt{8} \;\wedge\; \left(x = \frac{\pi}{2} + 2k\pi,\; k \in \mathbb{Z}\right)$$

$$\Updownarrow$$

$$\underline{z = \frac{\pi}{2} + 2k\pi - i\ln(3 \pm \sqrt{8}),\; k \in \mathbb{Z}.}$$

b)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{\sqrt{3}}{2}$$

$$\Updownarrow$$

$$e^{2iz} - \sqrt{3}e^{iz} + 1 = 0$$

$$\Updownarrow$$

$$e^{iz} = e^{(x+iy)i} = e^{-y}(\cos x + i \sin x) = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} =$$

$$= \frac{\sqrt{3}}{2} \pm \frac{i}{2} = \begin{cases} \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) \\ \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right) \end{cases}$$

$$\Updownarrow$$

$$e^{-y} = 1 \;\wedge\; \left(x = \pm\frac{\pi}{6} + 2k\pi,\; k \in \mathbb{Z}\right)$$

$$\Updownarrow$$

$$\underline{z = \pm\frac{\pi}{6} + 2k\pi,\; k \in \mathbb{Z}.}$$

c)

$$\frac{e^{iz}-e^{-iz}}{2i}+\frac{e^{iz}+e^{-iz}}{2}=2$$

$$\Updownarrow$$

$$e^{iz}-e^{-iz}+ie^{iz}+ie^{-iz}=4i$$

$$\Updownarrow$$

$$e^{2iz}(1+i)-4ie^{iz}+(i-1)=0$$

$$\Updownarrow$$

$$e^{iz}=\frac{4i\pm\sqrt{-16-4(1+i)(i-1)}}{2(1+i)}\quad=\frac{4i\pm\sqrt{2}i}{2(1+i)}=$$

$$=\frac{(2\pm\sqrt{2})i}{1+i}=\frac{(2\pm\sqrt{2})(1+i)}{2},$$

a odtud

$$iz=\operatorname{Ln}\frac{(2\pm\sqrt{2})(1+i)}{2}=\ln\frac{(2\pm\sqrt{2})\sqrt{2}}{2}+i\frac{\pi}{4}+2k\pi i,\ k\in\mathbb{Z},$$

$$\underline{z=\frac{\pi}{4}+2k\pi-i\ln(\sqrt{2}\pm 1),\ k\in\mathbb{Z}.}$$

d)

$$\frac{e^{iz}-e^{-iz}}{2i}-\frac{e^{iz}+e^{-iz}}{2}=3$$

$$\Updownarrow$$

$$e^{iz}-e^{-iz}-ie^{iz}-ie^{-iz}=6i$$

$$\Updownarrow$$

$$e^{2iz}(1-i)-6ie^{iz}-(1+i)=0$$

$$\Updownarrow$$

$$e^{iz}=\frac{6i\pm\sqrt{-36+4(1-i)(1+i)}}{2(1-i)}=\frac{(6\pm 2\sqrt{7})i}{2(1-i)}=$$

$$=\frac{3\pm\sqrt{7}}{2}\cdot(-1+i),$$

a proto

$$iz=\operatorname{Ln}\left(\frac{3\pm\sqrt{7}}{2}(-1+i)\right)=\ln\left(\frac{3\pm\sqrt{7}}{2}\cdot\sqrt{2}\right)+i\frac{3}{4}\pi+2k\pi i,\ k\in\mathbb{Z},$$

$$\underline{z=\frac{3}{4}\pi+2k\pi-i\ln\left(\frac{3\pm\sqrt{7}}{\sqrt{2}}\right),\ k\in\mathbb{Z}.}$$

e)

$$\underline{z =} \frac{-2 \pm \sqrt{4 - 4(9+6i)}}{2} =$$

$$= -1 \pm \sqrt{1 - (9+6i)} = -1 \pm \sqrt{-8-6i} =$$

$$= \underline{\begin{cases} -2+3i, \\ -3i, \end{cases}}$$

protože

$$\sqrt{-8-6i} = \sqrt{10\left(\cos\left(\pi + \operatorname{arctg}\frac{3}{4}\right) + i\sin\left(\pi + \operatorname{arctg}\frac{3}{4}\right)\right)} =$$

$$= \pm\sqrt{10}\cdot\left(\cos\left(\frac{\pi}{2} + \frac{1}{2}\operatorname{arctg}\frac{3}{4}\right) + i\sin\left(\frac{\pi}{2} + \frac{1}{2}\operatorname{arctg}\frac{3}{4}\right)\right) =$$

$$= \pm(-1+3i).$$

<u>PŘÍKLAD 16</u>.
Vypočtěte

a) 2^i;

b) $(-2)^{\sqrt{2}}$;

c) $\left(\frac{1-i}{\sqrt{2}}\right)^{1+i}$;

d) $i^{\frac{3}{4}}$;

e) $(-1)^{\sqrt{3}}$;

f) $(-\sqrt{3}i+1)^{-3}$.

<u>Řešení:</u>

a)

$$\underline{2^i =} \exp(i\,\mathrm{Ln}\,2) =$$

$$= \exp(i(\ln 2 + 2k\pi i)) = \exp(-2k\pi + i\ln 2) =$$

$$= \underline{e^{-2k\pi}\left(\cos(\ln 2) + i\sin(\ln 2)\right),\ k\in\mathbb{Z}}.$$

b)

$$\underline{(-2)^{\sqrt{2}} =} \exp\left(\sqrt{2}\,\mathrm{Ln}(-2)\right) = \exp\left(\sqrt{2}(\ln 2 + \pi i + 2k\pi i)\right) =$$

$$= \underline{2^{\sqrt{2}}\left(\cos\left((2k+1)\pi\sqrt{2}\right) + i\sin\left((2k+1)\pi\sqrt{2}\right)\right),\ k\in\mathbb{Z}}.$$

c)

$$\underline{\left(\frac{1-i}{\sqrt{2}}\right)^{1+i}} = \exp\left((1+i)\,\mathrm{Ln}\left(\frac{1-i}{\sqrt{2}}\right)\right) =$$

$$= \exp\left((1+i)(\ln 1 - i\frac{\pi}{4} + 2k\pi i)\right) = \exp\left(\frac{\pi}{4} - 2k\pi + (2k\pi - \frac{\pi}{4})i\right) =$$

$$= e^{\frac{\pi}{4}-2k\pi}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = e^{\frac{\pi}{4}-2k\pi}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) =$$

$$= \underline{\frac{1-i}{\sqrt{2}}e^{\frac{\pi}{4}-2k\pi},\ k\in\mathbb{Z}}.$$

d)

$$\underline{i^{\frac{3}{4}}} = e^{\frac{3}{4}\operatorname{Ln} i} = e^{\frac{3}{4}(\frac{\pi}{2}i+2k\pi i)} =$$

$$= \underline{\cos\left(\frac{3}{8}\pi + \frac{3}{2}k\pi\right) + i\sin\left(\frac{3}{8}\pi + \frac{3}{2}k\pi\right),\ k \in \{0,1,2,3\}.}$$

e)

$$\underline{(-1)^{\sqrt{3}}} = e^{\sqrt{3}\operatorname{Ln}(-1)} = e^{\sqrt{3}(\pi i+2k\pi i)} =$$

$$= \underline{\cos\left(\sqrt{3}\pi + 2k\pi\sqrt{3}\right) + i\sin\left(\sqrt{3}\pi + 2k\pi\sqrt{3}\right),\ k \in \mathbb{Z}.}$$

f)

$$\underline{(-\sqrt{3}i+1)^{-3}} = \left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]^{-3} =$$

$$= \frac{1}{\left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]^3} =$$

$$= \frac{1}{8\cos(-\pi) + i\sin(-\pi)} = \underline{-\frac{1}{8}}.$$

Jinak:

$$\underline{(-\sqrt{3}i+1)^{-3}} = e^{-3\operatorname{Ln}\left(-\sqrt{3}i+1\right)} = e^{-3\left(\ln 2 - \frac{\pi}{3}i + 2k\pi i\right)} =$$

$$= \frac{1}{8}(\cos\pi + i\sin\pi) = \underline{-\frac{1}{8}}.$$

<u>Příklad 17.</u>

Najděte reálnou a imaginární část funkce $f\colon\ \mathbb{C} \to \mathbb{C}$ definované předpisem

a) $f(z) := \sin z$;

b) $f(z) := z^2 \cos z$;

c) $f(z) := z^3 + 5z - 1$;

d) $f(z) := |z|\,\overline{z}$;

e) $f(z) := z^2\,\overline{z}$;

f) $f(z) := \frac{1}{z}$.

<u>Řešení:</u>

a)

$$f(z) = f(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} =$$

$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} =$$

$$= \frac{e^y + e^{-y}}{2}\sin x + i\,\frac{e^y - e^{-y}}{2}\cos x.$$

Odtud

$$\underline{(\operatorname{Re} f)(x,y) = \cosh y\ \sin x},$$
$$\underline{(\operatorname{Im} f)(x,y) = \sinh y\ \cos x}.$$

b)

$$
\begin{aligned}
f(x+iy) &= (x^2-y^2+2xyi)\frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2} = \\
&= (x^2-y^2+2xyi)\frac{e^{-y}\left(\cos x + i\sin x\right)+e^{y}\left(\cos x - i\sin x\right)}{2},
\end{aligned}
$$

a proto

$$
\begin{aligned}
&\underline{(\operatorname{Re} f)(x,y) = (x^2-y^2)\cosh y\ \cos x + 2xy\sinh y\ \sin x,} \\
&\underline{(\operatorname{Im} f)(x,y) = -(x^2-y^2)\sinh y\ \sin x + 2xy\cosh y\ \cos x.}
\end{aligned}
$$

c)

$$
\begin{aligned}
f(x+iy) &= (x^2-y^2+2xyi)(x+iy)+5(x+iy)-1 = \\
&= (x^3-xy^2-2xy^2+5x-1)+i\left(2x^2y+x^2y-y^3+5y\right).
\end{aligned}
$$

Tedy

$$
\begin{aligned}
&\underline{(\operatorname{Re} f)(x,y) = x^3-3xy^2+5x-1,} \\
&\underline{(\operatorname{Im} f)(x,y) = 3x^2y-y^3+5y.}
\end{aligned}
$$

d)

$$
\begin{aligned}
f(x+iy) &= \sqrt{x^2+y^2}\,(x-iy) = \\
&= x\sqrt{x^2+y^2}-iy\sqrt{x^2+y^2}.
\end{aligned}
$$

Zjistili jsme, že

$$
\begin{aligned}
&\underline{(\operatorname{Re} f)(x,y) = x\sqrt{x^2+y^2},} \\
&\underline{(\operatorname{Im} f)(x,y) = -y\sqrt{x^2+y^2}.}
\end{aligned}
$$

e)

$$
\begin{aligned}
f(x+iy) &= (x^2-y^2+2xyi)(x-iy) = \\
&= x^3-xy^2+2xy^2+i(2x^2y-x^2y+y^3).
\end{aligned}
$$

Tudíž

$$
\begin{aligned}
&\underline{(\operatorname{Re} f)(x,y) = x^3+xy^2,} \\
&\underline{(\operatorname{Im} f)(x,y) = x^2y+y^3.}
\end{aligned}
$$

f)

$$
f(x+iy) = \frac{x-iy}{x^2+y^2},
$$

a proto

$$
\begin{aligned}
&\underline{(\operatorname{Re} f)(x,y) = \frac{x}{x^2+y^2},} \\
&\underline{(\operatorname{Im} f)(x,y) = -\frac{y}{x^2+y^2}.}
\end{aligned}
$$

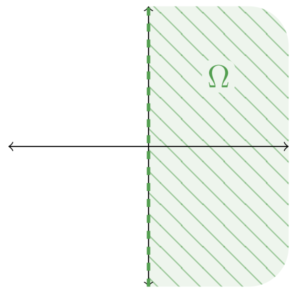
Příklad 18.
Zjistěte, zda je funkce $f(z) := z^3$ prostá na množině Ω, je-li

a) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\}$;

b) $\Omega = \{z \in \mathbb{C}\colon\ \arg z \in \langle 0, \frac{\pi}{4})\}$.

Řešení:

a)



Volme

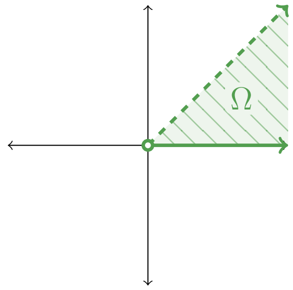
$$z_1 = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right) \in \Omega,$$
$$z_2 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \in \Omega.$$

Potom

$$\begin{aligned} z_1^3 &= \cos -\pi + i\sin -\pi &= -1, \\ z_2^3 &= \cos\pi + i\sin\pi &= -1, \end{aligned}$$

a proto f není na Ω prostá.

b)



Buď

$$z_1 = |z_1|\left(\cos\left(\varphi_1\right) + i\sin\left(\varphi_1\right)\right),$$
$$z_2 = |z_2|\left(\cos\left(\varphi_2\right) + i\sin\left(\varphi_2\right)\right),$$

kde $\varphi_1,\ \varphi_2 \in \left\langle 0, \frac{\pi}{4}\right)$. Pak

$$z_1^3 = z_2^3$$
$$\Updownarrow$$
$$|z_1|^3\left(\cos\left(3\varphi_1\right) + i\sin\left(3\varphi_1\right)\right) = |z_2|\left(\cos\left(3\varphi_2\right) + i\sin\left(3\varphi_2\right)\right)$$
$$\Updownarrow$$
$$\left(|z_1| = |z_2|\right) \wedge \left(\exists k \in \mathbb{Z}\colon\ 3\varphi_1 = 3\varphi_2 + 2k\pi\right).$$

Odtud plyne (využíváme předpokladu, že $\varphi_1,\ \varphi_2 \in \left\langle 0, \frac{\pi}{4}\right)$):

$$\begin{matrix} z_1,\ z_2 \in \Omega \\ z_1^3 = z_2^3 \end{matrix} \Rightarrow \begin{matrix} |z_1| = |z_2| \\ \varphi_1 = \varphi_2 \end{matrix} \Rightarrow z_1 = z_2,$$

tzn. že funkce f je na Ω prostá.

PŘÍKLAD 19.

Určete, zda daná limita existuje, a pokud ano, vypočtěte ji

a) $\lim\limits_{z\to 0} \frac{\operatorname{Re} z}{z}$;

b) $\lim\limits_{z\to 0} \frac{\operatorname{Im}(z^2)}{z\bar{z}}$;

c) $\lim\limits_{z\to 0} \frac{z \operatorname{Im} z}{|z|}$;

d) $\lim\limits_{z\to 0} \frac{z^2}{|z|^2}$;

e) $\lim\limits_{z\to 0} \frac{z^3}{|z|^2}$;

f) $\lim\limits_{z\to i} \frac{z^2+z(2-i)-2i}{z^2+1}$;

g) $\lim\limits_{z\to 0} \frac{\operatorname{Re} z}{1+|z|}$.

Řešení:

a) $\lim\limits_{z\to 0} \frac{\operatorname{Re} z}{z}$ neexistuje, protože

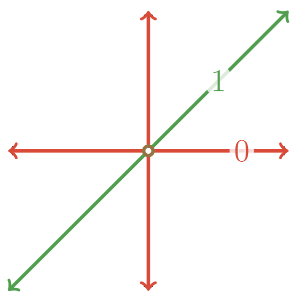
$$0 \neq \frac{1}{n} \to 0 \ \wedge\ \frac{\operatorname{Re}\left(\frac{1}{n} + 0i\right)}{\frac{1}{n}} = 1 \to 1$$

a současně

$$0 \neq i\frac{1}{n} \to 0 \ \wedge\ \frac{\operatorname{Re}\left(i\frac{1}{n}\right)}{\frac{1}{n}} = 0 \to 0.$$

b) $\lim\limits_{z\to 0} \frac{\operatorname{Im} z^2}{z\cdot\bar{z}}$ neexistuje, protože pro $0 \neq z = x + iy$ platí

$$\frac{\operatorname{Im} z^2}{z\cdot\bar{z}} = \frac{2xy}{x^2+y^2} = \begin{cases} 1, & x = y \neq 0, \\ 0, & x\cdot y = 0,\ x^2+y^2 \neq 0. \end{cases}$$

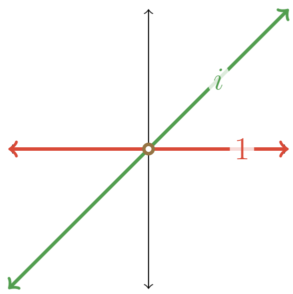


c) $\underline{\lim_{z\to 0} \frac{z \operatorname{Im} z}{|z|} = 0}$, protože

$$0 \neq z_n \to 0 \;\Rightarrow\; \left|\frac{z_n \operatorname{Im} z_n}{|z_n|}\right| = |\operatorname{Im} z_n| \to 0 \;\Rightarrow\; \frac{z_n \operatorname{Im} z_n}{|z_n|} \to 0.$$

d) $\underline{\lim_{z\to 0} \frac{z^2}{|z|^2} \text{ neexistuje}}$, protože pro $0 \neq z = x + iy$ platí

$$\frac{z^2}{|z|^2} = \frac{x^2 - y^2 + 2ixy}{x^2 + y^2} = \begin{cases} i, & x = y \neq 0, \\ 1, & y = 0 \neq x. \end{cases}$$



e) $\underline{\lim_{z\to 0} \frac{z^3}{|z|^2} = 0}$, protože

$$\lim_{z\to 0} \left|\frac{z^3}{|z|^2}\right| = \lim_{z\to 0} |z| = 0.$$

f)

$$\underline{\lim_{z\to i} \frac{z^2 + z(2-i) - 2i}{z^2+1}} = \lim_{z\to i} \frac{(z-i)(z+2)}{(z-i)(z+i)} = \underline{\lim_{z\to i} \frac{z+2}{z+i}} =$$

$$= \lim_{x+iy\to i} \frac{x+2+iy}{x+i(y+1)} =$$

$$= \lim_{(x,y)\to(0,1)} \frac{x(x+2)+y(y+1)}{x^2+(y+1)^2} +$$

$$+ i \lim_{(x,y)\to(0,1)} \frac{xy-(x+2)(y+1)}{x^2+(y+1)^2} =$$

$$= \frac{1\cdot 2}{2^2} + i\frac{-2\cdot 2}{4} = \underline{\frac{1}{2} - i},$$

případně pomocí spojitosti funkce $f(z) := \frac{z+2}{z+i}$ v bodě i:

$$\underline{\lim_{z\to i} \frac{z+2}{z+i}} = \frac{2+i}{2i} = \underline{\frac{1}{2} - i}.$$

g)

$$\underline{\lim_{z\to 0} \frac{\operatorname{Re} z}{1+|z|}} = \lim_{(x,y)\to(0,0)} \frac{x}{1+\sqrt{x^2+y^2}} = \frac{0}{1} = \underline{0}.$$

PŘÍKLAD 20.
Znázorněte množinu $\langle\varphi\rangle := \{\varphi(t)\colon\ t \in D\varphi\}$, je-li

a) $\varphi(t) := 1 - it,\ D\varphi = \langle 0, 2\rangle$;

b) $\varphi(t) := t - it^2,\ D\varphi = \langle -1, 2\rangle$;

c) $\varphi(t) := 1 + e^{-it},\ D\varphi = \langle 0, 2\pi\rangle$;

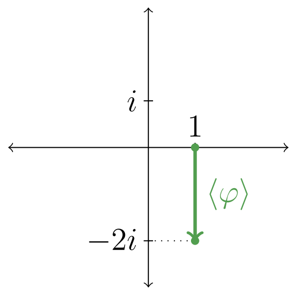
d) $\varphi(t) := e^{2it} - 1,\ D\varphi = \langle 0, 2\pi\rangle$;

e) $\varphi(t) := \begin{cases} e^{i\pi t}, & t \in \langle 0, 1), \\ t - 2, & t \in \langle 1, 3\rangle; \end{cases}$

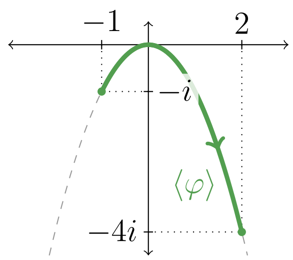
f) $\varphi(t) := \begin{cases} e^{it}, & t \in \langle -\frac{\pi}{2}, \pi), \\ \frac{3t}{\pi} - 4, & t \in \langle \pi, 2\pi\rangle. \end{cases}$

Řešení:

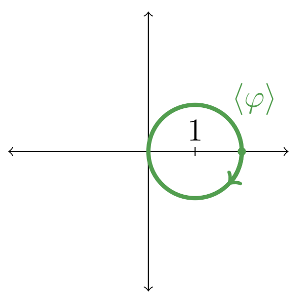
a) $\varphi(t) := 1 - it,\ D\varphi = \langle 0, 2\rangle$.



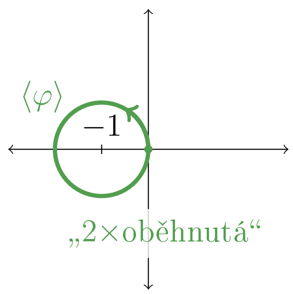
b) $\varphi(t) := t - it^2,\ D\varphi = \langle -1, 2\rangle$.



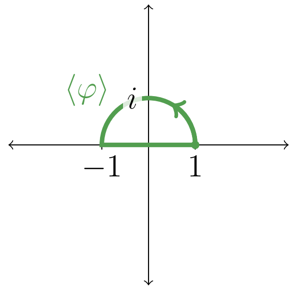
c) $\varphi(t) := 1 + e^{-it},\ D\varphi = \langle 0, 2\pi\rangle$.



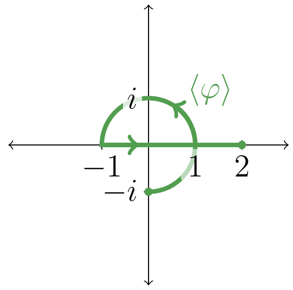
d) $\varphi(t) := e^{2it} - 1,\ D\varphi = \langle 0, 2\pi\rangle$.



e) $\varphi(t) := \begin{cases} e^{i\pi t}, & t \in \langle 0, 1), \\ t - 2, & t \in \langle 1, 3 \rangle. \end{cases}$



f) $\varphi(t) := \begin{cases} e^{it}, & t \in \langle -\frac{\pi}{2}, \pi), \\ \frac{3t}{\pi} - 4, & t \in \langle \pi, 2\pi \rangle. \end{cases}$



<u>Příklad 21</u>.
Parametrizujte množinu Ω (tzn. najděte křivku φ takovou, aby $\langle \varphi \rangle = \Omega$), je-li

a) $\Omega = \{z \in \mathbb{C}\colon\ |z - 2 + 3i| = 2\}$;

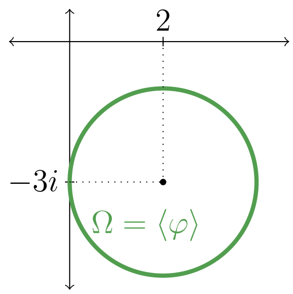
b) Ω úsečka s krajními body $a, b \in \mathbb{C}$, $a \neq b$;

c) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z = 2 \operatorname{Im} z\}$;

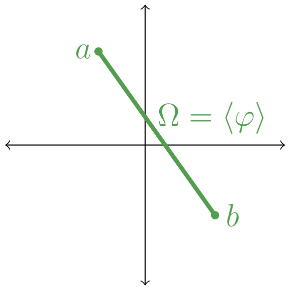
d) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re}\left(\frac{1}{z}\right) = 2\}$.

<u>Řešení:</u>

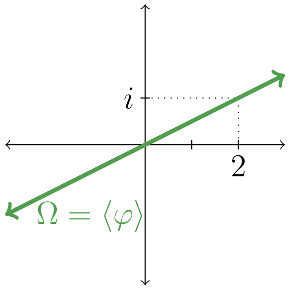
a) $\Omega = \{z \in \mathbb{C}\colon\ |z - 2 + 3i| = 2\}$; $\underline{\varphi(t) := 2 - 3i + 2e^{it},\ t \in \langle 0, 2\pi \rangle}$.



b) Ω je úsečka s krajními body $a, b \in \mathbb{C}$, $a \neq b$; $\underline{\varphi(t) := a + (b-a)t,\ t \in \langle 0, 1\rangle}$.



c) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z = 2\operatorname{Im} z\}$; $\underline{\varphi(t) := t + \frac{t}{2}i,\ t \in \mathbb{R}}$.



d)

$$\Omega = \left\{z \in \mathbb{C}\colon\ \operatorname{Re}\left(\frac{1}{z}\right) = 2\right\} =$$

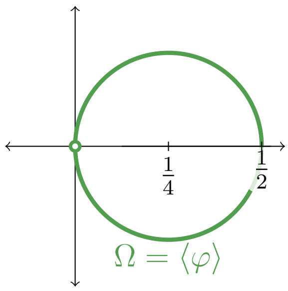
$$= \left\{x + iy\colon\ \operatorname{Re}\left(\frac{1}{x+iy}\right) = \frac{x}{x^2+y^2} = 2\right\} =$$

$$= \left\{x + iy \in \mathbb{C}\setminus\{0\}\colon\ 2\left(x^2 - \frac{x}{2} + y^2\right) = 0\right\} =$$

$$= \left\{x + iy \in \mathbb{C}\setminus\{0\}\colon\ 2\left((x - \frac{1}{4})^2 + y^2 - \frac{1}{16}\right) = 0\right\} =$$

$$= \left\{x + iy \in \mathbb{C}\setminus\{0\}\colon\ \left(x - \frac{1}{4}\right)^2 + y^2 = \frac{1}{16}\right\};$$

$$\underline{\varphi(t) := \frac{1}{4} + \frac{1}{4}e^{it},\ t \in (-\pi, \pi).}$$



Příklad 22.
Znázorněte množinu Ω a rozhodněte, zda je Ω oblastí a zda je Ω otevřenou množinou, je-li

a) $\Omega = \{z \in \mathbb{C}\colon\ |z-i| < 1 \ \vee\ |z+i| < 1\}$;

b) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| < 1 \ \wedge\ |z-2| < 2\}$;

c) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| < |z+1|\}$;

d) $\Omega = \{z \in \mathbb{C}\colon\ |z+1| > 2|z|\}$;

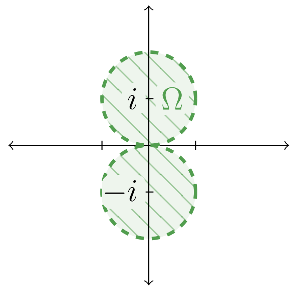
e) $\Omega = \{z \in \mathbb{C}\colon\ 1 < |z| < 2\}$;

f) $\Omega = \big\{z \in \mathbb{C}\colon\ |z| < 1 \ \wedge\ \arg z \in (-\pi, \pi\rangle \setminus \{0\}\big\}$;

g) $\Omega = \{z \in \mathbb{C}\colon\ |2z| < |1+z^2|\}$.

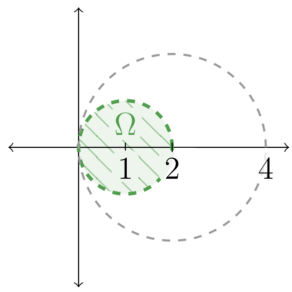
Řešení:

a) $\Omega = \{z \in \mathbb{C}\colon\ |z-i| < 1 \ \vee\ |z+i| < 1\}$.



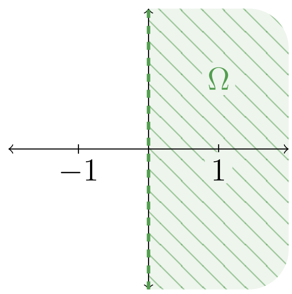
Ω je otevřená, ale ne souvislá množina, a proto Ω není oblast.

b) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| < 1 \ \wedge\ |z-2| < 2\}$.



Ω je otevřená i souvislá množina, a proto Ω je oblast.

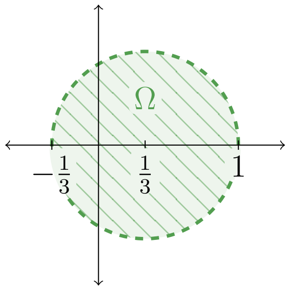
c) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| < |z+1|\}$.



Ω je otevřená i souvislá množina, a proto Ω je oblast.

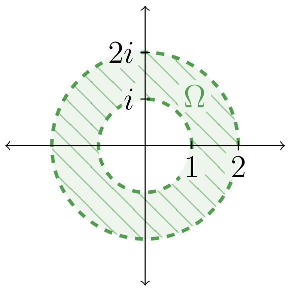
d)

$$\begin{aligned}\Omega = \{z \in \mathbb{C}\colon\ |z+1| > 2|z|\} &= \\ = \big\{x+iy\colon\ (x+1)^2+y^2 > 4(x^2+y^2)\big\} &= \\ = \big\{x+iy\colon\ 3x^2+3y^2-2x-1<0\big\} &= \\ = \left\{x+iy\colon\ x^2+y^2-\frac{2}{3}x-\frac{1}{3}<0\right\} &= \\ = \left\{x+iy\colon\ \left(x-\frac{1}{3}\right)^2+y^2<\frac{4}{9}\right\}&.\end{aligned}$$



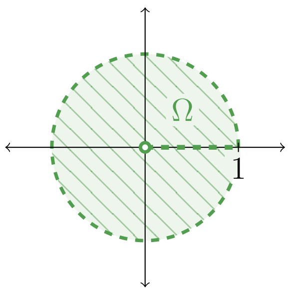
<u>Ω je otevřená</u> i souvislá množina, a tedy <u>Ω je oblast</u>.

e) $\Omega = \{z \in \mathbb{C}\colon\ 1 < |z| < 2\}$.



<u>Ω je otevřená</u> i souvislá množina, a proto <u>Ω je oblast</u>.

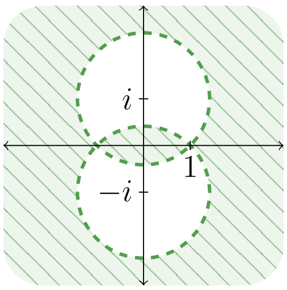
f) $\Omega = \big\{z \in \mathbb{C}\colon\ |z| < 1\ \wedge\ \arg z \in (-\pi, \pi\rangle \setminus \{0\}\big\}$.



<u>Ω je otevřená</u> i souvislá množina, a proto <u>Ω je oblast</u>.

g)

$$\begin{aligned}
\Omega = \{z \in \mathbb{C}\colon\ |2z| < |1+z^2|\} &= \\
&= \big\{x+iy\colon\ 4(x^2+y^2) < (1+x^2-y^2)^2 + 4x^2y^2\big\} = \\
&= \big\{x+iy\colon\ 4x^2+4y^2 < 1+x^4+y^4+2x^2-2y^2-2x^2y^2+4x^2y^2\big\} = \\
&= \big\{x+iy\colon\ 0 < 1+x^4+y^4-2x^2-6y^2+2x^2y^2\big\} = \\
&= \big\{x+iy\colon\ (x^2+y^2-1)^2-4y^2 > 0\big\} = \\
&= \big\{x+iy\colon\ (x^2+y^2-1+2y)(x^2+y^2-1-2y) > 0\big\} = \\
&= \big\{x+iy\colon\ [x^2+(y+1)^2-2][x^2+(y-1)^2-2] > 0\big\}\,.
\end{aligned}$$



<u>Ω je otevřená</u>, ale ne souvislá množina, a proto <u>Ω není oblast</u>.

<u>PŘÍKLAD 23.</u>
Zjistěte, ve kterých bodech má funkce f derivaci a ve kterých bodech je funkce f holomorfní, je-li

a) $f(z) := \operatorname{Re} z$;

b) $f(z) := |z^2|$;

c) $f(z) := ze^z$;

d) $f(z) := \overline{z}|z|$;

e) $f(z) := \frac{\operatorname{Re} z}{z}$;

f) $f(z) := z^2\overline{z}$;

g) $f(z) := z^2 + 2z - 1$.

<u>Řešení:</u>

a)

$$f(x+iy) = \underbrace{x}_{=:u(x,y)} + \underbrace{0}_{=:v(x,y)} \cdot i.$$

Pro každé $(x,y) \in \mathbb{R}^2$ platí, že

$$\frac{\partial u}{\partial x}(x,y) = 1 \neq 0 = \frac{\partial v}{\partial y}(x,y),$$

a proto <u>f nemá nikde derivaci</u> a <u>f není v žádném bodě holomorfní</u>.

b) $f(x+iy) = |(x+iy)^2| = (|x+iy|)^2 = x^2+y^2$. Takže $f = u+iv$, kde $u(x,y) := x^2+y^2$ a $v(x,y) := 0$.

$$\left.\begin{aligned} \frac{\partial u}{\partial x}(x,y) = 2x \quad &= \quad \frac{\partial v}{\partial y}(x,y) = 0 \\ \frac{\partial u}{\partial y}(x,y) = 2y \quad &= \quad -\frac{\partial v}{\partial x}(x,y) = 0 \end{aligned}\right\} \Leftrightarrow (x,y) = (0,0),$$

a současně funkce u a v jsou diferencovatelné v $\mathbb{R}^2$, a proto f má derivaci (jen) v bodě 0 a holomorfní není v žádném bodě.

c)

$$\begin{aligned} f(x+iy) &= (x+iy)e^x(\cos y + i\sin y) = \\ &= \underbrace{xe^x\cos y - ye^x\sin y}_{=:u(x,y)} + i\underbrace{(xe^x\sin y + ye^x\cos y)}_{=:v(x,y)}. \end{aligned}$$

Funkce u a v jsou diferencovatelné v $\mathbb{R}^2$ a navíc

$$\frac{\partial u}{\partial x}(x,y) = e^x\cos y + xe^x\cos y - ye^x\sin y,$$

$$\frac{\partial v}{\partial y}(x,y) = xe^x\cos y + e^x\cos y - ye^x\sin y$$

a

$$\frac{\partial u}{\partial y}(x,y) = -xe^x\sin y - e^x\sin y - ye^x\cos y,$$

$$-\frac{\partial v}{\partial x}(x,y) = -\left[e^x\sin y + xe^x\sin y + ye^x\cos y\right].$$

Odtud plyne, že $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ a $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ v $\mathbb{R}^2$, a proto funkce f je holomorfní ve všech bodech $\mathbb{C}$ a $f'(z)$ existuje pro každé $z \in \mathbb{C}$.

$$\left(f'(z) = f'(x+iy) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(x,y) = \cdots = e^z + ze^z.\right)$$

d)

$$f(x+iy) = (x-iy)\sqrt{x^2+y^2} = \underbrace{x\sqrt{x^2+y^2}}_{=:u(x,y)} + i\underbrace{(-y\sqrt{x^2+y^2})}_{=:v(x,y)}.$$

Odtud plyne, že pro každé $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ je

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}} > 0, \\ \frac{\partial v}{\partial y}(x,y) &= -\sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}} < 0, \end{aligned}$$

a proto: je-li $z \neq 0$, tak $f'(z)$ neexistuje.

Zbývá vyšetřit existenci derivace v bodě 0:

$$\underline{f'(0)} = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{\overline{z}\cdot|z|}{z} =$$

$$= \lim_{z\to 0} \frac{|z|\,(\cos(\arg z) - i\sin(\arg z))\cdot|z|}{|z|\,(\cos(\arg z) + i\sin(\arg z))} =$$

$$= \lim_{z\to 0} \left[|z|\cdot(\cos(-2\arg z) + i\sin(-2\arg z))\right] = \underline{0},$$

$$\text{protože } \forall z \neq 0\colon\ |\cos(-2\arg z) + i\sin(-2\arg z)| = 1.$$

Shrnutí: funkce <u>f má derivaci pouze v bodě 0</u>, a proto <u>není holomorfní v žádném bodě</u>.

e)

$$f(x+iy) = \frac{x}{x+iy} = \frac{x(x-iy)}{x^2+y^2} = \underbrace{\frac{x^2}{x^2+y^2}}_{=:u(x,y)} + i\underbrace{\left(-\frac{xy}{x^2+y^2}\right)}_{=:v(x,y)}.$$

Pro každé $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ platí

$$\frac{\partial u}{\partial x}(x,y) = \frac{2x(x^2+y^2)-x^2 2x}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial y}(x,y) = \frac{-x(x^2+y^2)+xy2y}{(x^2+y^2)^2} = \frac{-x^3+xy^2}{(x^2+y^2)^2},$$

$$\frac{\partial u}{\partial y}(x,y) = \frac{-x^2 2y}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial x}(x,y) = \frac{-y(x^2+y^2)+xy2x}{(x^2+y^2)^2} = \frac{x^2y-y^3}{(x^2+y^2)^2},$$

a proto derivace může existovat pouze v takových bodech $x+iy$, v nichž platí

$$\left(\frac{2xy^2}{(x^2+y^2)^2} = \frac{x(-x^2+y^2)}{(x^2+y^2)^2}\right)\wedge\left(\frac{2x^2y}{(x^2+y^2)^2} = \frac{y(x^2-y^2)}{(x^2+y^2)^2}\right),$$

neboli

$$\left(\frac{xy^2}{(x^2+y^2)^2} = \frac{-x^3}{(x^2+y^2)^2}\right)\wedge\left(\frac{x^2y}{(x^2+y^2)^2} = \frac{-y^3}{(x^2+y^2)^2}\right).$$

Není těžké si všimnout, že tato soustava nemá žádné řešení.

Závěr: funkce <u>f nemá v žádném bodě derivaci</u>, a proto taky <u>není v žádném bodě holomorfní</u>.

f)

$$f(x+iy) = (x^2-y^2+2ixy)(x-iy) =$$

$$= x^3 - xy^2 + 2xy^2 + i(-x^2y + y^3 + 2x^2y) =$$

$$= \underbrace{x^3+xy^2}_{=:u(x,y)} + i\underbrace{(y^3+x^2y)}_{=:v(x,y)}.$$

Funkce u a v jsou diferencovatelné v $\mathbb{R}^2$ a pro každé $(x,y) \in \mathbb{R}^2$ platí

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= 3x^2 + y^2, & \frac{\partial u}{\partial y}(x,y) &= \quad 2xy, \\ \frac{\partial v}{\partial y}(x,y) &= 3y^2 + x^2, & -\frac{\partial v}{\partial x}(x,y) &= -2xy. \end{aligned}$$

Odtud plyne, že derivace existuje v těch bodech $x + iy$, v nichž platí současně $2x^2 = 2y^2$ a $4xy = 0$. Takový bod je pouze jeden, a to $z = 0 + i0 = 0$.

Shrnutí: funkce f má derivaci pouze v bodě 0 ($f'(0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0$), holomorfní není v žádném bodě.

g)

$$f(x+iy) = x^2 - y^2 + 2ixy + 2x + 2iy - 1 = \underbrace{x^2 - y^2 + 2x - 1}_{=:u(x,y)} + i\underbrace{(2xy + 2y)}_{=:v(x,y)}.$$

Pro každé $(x,y) \in \mathbb{R}^2$ platí, že

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \quad 2x+2 \quad = \frac{\partial v}{\partial y}(x,y), \\ \frac{\partial u}{\partial y}(x,y) &= \quad -2y \quad = -\frac{\partial v}{\partial x}(x,y), \end{aligned}$$

a protože funkce u a v jsou navíc zřejmě diferencovatelné v $\mathbb{R}^2$, je pro každé $z = x + iy$

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \cdots = 2z + 2.$$

Funkce f má derivaci v každém bodě $z \in \mathbb{C}$ a v každém bodě $z \in \mathbb{C}$ je holomorfní.

Příklad 24.

Zjistěte, zda je funkce Φ harmonická na oblasti Ω, je-li

a) $\Phi(x,y) := x^2 - y^2 + 2011$, $\Omega = \mathbb{C}$;

b) $\Phi(x,y) := \frac{x}{x^2+y^2} + x^2 - y^2 + x - y$, $\Omega = \mathbb{C} \setminus \{0\}$.

Řešení:

a) Protože $\Phi \in \mathcal{C}^\infty(\mathbb{R}^2)$ a pro každé $(x,y) \in \mathbb{R}^2$ platí, že

$$\Delta\Phi(x,y) = \frac{\partial^2 \Phi}{\partial x^2}(x,y) + \frac{\partial^2 \Phi}{\partial y^2}(x,y) = 2 - 2 = 0,$$

je Φ harmonická na $\mathbb{C}$.

b) $\Phi \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$ a pro každé $(x,y) \neq (0,0)$ platí

$$\frac{\partial \Phi}{\partial x}(x,y) = \frac{x^2+y^2-x2x}{(x^2+y^2)^2} + 2x + 1,$$

$$\frac{\partial^2 \Phi}{\partial x^2}(x,y) = \frac{-2x(x^2+y^2)^2 - (-x^2+y^2)2(x^2+y^2)^2 2x}{(x^2+y^2)^4} + 2,$$

$$\frac{\partial \Phi}{\partial y}(x,y) = \frac{-x2y}{(x^2+y^2)^2} - 2y - 1,$$

$$\frac{\partial^2 \Phi}{\partial y^2}(x,y) = \frac{-2x(x^2+y^2)^2 + 2xy2(x^2+y^2)2y}{(x^2+y^2)^4} - 2.$$

Odtud plyne, že

$$\Delta\Phi(x,y) = \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3} + \frac{-2x(x^2+y^2) + 8xy^2}{(x^2+y^2)^3} = 0,$$

a proto Φ je harmonická na $\mathbb{C} \setminus \{0\}$.

PŘÍKLAD 25.

Najděte (existuje-li) na oblasti Ω holomorfní funkci $f = u + iv$, je-li

a) $u(x,y) := x^3 - 3xy^2 - 2y$, $\Omega = \mathbb{C}$;

b) $u(x,y) := \frac{x}{x^2+y^2}$, $\Omega = \mathbb{C} \setminus \{0\}$;

c) $u(x,y) := 3x^2 - y^2 + 3x + y$, $\Omega = \mathbb{C}$;

d) $u(x,y) := x^2 - y^2 + 5x + y - \frac{y}{x^2+y^2}$, $\Omega = \mathbb{C} \setminus \{0\}$.

Řešení:

a)

$$\frac{\partial u}{\partial x}(x,y) = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}(x,y) \;\Rightarrow\; v(x,y) = 3x^2y - y^3 + \varphi(x),$$

$$\frac{\partial u}{\partial y}(x,y) = -6xy - 2 = -\frac{\partial v}{\partial x}(x,y) = -6xy - \varphi'(x) \;\Rightarrow\; \varphi(x) = 2x + c,\; c \in \mathbb{R},$$

a proto

$$f(x+iy) = (x^3 - 3xy^2 - 2y) + i(3x^2y - y^3 + 2x + c),\; c \in \mathbb{R}.$$

(Můžete si všimnout, že $f(z) = z^3 + 2zi + ci,\; c \in \mathbb{R}$.)

b)

$$-\frac{\partial u}{\partial y}(x,y) = \frac{2xy}{(x^2+y^2)^2} = \frac{\partial v}{\partial x}(x,y) \;\Rightarrow\; v(x,y) = \int \frac{2xy}{(x^2+y^2)^2}\,\mathrm{d}x.$$

Po substituci $x^2 + y^2 = t$ ($2x\,\mathrm{d}x = \mathrm{d}t$) dostaneme

$$\int \frac{2xy}{(x^2+y^2)^2}\,\mathrm{d}x = y\int \frac{\mathrm{d}t}{t^2} = y\frac{-1}{t} = \frac{-y}{x^2+y^2},$$

a proto

$$v(x,y) = \frac{-y}{x^2+y^2} + \varphi(y).$$

Dosazením do druhé z Cauchyho-Riemannových podmínek dostaneme

$$\frac{\partial u}{\partial x}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} + \varphi'(y) \;\Rightarrow\; \varphi(y) = c,\; c \in \mathbb{R},$$

a proto

$$\underline{f(z) = f(x+iy) =} \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2} + c\right) = \underline{\frac{1}{z} + ci,\; c \in \mathbb{R}.}$$

c)

$$\frac{\partial u}{\partial x}(x,y) = 6x + 3, \qquad \frac{\partial u}{\partial y}(x,y) = -2y + 1,$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) = 6, \qquad \frac{\partial^2 u}{\partial y^2}(x,y) = -2,$$

a tudíž

$$\Delta u(x,y) \neq 0 \text{ pro každé } (x,y) \text{ v } \mathbb{R}^2.$$

Funkce u není v Ω harmonická, a proto <u>hledaná funkce f neexistuje</u>.

d)

$$\frac{\partial u}{\partial x}(x,y) = 2x + 5 + \frac{2xy}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}(x,y)$$

$$\Downarrow$$

$$v(x,y) = 2xy + 5y - \frac{x}{x^2+y^2} + \varphi(x).$$

Navíc

$$\frac{\partial u}{\partial y}(x,y) = -2y + 1 + \frac{y^2-x^2}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}(x,y) = -2y + \frac{y^2-x^2}{(x^2+y^2)^2} - \varphi'(x)$$

$$\Downarrow$$

$$\varphi'(x) = -1.$$

Odtud již snadno dostaneme, že $v(x,y) = 2xy + 5y - \frac{x}{x^2+y^2} - x + c,\; c \in \mathbb{R}$.
Hledanou funkcí je

$$\underline{f(x+iy) = \left(x^2 - y^2 + 5x + y - \frac{y}{x^2+y^2}\right) + i\left(2xy + 5y - \frac{x}{x^2+y^2} - x + c\right),\; c \in \mathbb{R},}$$

neboli

$$\underline{f(z) = z^2 + (5-i)z - \frac{i}{z} + ci,\; c \in \mathbb{R}.}$$

Příklad 26.
Buď $u(x,y) := x^3 - 3xy^2 - 2y + 2$. Najděte (existuje-li) na $\mathbb{C}$ holomorfní funkci $f = u + iv$, pro niž platí

a) $f(0) = i$;

b) $f(1) = 3 - i$.

Řešení:
Podobně jako u Příkladu 25 a) zjistíme, že

$$f(x+iy) = (x^3 - 3xy^2 - 2y + 2) + i(3x^2y - y^3 + 2x + c), \text{ kde } c \in \mathbb{R}.$$

a) Požadavek

$$f(0) = f(0 + 0i) = 2 + ic = i$$

zřejmě nelze žádnou volbou $c \in \mathbb{R}$ splnit. Funkce f požadovaných vlastností neexistuje.

b) Chceme splnit podmínku

$$f(1) = f(1 + 0i) = 3 + i(2 + c) = 3 - i,$$

a proto $2 + c = -1$, neboli $c = -3$. Hledaná funkce existuje, je to funkce

$$\underline{f(x+iy) = (x^3 - 3xy^2 - 2y + 2) + i(3x^2y - y^3 + 2x - 3)}.$$

Příklad 27.
Najděte (existuje-li) na oblasti Ω holomorfní funkci $f = u + iv$, je-li

a) $v(x,y) := -3xy^2 + x^3 + 5$, $\Omega = \mathbb{C}$;

b) $v(x,y) := \operatorname{arctg} \frac{y}{x}$, $\Omega = \{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$.

Řešení:

a)

$$\frac{\partial v}{\partial y}(x,y) = -6xy = \frac{\partial u}{\partial x}(x,y) \;\Rightarrow\; u(x,y) = -3x^2y + \varphi(y),$$

$$-\frac{\partial v}{\partial x}(x,y) = 3y^2 - 3x^2 = \frac{\partial u}{\partial y}(x,y) = -3x^2 + \varphi'(y) \;\Rightarrow\; \varphi(y) = y^3 + c,\; c \in \mathbb{R},$$

a proto

$$\underline{f(x+iy) = (-3x^2y + y^3 + c) + i(-3xy^2 + x^3 + 5),\; c \in \mathbb{R}},$$

neboli

$$\underline{f(z) = c + iz^3 + 5i,\; c \in \mathbb{R}}.$$

b)

$$\frac{\partial v}{\partial y}(x,y) = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{\partial u}{\partial x}(x,y),$$

a proto

$$u(x,y) = \frac{1}{2}\ln(x^2 + y^2) + \varphi(y).$$

Dosazením do druhé C-R podmínky

$$-\frac{\partial v}{\partial x}(x,y) = \frac{-1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{y}{x^2+y^2} = \frac{\partial u}{\partial y}(x,y) = \frac{y}{x^2+y^2} + \varphi'(y)$$

zjistíme, že

$$\varphi(y) = c,\ c \in \mathbb{R}.$$

Hledanou funkcí na oblasti Ω je funkce

$$\underline{f(x+iy) = \ln\sqrt{x^2+y^2} + c + i\arg(x+iy),\ c \in \mathbb{R}},$$

neboli

$$\underline{f(z) = c + \ln z,\ c \in \mathbb{R}}.$$

Příklad 28.
Buď $v(x,y) := 1 + \operatorname{arctg}\frac{y}{x}$. Najděte (existuje-li) na oblasti $\Omega := \{z \in \mathbb{C} :\ \operatorname{Re} z > 0\}$ holomorfní funkci $f = u + iv$, pro niž platí

a) $f(3) = \ln 3 + 6 + i$;

b) $f(e) = 1 - i$.

Řešení:
Podobně jako v Příkladu 27 b) je

$$f(x+iy) = \ln\sqrt{x^2+y^2} + c + i\left(\operatorname{arctg}\frac{y}{x} + 1\right),\ c \in \mathbb{R}.$$

a)

$$f(3+0i) = \ln 3 + c + i = \ln 3 + 6 + i \ \Rightarrow\ c = 6,$$

a proto

$$\underline{f(x+iy) = \ln\sqrt{x^2+y^2} + 6 + i\left(\operatorname{arctg}\frac{y}{x} + 1\right) \text{ v } \Omega}.$$

b)

$$\forall c \in \mathbb{R}\colon\ f(e+0i) = 1 + c + i \neq 1 - i.$$

<u>Hledaná funkce f neexistuje</u>.

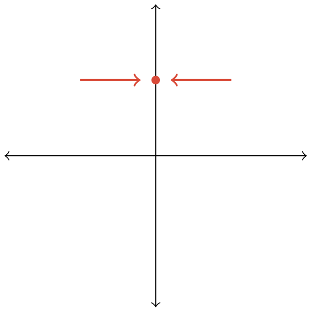
<u>PŘÍKLAD 29.</u>
Dokažte, že je funkce
$$v(x,y) := \ln(x^2+y^2)$$
harmonická na (*dvojnásobně souvislé*) oblasti $\mathbb{C}\setminus\{0\}$ a že přesto neexistuje taková funkce $u:\ \mathbb{R}^2\to\mathbb{R}$, aby funkce $f := u+iv$ byla holomorfní na $\mathbb{C}\setminus\{0\}$.

<u>Řešení:</u>
Zřejmě $v\in\mathcal{C}^\infty(\mathbb{R}^2\setminus\{(0,0)\})$, navíc pro každé $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$:
$$\frac{\partial v}{\partial x}(x,y) = \frac{2x}{x^2+y^2},$$
$$\frac{\partial^2 v}{\partial x^2}(x,y) = \frac{2(x^2+y^2)-2x2x}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2},$$
a proto (derivace podle y jsou analogické)
$$\Delta v(x,y) = \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2} = 0.$$
Dokázali jsme, že funkce <u>v je na $\mathbb{C}\setminus\{0\}$ harmonická</u>.

Předpokládejme nyní pro spor, že výše charakterizovaná funkce u existuje. Pak pro $(x,y)\in\mathbb{R}^2$ takové, že $x+iy\in\Omega_1 := \{z\in\mathbb{C}:\ \operatorname{Re} z>0\}$ platí, že
$$-\frac{\partial v}{\partial x}(x,y) = -\frac{2x}{x^2+y^2} = \frac{\partial u}{\partial y}(x,y),$$
a proto taky (volíme substituci $\frac{y}{x}=t,\ \frac{1}{x}\,\mathrm{d}y=\,\mathrm{d}t$)
$$u(x,y) = \int -\frac{2x}{x^2+y^2}\,\mathrm{d}y =$$
$$= \int -\frac{2}{x}\frac{\mathrm{d}y}{1+\left(\frac{y}{x}\right)^2} = -2\operatorname{arctg}\frac{y}{x} + \varphi(x).$$
Dosaďme do druhé C-R podmínky
$$\frac{\partial v}{\partial y}(x,y) = \frac{2y}{x^2+y^2} = \frac{\partial u}{\partial x}(x,y) = \frac{2y}{x^2+y^2} + \varphi'(x)$$
a zjistíme, že
$$u(x,y) = -2\operatorname{arctg}\frac{y}{x} + c_1 \text{ pro nějaké } c_1\in\mathbb{R}.$$
Podobně musí existovat $c_2\in\mathbb{R}$ takové, že pro každé $x+iy\in\Omega_2 := \{z\in\mathbb{C}:\ \operatorname{Re} z<0\}$ je
$$u(x,y) = -2\operatorname{arctg}\frac{y}{x} + c_2.$$

Současně ale musí být funkce u spojitá na $\mathbb{R}^2 \setminus \{(0,0)\}$ (v každém bodě $\mathbb{R}^2 \setminus \{(0,0)\}$ musí být diferencovatelná), a proto

$$\begin{array}{ccccc} \lim\limits_{x\to 0-} u(x,1) & = & u(0,1) & = & \lim\limits_{x\to 0+} u(x,1). \\ \| & & & & \| \\ \pi + c_2 & & & & -\pi + c_1 \end{array}$$

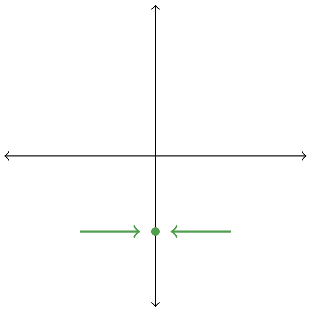


Odtud plyne, že

$$2\pi = c_1 - c_2.$$

Analogicky

$$\begin{array}{ccccc} \lim\limits_{x\to 0-} u(x,-1) & = & u(0,-1) & = & \lim\limits_{x\to 0+} u(x,-1), \\ \| & & & & \| \\ -\pi + c_2 & & & & \pi + c_1 \end{array}$$



a proto

$$2\pi = c_2 - c_1.$$

To nás ale vede ke vztahům

$$2\pi = c_1 - c_2 = -(c_2 - c_1) = -2\pi,$$

což je spor. Funkce u daných vlastností neexistuje.

Příklad 30.
Určete úhel otočení a koeficient roztažnosti funkce f v bodě z_0, je-li

a) $f(z) := e^z$, $z_0 = -1 - \frac{\pi}{2}i$;

b) $f(z) := z^3$, $z_0 = -3 + 4i$;

c) $f(z) := \frac{z+i}{z-i}$, $z_0 = 2i$.

Řešení:

a)

$$|f'(z_0)| = |e^{z_0}| = |e^{-1-i\frac{\pi}{2}}| = \underline{\frac{1}{e}},$$

což je koeficient roztažnosti funkce f v bodě z_0 (a protože $\frac{1}{e} < 1$, jedná se o *kontrakci*).

$$\arg f'(z_0) = \arg\left(\frac{1}{e}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)\right) = \arg\left(-\frac{i}{e}\right) = \underline{-\frac{\pi}{2}},$$

což je úhel otočení funkce f v bodě z_0.

b) $z_0 = 5\left(\cos\left(\frac{\pi}{2} + \operatorname{arctg}\frac{3}{4}\right) + i\sin\left(\frac{\pi}{2} + \operatorname{arctg}\frac{3}{4}\right)\right)$, a proto

$$f'(z_0) = 3z_0^2 = 3 \cdot 25\left(\cos\left(\pi + 2\operatorname{arctg}\frac{3}{4}\right) + i\sin\left(\pi + 2\operatorname{arctg}\frac{3}{4}\right)\right).$$

Odtud

$|f'(z_0)| = \underline{75}$ ... koeficient roztažnosti funkce f v bodě z_0 ($75 > 1$, proto jde o *dilataci*),

$\arg f'(z_0) = \underline{-\pi + 2\operatorname{arctg}\frac{3}{4}}$ ... úhel otočení funkce f v bodě z_0.

c)

$$f'(z) = \frac{z - i - (z+i)}{(z-i)^2} = \frac{-2i}{(z-i)^2},$$

$$f'(z_0) = \frac{-2i}{i^2} = 2i,$$

a proto

$1 < |f'(z_0)| = \underline{2}$ ... koeficient roztažnosti f v z_0 (*dilatace*),

$\arg f'(z_0) = \underline{\frac{\pi}{2}}$ ... úhel otočení f v z_0.

PŘÍKLAD 31.
Určete, ve kterých bodech Gaussovy roviny dochází při daném zobrazení ke kontrakci:

a) $f(z) := \frac{2}{z}$;

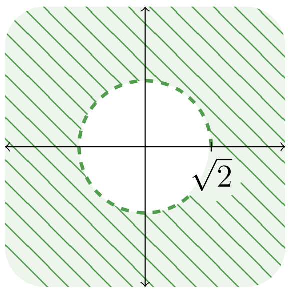
b) $f(z) := \ln(z+4)$.

Řešení:

a) $f'(z) = \frac{-2}{z^2}$. Tedy pro $z \in \mathbb{C}$:

$$0 < |f'(z)| < 1 \Leftrightarrow \left|\frac{-2}{z^2}\right| < 1 \Leftrightarrow 2 < |z|^2 \Leftrightarrow \sqrt{2} < |z|.$$

Ke kontrakci dochází v každém bodě množiny $\left\{z \in \mathbb{C}\colon\ |z| > \sqrt{2}\right\}$.

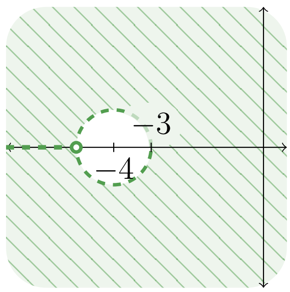


b) $f'(z)$ existuje v $\mathbb{C} \setminus \{x+iy\colon\ y = 0 \ \wedge\ x \leq -4\} =: \Omega$. Pro každé $z \in \Omega$ platí, že

$$\begin{aligned} |f'(z)| &= \left|\frac{1}{z+4}\right|, \\ 0 < \frac{1}{|z+4|} < 1 &\Leftrightarrow 1 < |z+4|. \end{aligned}$$

Ke kontrakci dochází v každém bodě množiny

$$\{z \in \mathbb{C}\colon\ |z+4| > 1\} \setminus \{x+iy\colon\ y = 0 \ \wedge\ x \leq -4\}.$$



PŘÍKLAD 32.
Znázorněte množiny Ω a $f(\Omega) = \{f(z)\colon\ z \in \Omega\}$, je-li[2]

a) $\Omega = U(1,2),\ f(z) := 1 - 2iz;$

b) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\},\ f(z) := (1+i)z + 1;$

c) $\Omega = U(1,2),\ f(z) := \frac{1}{z};$

d) $\Omega = U(1,2),\ f(z) := \frac{2iz}{z+3};$

e) $\Omega = U(1,2),\ f(z) := \frac{z-1}{2z-6};$

f) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\},\ f(z) := \frac{1}{z};$

g) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\},\ f(z) := \frac{z}{z-1+i};$

h) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\},\ f(z) := \frac{z}{z-2};$

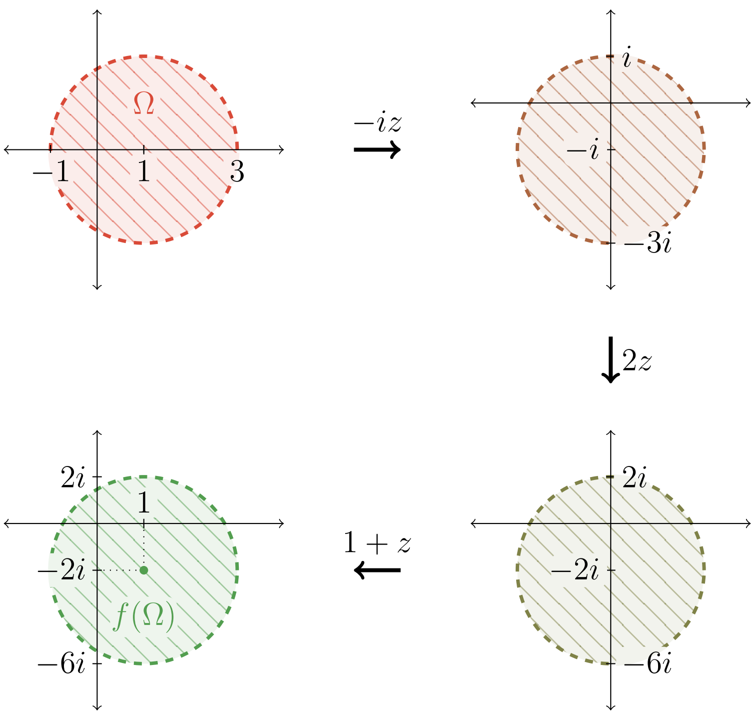
i) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z < 0\},\ f(z) := \frac{1}{z};$

j) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\ \wedge\ \operatorname{Im} z > 0\},\ f(z) := \frac{z-1}{z+1};$

k) $\Omega = \{z \in \mathbb{C}\colon\ -1 < \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z < 0\},\ f(z) := \frac{z-i}{z+i};$

l) $\Omega = \{z \in \mathbb{C}\colon\ |z| < 1\ \wedge\ \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z > 0\},\ f(z) := \frac{z}{z-i}.$

Řešení:
a) $\Omega = U(1,2),\ f(z) := 1 - 2iz,\ \underline{f(\Omega) = U(1-2i, 4)}.$



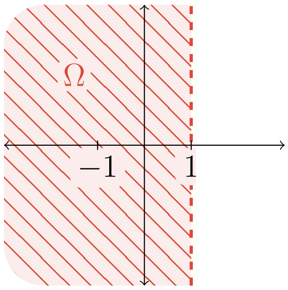
[2] *Nápověda k některým z níže uvedených příkladů.* Uvědomte si (a dokažte), že platí tvrzení:

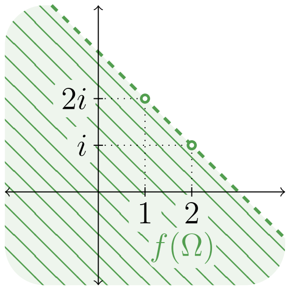
$$\left.\begin{array}{c} f \text{ je konformní na oblasti } \Omega \subset \mathbb{C}_\infty, \\ A,\, B \subset \Omega \end{array}\right\} \Rightarrow\ f(A \cap B) = f(A) \cap f(B).$$

b) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\},\ f(z) := (1+i)z + 1,$

$$\begin{aligned} f(1) &= 1 + i + 1 = 2 + i, \\ f(1+i) &= 2i + 1, \\ f(0) &= 1, \end{aligned}$$

a proto (rozmyslete si to!) $\underline{f(\Omega) = \{z \in \mathbb{C}\colon\ \operatorname{Re} z + \operatorname{Im} z < 3\}}$.

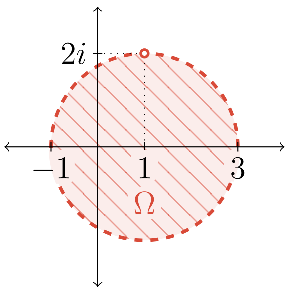


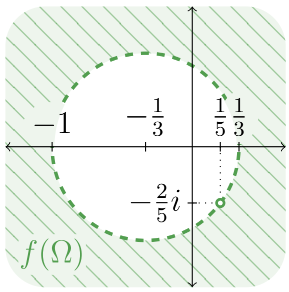


c) $\Omega = U(1,2),\ f(z) := \frac{1}{z},$

$$\begin{aligned} f(0) &= \infty, \\ f(-1) &= -1, \\ f(3) &= \frac{1}{3}, \\ f(1+2i) &= \frac{1}{1+2i} = \frac{1-2i}{5}, \end{aligned}$$

a proto $\underline{f(\Omega) = \mathbb{C}_\infty \setminus \overline{U\left(-\frac{1}{3}, \frac{2}{3}\right)}}$.

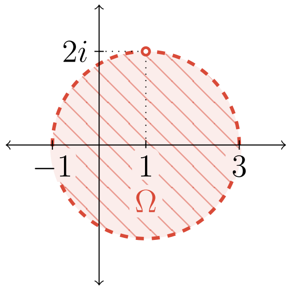


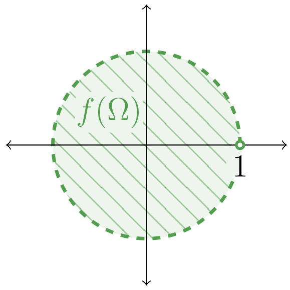


d) $\Omega = U(1,2),\ f(z) := \frac{2iz}{z+3},$

$$\begin{aligned} f(-3) &= \infty, \\ f(-1) &= \frac{-2i}{2} = -i, \\ f(3) &= i, \\ f(1+2i) &= \frac{2i(1+2i)}{4+2i} = -\frac{4-2i}{4+2i} = -\frac{3}{5} + \frac{4}{5}i, \end{aligned}$$

a proto $\underline{f(\Omega) = U(0,1)}$.

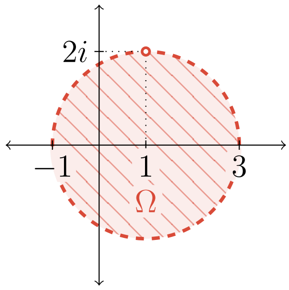


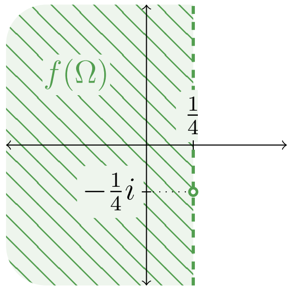


e) $\Omega = U(1,2),\ f(z) := \frac{z-1}{2z-6}$,

$$f(0) = \frac{1}{6},$$
$$f(3) = \infty,$$
$$f(-1) = \frac{-2}{-8} = \frac{1}{4},$$
$$f(1+2i) = \frac{2i}{2+4i-6} = \frac{1}{4} - \frac{1}{4}i,$$

$$\text{a proto } \underline{f(\Omega) = \left\{z \in \mathbb{C}\colon\ \operatorname{Re} z < \frac{1}{4}\right\}}.$$

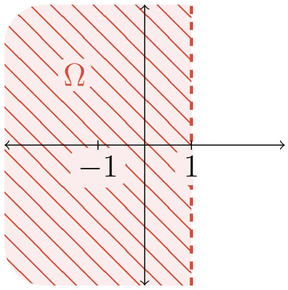


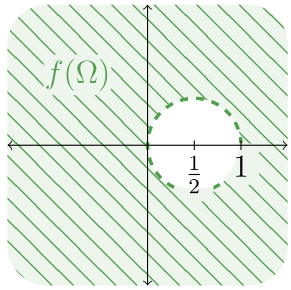


f) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\},\ f(z) := \frac{1}{z}$,

$$f(1) = 1,$$
$$f(0) = \infty,$$
$$f(1+i) = \frac{1}{1+i} = \frac{1-i}{2},$$
$$f(1-i) = \frac{1}{1-i} = \frac{1+i}{2},$$
$$f(\infty) = 0,$$

$$\text{a proto } \underline{f(\Omega) = \mathbb{C}_\infty \setminus \overline{U\left(\frac{1}{2}, \frac{1}{2}\right)}}.$$

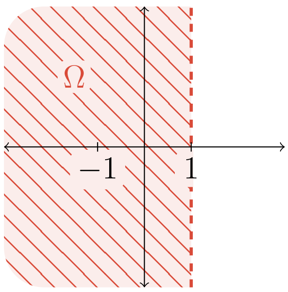


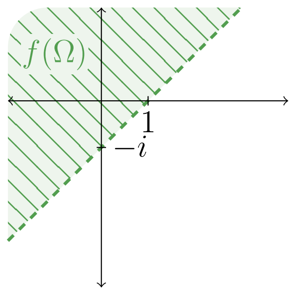


g) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\}$, $f(z) := \frac{z}{z-1+i}$,

$$\begin{aligned} f(0) &= 0, \\ f(1) &= \frac{1}{i} = -i, \\ f(1-i) &= \infty, \\ f(1+i) &= \frac{1+i}{2i} = \frac{1}{2} - \frac{1}{2}i, \end{aligned}$$

$$\text{a proto } \underline{f(\Omega) = \{z \in \mathbb{C}\colon\ \operatorname{Im} z > \operatorname{Re} z - 1\}}\,.$$

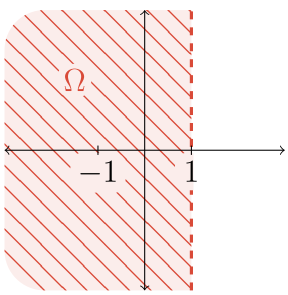


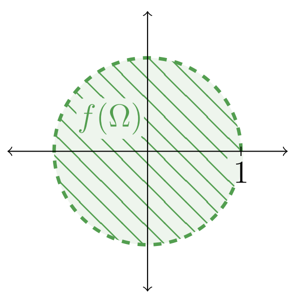


h) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\}$, $f(z) := \frac{z}{z-2}$,

$$\begin{aligned} f(2) &= \infty, \\ f(1) &= -1, \\ f(1+i) &= \frac{-2i}{2} = -i, \\ f(\infty) &= 1, \end{aligned}$$

$$\text{a proto } \underline{f(\Omega) = U(0,1)}.$$

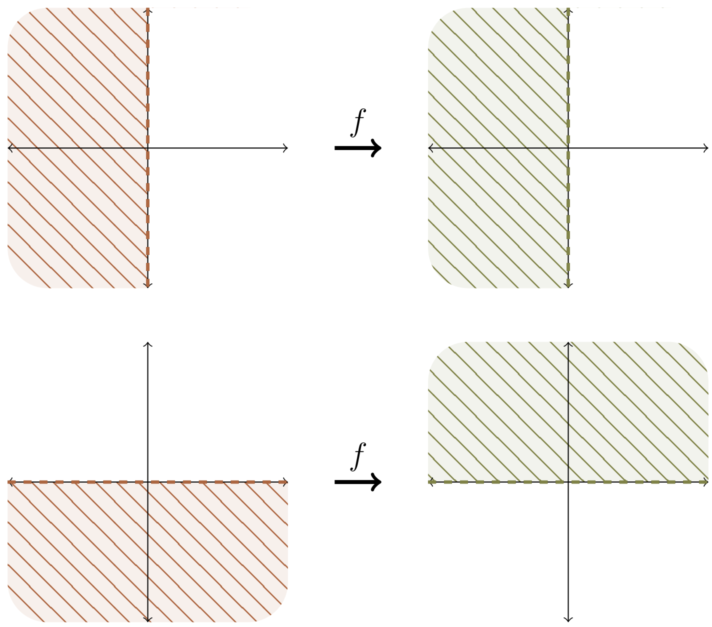




i) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z < 0\}$, $f(z) := \frac{1}{z}$,

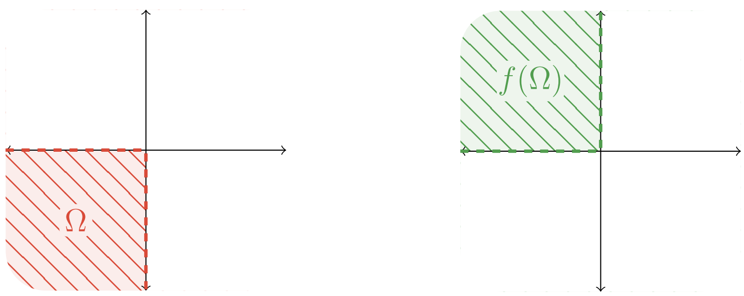
$$\begin{aligned} f(0) &= \infty, \\ f(-1) &= -1, \\ f(1) &= 1, \\ f(i) &= -i, \\ f(-i) &= i, \end{aligned}$$

a protože $\Omega = \Omega_1 \cap \Omega_2$, kde $\Omega_1 := \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 0\}$, $\Omega_2 := \{z \in \mathbb{C}\colon\ \operatorname{Im} z < 0\}$,



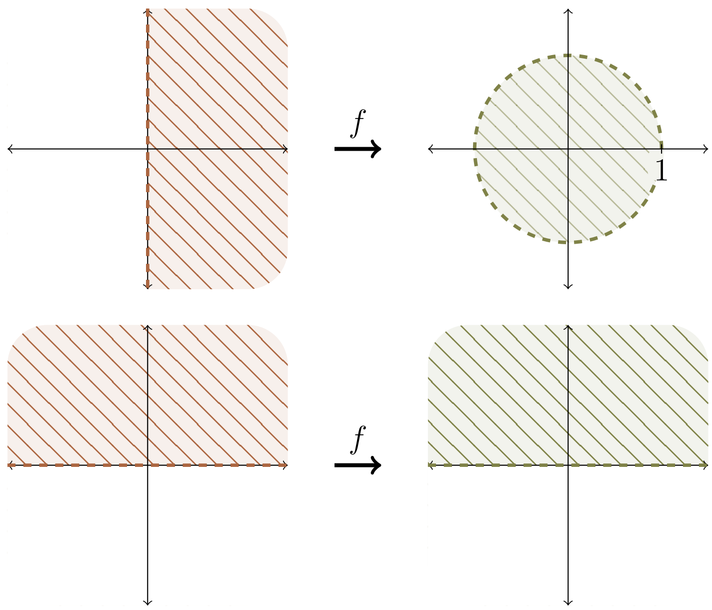
je

$$\underline{f(\Omega) =} f(\Omega_1) \cap f(\Omega_2) = \underline{\{z \in \mathbb{C}\colon\ \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z > 0\}}.$$



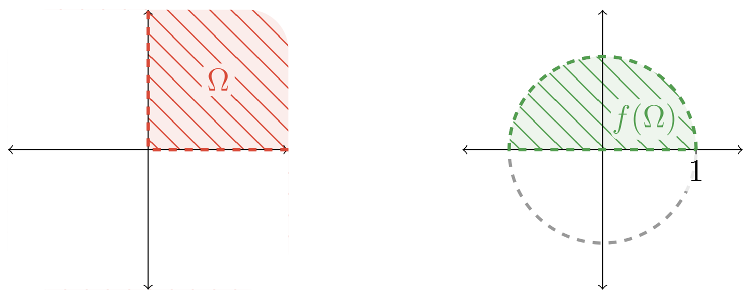
j) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\ \wedge\ \operatorname{Im} z > 0\}$, $f(z) := \frac{z-1}{z+1}$,

$$\begin{aligned} f(0) &= -1, \\ f(1) &= 0, \\ f(i) &= \frac{i-1}{i+1} = i, \\ f(-i) &= -i, \\ f(-1) &= \infty, \end{aligned}$$



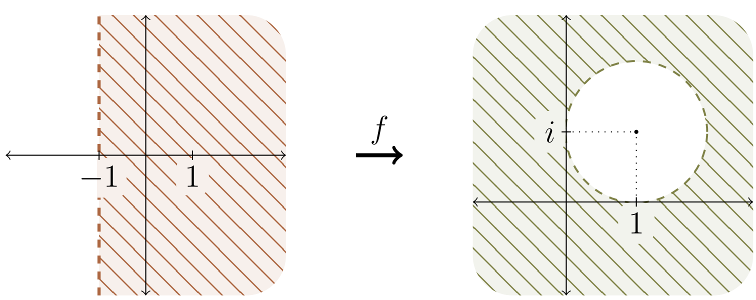
a proto

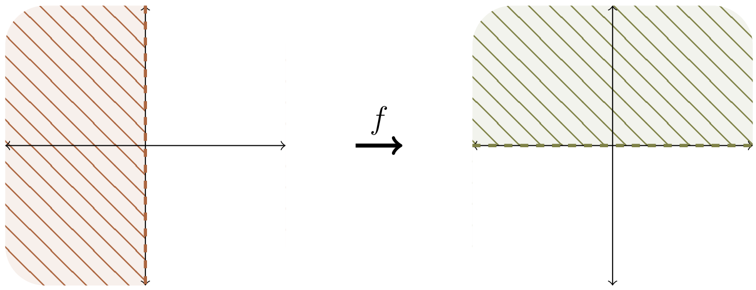
$$\underline{f(\Omega) = \{z \in \mathbb{C}\colon\ |z| < 1 \ \wedge\ \operatorname{Im} z > 0\}}.$$

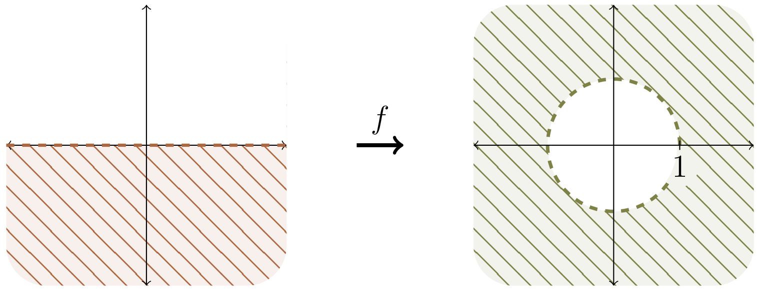


k) $\Omega = \{z \in \mathbb{C}\colon\ -1 < \operatorname{Re} z < 0 \ \wedge\ \operatorname{Im} z < 0\},\ f(z) := \frac{z-i}{z+i}$,

$$\begin{aligned} f(0) &= -1,\\ f(i) &= 0,\\ f(-i) &= \infty,\\ f(1) &= \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = -i,\\ f(-1) &= \frac{-1-i}{-1+i} = \frac{(-1-i)^2}{2} = i,\\ f(-1+i) &= \frac{-1}{-1+2i} = \frac{1+2i}{5},\\ f(-1-i) &= 1+2i, \end{aligned}$$

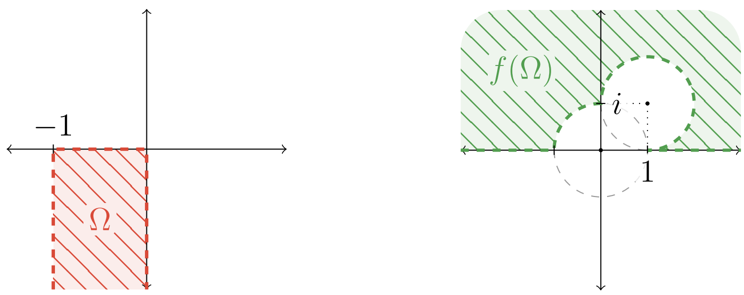






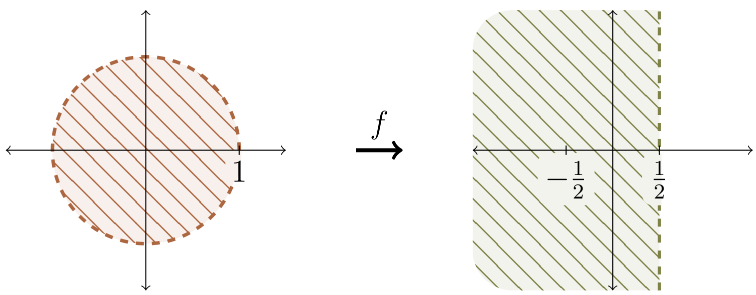
a proto

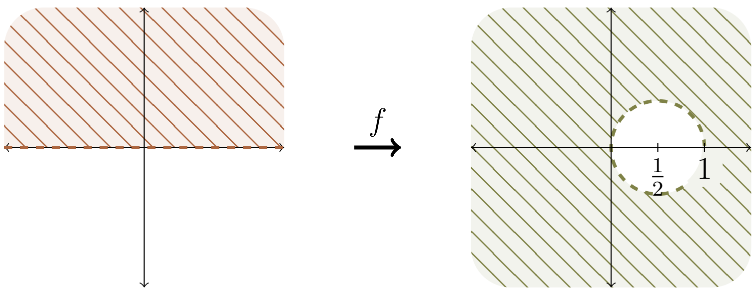
$$\underline{f(\Omega) = \{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\ \wedge\ |z-(1+i)| > 1\ \wedge\ |z| > 1\}.}$$

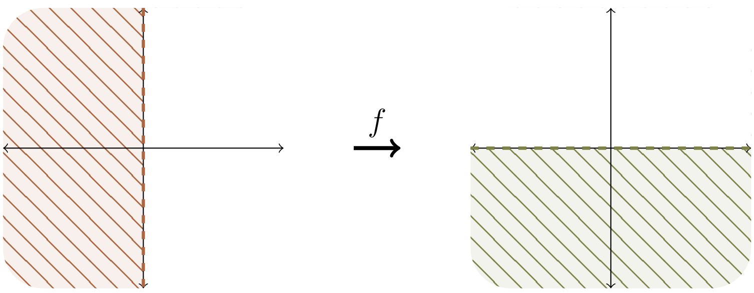


l) $\Omega = \{z \in \mathbb{C}\colon\ |z| < 1\ \wedge\ \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z > 0\}$, $f(z) := \frac{z}{z-i}$,

$$\begin{aligned} f(0) &= 0, \\ f(i) &= \infty, \\ f(-1) &= \frac{1-i}{2}, \\ f(1) &= \frac{1+i}{2}, \\ f(-i) &= \frac{1}{2}, \end{aligned}$$

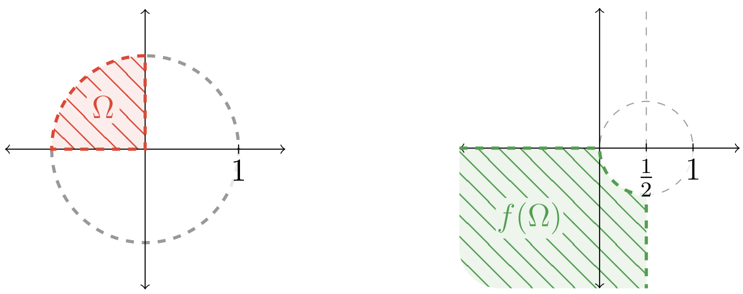






a proto

$$f(\Omega) = \left\{ z \in \mathbb{C}\colon\ \operatorname{Re} z < \frac{1}{2}\ \wedge\ \operatorname{Im} z < 0\ \wedge\ \left| z - \frac{1}{2} \right| > \frac{1}{2} \right\}.$$



PŘÍKLAD 33.
Najděte lineární lomenou funkci f takovou, aby

a) $f(-1) = 0,\ f(i) = 2i,\ f(1+i) = 1-i;$

b) $f(i) = \infty,\ f(6) = 0,\ f(\infty) = 3;$

c) $f(0) = i,\ f(i) = 0,\ f(-1) = -i.$

Řešení:

a) Hledejme funkci f ve tvaru

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{pro } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{pro } z = \infty, \end{cases}$$

kde $a, b, c, d \in \mathbb{C}$, $ad \neq bc$.

Ze zadaných podmínek dostaneme soustavu rovnic

$$\frac{-a+b}{-c+d} = 0,$$
$$\frac{ai+b}{ci+d} = 2i,$$
$$\frac{a(1+i)+b}{c(1+i)+d} = 1-i,$$

přičemž z první z nich plyne, že $a = b$. Můžeme zvolit (rozmyslete si proč!) $a = b = 1$. Další dvě rovnice pak získají tvar

$$\frac{i+1}{ci+d} = 2i,$$
$$\frac{2+i}{c(1+i)+d} = 1-i,$$

a odtud

$$\begin{aligned} i+1 &= -2c+2id, \\ 2+i &= 2c+d-di. \end{aligned}$$

Sečtením těchto rovnic dostaneme $3+2i = d+id$, a proto

$$d = \frac{3+2i}{1+i} = \frac{5-i}{2}.$$

Zbývá dopočítat c:

$$2c = 2+i-d(1-i) = 2+i-\frac{1}{2}(4-6i) = 4i,$$

tudíž $c = 2i$.

Shrnutí:

$$f(z) = \begin{cases} \dfrac{z+1}{2iz+\frac{5-i}{2}} = \dfrac{2z+2}{4iz+5-i}, & z \in \mathbb{C}, \\ \dfrac{1}{2i} = -\dfrac{i}{2}, & z = \infty. \end{cases}$$

b) Buď

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{pro } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{pro } z = \infty. \end{cases}.$$

Z podmínky

$$f(\infty) = \frac{a}{c} = 3$$

plyne, že lze volit $a = 3$ a $c = 1$. A pak je to už snadné:

$$f(6) = \frac{6a + b}{6z + d} = 0 \;\Rightarrow\; b = -6a = -18,$$

$$f(i) = \infty \;\Rightarrow\; ci + d = 0 \;\Rightarrow\; d = -ci = -i.$$

Shrnutí:

$$f(z) = \begin{cases} \dfrac{3z - 18}{z - i}, & z \in \mathbb{C}, \\ 3, & z = \infty. \end{cases}$$

c) Buď

$$f(z) = \begin{cases} \dfrac{az + b}{cz + d} & \text{pro } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{pro } z = \infty. \end{cases}.$$

Postupnou analýzou předepsaných podmínek zjistíme, že

$$f(0) = \frac{b}{d} = i \;\Rightarrow\; \text{lze volit } b = 1,\; d = -i,$$

$$f(i) = \frac{ai + b}{ci + d} = 0 \;\Rightarrow\; ai + b = 0 \;\Rightarrow\; ai + 1 = 0 \;\Rightarrow\; a = i,$$

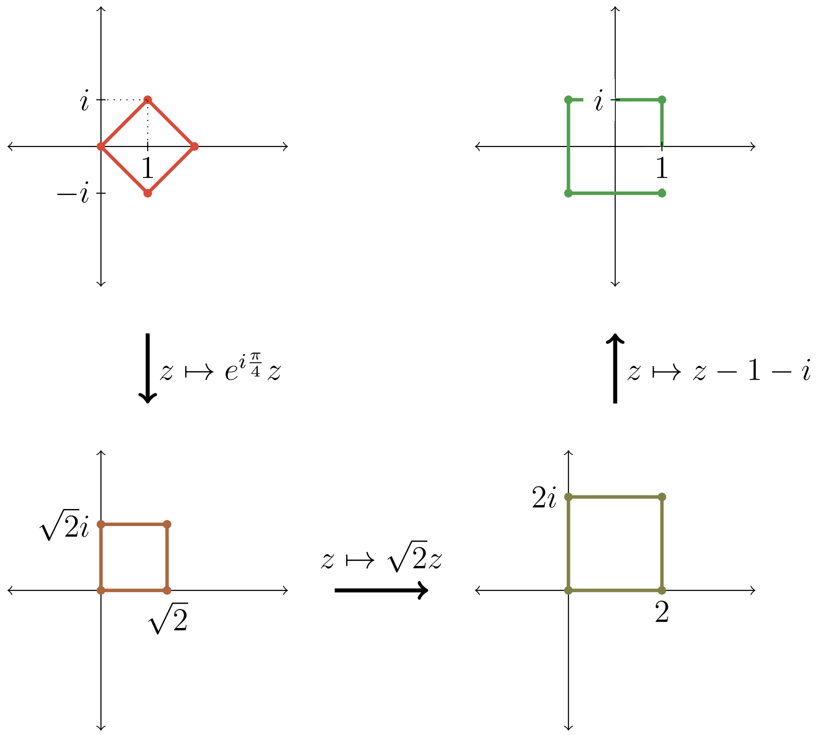
$$f(-1) = \frac{-a + b}{-c + d} = -i \;\Rightarrow\; -a + b = -i(-c + d) \;\Rightarrow\; c = -1 - 2i,$$

a proto

$$f(z) = \begin{cases} \dfrac{iz + 1}{(-1 - 2i)z - i}, & z \in \mathbb{C}, \\ \dfrac{i}{-1 - 2i} = -\dfrac{2}{5} - \dfrac{i}{5}, & z = \infty. \end{cases}$$

PŘÍKLAD 34.
Najděte lineární funkci, která zobrazí čtverec s vrcholy $0,\ 1-i,\ 2,\ 1+i$ na čtverec s vrcholy $1+i,\ -1+i,\ -1-i,\ 1-i$.

Řešení:



Postupným skládáním funkcí $z \mapsto e^{i\frac{\pi}{4}}$, $z \mapsto \sqrt{2}z$ a $z \mapsto z-1-i$ dostaneme

$$\begin{aligned} f(z) &= \left(\sqrt{2}\left(e^{i\frac{\pi}{4}}z\right)\right) - 1 - i = \\ &= (1+i)z - 1 - i, \end{aligned}$$

neboli

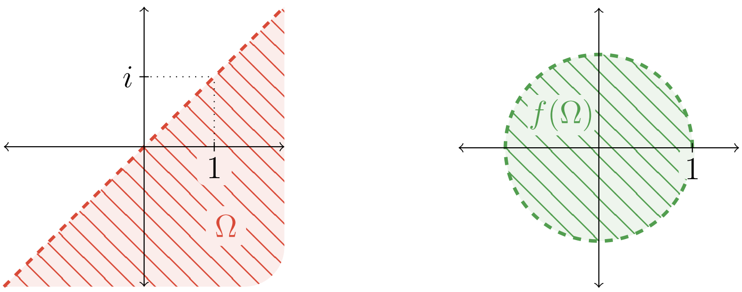
$$\underline{f(z) = (1+i)(z-1)}.$$

PŘÍKLAD 35.
Buď

$$\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > \operatorname{Im} z\}.$$

Najděte lineární lomenou funkci f takovou, aby $f(\Omega) = U(0,1)$.

Řešení:

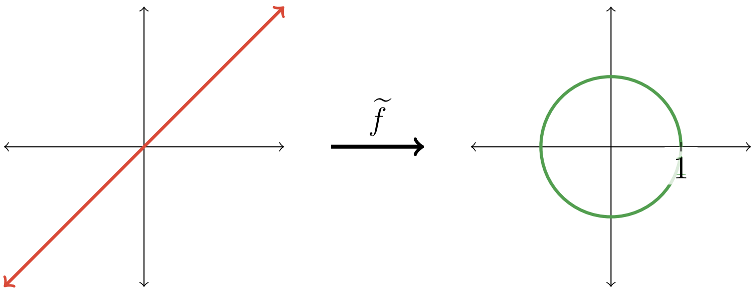


Nejdříve najdeme lineární lomenou funkci

$$\widetilde{f}(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{pro } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{pro } z = \infty, \end{cases}$$

takovou, aby $\widetilde{f}(0) = -1$, $\widetilde{f}(1+i) = i$ a $\widetilde{f}(\infty) = 1$.

Potom



a $\widetilde{f}(\Omega)$ je buď $U(0,1)$ (pak bychom volili $f := \widetilde{f}$), nebo $\widetilde{f}(\Omega) = \mathbb{C}_\infty \setminus \overline{U(0,1)}$ (pak bychom volili $f := \frac{1}{\widetilde{f}}$).

Řešením soustavy rovnic

$$\frac{b}{d} = -1,$$
$$\frac{a(1+i)+b}{c(1+i)+d} = i,$$
$$\frac{a}{c} = 1$$

zjistíme, že

$$\widetilde{f}(z) = \begin{cases} \dfrac{z-1+i}{z+1-i}, & z \in \mathbb{C}, \\ 1, & z = \infty, \end{cases}$$

a protože

$$\left|\widetilde{f}(1)\right| = \left|\frac{i}{2-i}\right| = \left|\frac{-1+2i}{5}\right| = \frac{1}{5}\sqrt{5} < 1,$$

volíme $f := \widetilde{f}$, tzn.

$$f(z) = \begin{cases} \dfrac{z-1+i}{z+1-i} & \text{pro } z \in \mathbb{C}, \\ 1 & \text{pro } z = \infty. \end{cases}$$

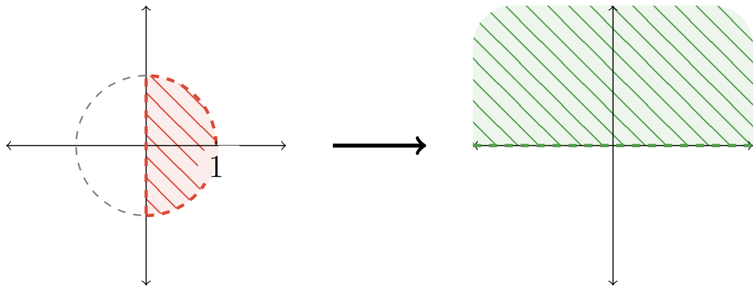
Příklad 36.
Najděte konformní zobrazení, které zobrazí oblast

$$\Omega = \{z \in \mathbb{C}\colon\ |z| < 1 \ \wedge\ \operatorname{Re} z > 0\}$$

na oblast

$$\{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\}.$$

Řešení:

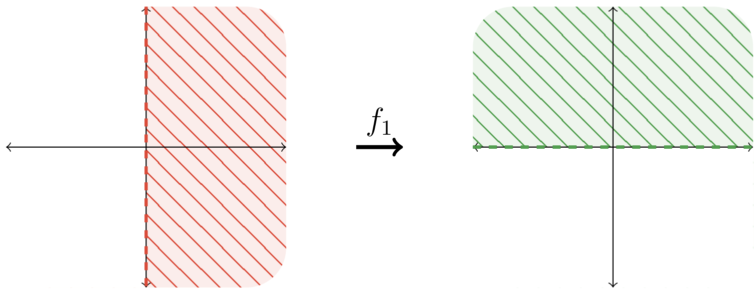


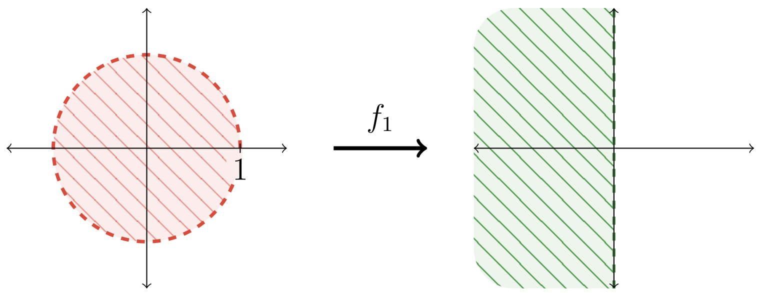
Uvažujme lineární lomenou funkci f_1 takovou, že $f_1(i) = \infty$ a $f_1(-i) = 0$. Pak zřejmě obrazem kružnice $\{z \in \mathbb{C}\colon\ |z| = 1\}$ i přímky $\{z \in \mathbb{C}\colon\ \operatorname{Re} z = 0\}$ (při f_1) budou přímky protínající se v bodě 0 „pod úhlem“ $\frac{\pi}{2}$. Volme například

$$f_1(z) := \begin{cases} \dfrac{z+i}{z-i}, & z \in \mathbb{C}, \\ 1, & z = \infty. \end{cases}$$

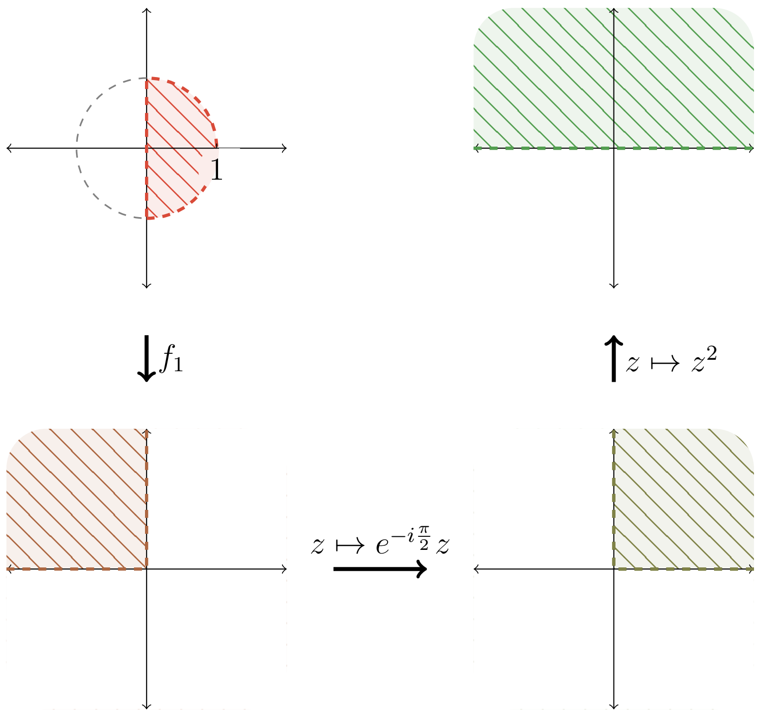
Pak

$$f_1(0) = -1,\ \ f_1(i) = \infty,\ \ f_1(-i) = 0 \text{ a } f_1(1) = i,$$





a proto



Shrnutí: jednou z funkcí majících požadované vlastnosti je funkce <u>definovaná na Ω</u> předpisem

$$\underline{f(z)} := \left(e^{-i\frac{\pi}{2}} \cdot \frac{z+i}{z-i}\right)^2 = \left(-i\frac{z+i}{z-i}\right)^2 = \underline{-\left(\frac{z+i}{z-i}\right)^2}.$$

<u>Příklad 37.</u>

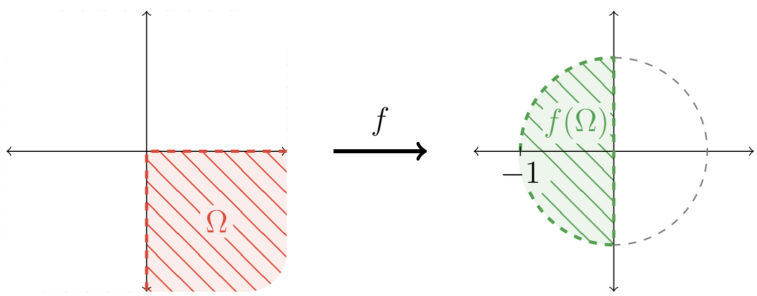
Buď

$$\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0 \ \wedge\ \operatorname{Im} z < 0\}.$$

Najděte lineární lomenou funkci f takovou, aby

$$f(\Omega) = \{z \in \mathbb{C}\colon\ |z| < 1 \ \wedge\ \operatorname{Re} z < 0\}.$$

<u>Řešení:</u>



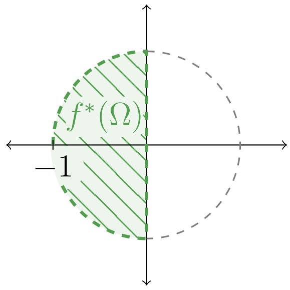
Nejdříve najdeme lineární lomenou funkci f^*, pro niž platí

$$f^*(0) = i,$$
$$f^*(\infty) = -i,$$
$$f^*(-1) = \infty,$$

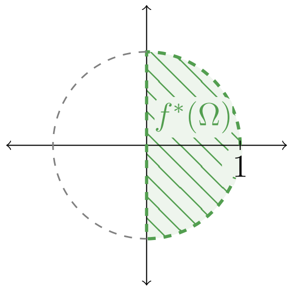
tj. funkci

$$f^*(z) = \begin{cases} \dfrac{-iz+i}{z+1}, & z \in \mathbb{C}, \\ -i, & z = \infty. \end{cases}$$

Pak zřejmě platí buď



(pak bychom definovali $f := f^*$), a nebo



(což by nás vedlo k definici $f := -f^*$).

Protože $f^*(i) = \frac{1+i}{i+1} = 1$ (nastala první z možností), volíme

$$f(z) := f^*(z) = \begin{cases} \dfrac{-iz+i}{z+1}, & z \in \mathbb{C}, \\ -i, & z = \infty. \end{cases}$$

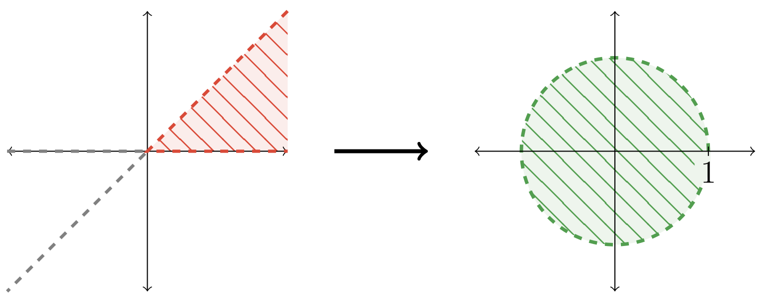
PŘÍKLAD 38.

Najděte konformní zobrazení, které zobrazí oblast

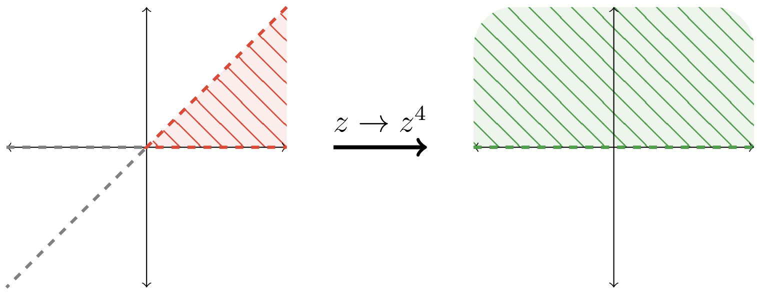
$$\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > \operatorname{Im} z > 0\}$$

na oblast $U(0,1)$.

Řešení:



Nejprve uvažme zobrazení $z \mapsto z^4$



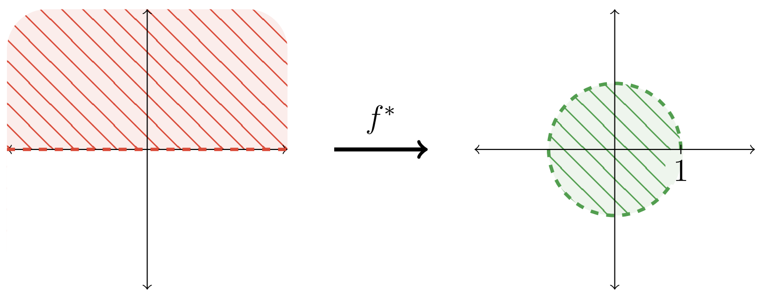
a potom lineární lomenou funkci f^* takovou, že

$$
\begin{aligned}
f^*(-1) &= -1,\\
f^*(0) &= i,\\
f^*(1) &= 1,
\end{aligned}
$$

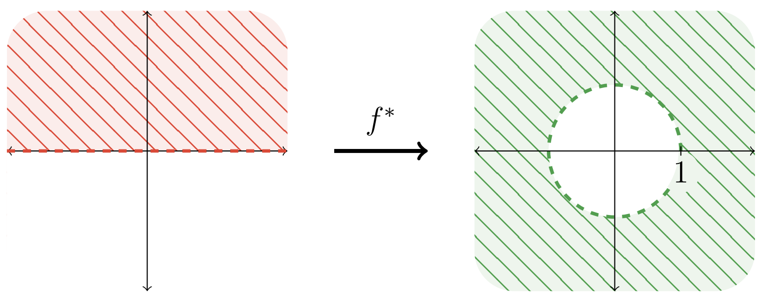
tzn.

$$
f^*(z) = \begin{cases} \dfrac{z+i}{iz+1}, & z \in \mathbb{C},\\[2mm] \dfrac{1}{i} = -i, & z = \infty. \end{cases}
$$

Zřejmě platí buď



(v takovém případě bychom pro $z \in \Omega$ definovali $f(z) := f^*(z^4)$), a nebo[3]



(to bychom definovali $f(z) := \frac{1}{f^*(z^4)}$ v Ω).

Protože $f^*(i) = \infty$, nastala druhá z možností. Volíme (pro $z \in \Omega$)

$$
\underline{f(z) := \frac{1}{f^*(z^4)} = \frac{iz^4+1}{z^4+i}.}
$$

[3]K obrázku vpravo si je třeba přimyslet $\infty = f^*(i)$.

Příklad 39.
Nalezněte obrazy přímek rovnoběžných s reálnou resp. imaginární osou při zobrazení $f(z) := \frac{1}{z}$ (přímky uvažujte včetně bodu ∞).

Řešení:
Pro $\underline{0 < c \in \mathbb{R}}$ platí, že

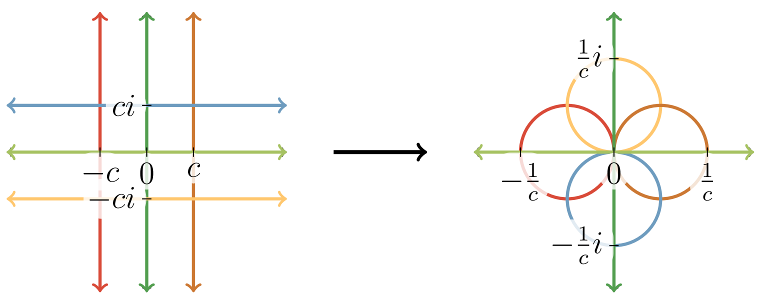
$$\begin{aligned} f(0) &= \infty, & f(\infty) &= 0, \\ f(i) &= -i, & f(c) &= \frac{1}{c}, \\ f(-i) &= i, & f(-c) &= -\frac{1}{c}, \\ f(1) &= 1, & f(ci) &= -\frac{1}{c}i, \\ f(-1) &= -1, & f(-ci) &= \frac{1}{c}i, \end{aligned}$$

a proto taky

$$\underline{\{z \in \mathbb{C}\colon \ \operatorname{Re} z = 0\} \cup \{\infty\} \to \{z \in \mathbb{C}\colon \ \operatorname{Re} z = 0\} \cup \{\infty\}},$$
$$\underline{\{z \in \mathbb{C}\colon \ \operatorname{Im} z = 0\} \cup \{\infty\} \to \{z \in \mathbb{C}\colon \ \operatorname{Im} z = 0\} \cup \{\infty\}},$$

$$\underline{\{z \in \mathbb{C}\colon \ \operatorname{Re} z = c\} \cup \{\infty\} \to \left\{z \in \mathbb{C}\colon \ \left|z - \frac{1}{2c}\right| = \frac{1}{2c}\right\}},$$
$$\underline{\{z \in \mathbb{C}\colon \ \operatorname{Im} z = c\} \cup \{\infty\} \to \left\{z \in \mathbb{C}\colon \ \left|z + \frac{1}{2c}i\right| = \frac{1}{2c}\right\}},$$

$$\underline{\{z \in \mathbb{C}\colon \ \operatorname{Re} z = -c\} \cup \{\infty\} \to \left\{z \in \mathbb{C}\colon \ \left|z + \frac{1}{2c}\right| = \frac{1}{2c}\right\}},$$
$$\underline{\{z \in \mathbb{C}\colon \ \operatorname{Im} z = -c\} \cup \{\infty\} \to \left\{z \in \mathbb{C}\colon \ \left|z - \frac{1}{2c}i\right| = \frac{1}{2c}\right\}}.$$



Příklad 40.
Nalezněte obrazy množin

$$M_\alpha = \{z \in \mathbb{C}\colon\ \arg z = \alpha\} \text{ a } N_r = \{z \in \mathbb{C}\colon\ |z| = r\},$$

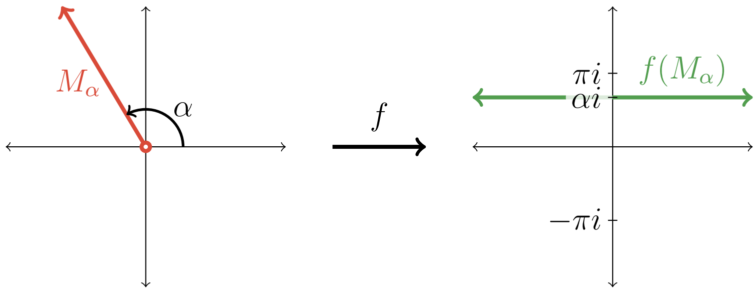
kde $\alpha \in (-\pi, \pi\rangle$ a $r \in \mathbb{R}^+$, při zobrazení $f(z) := \ln z$.

Řešení:

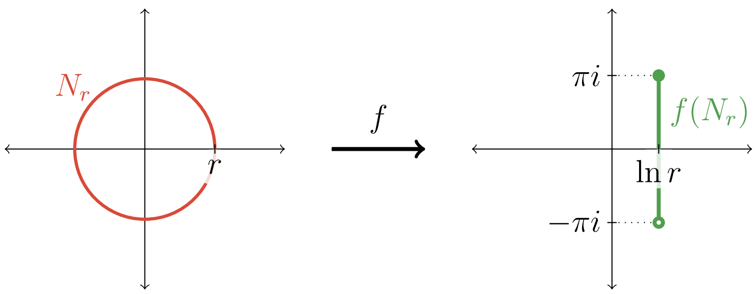
$$\ln z = \ln |z| + i \arg z,$$

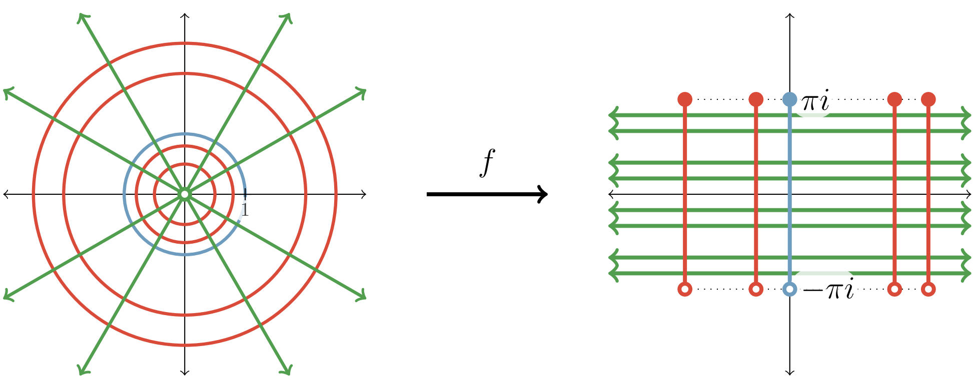
a proto

$$f(M_\alpha) = \{z \in \mathbb{C}\colon\ \operatorname{Im} z = \alpha\}\,,$$



$$f(N_r) = \{\ln r + ik\colon\ k \in (-\pi, \pi\rangle\}\,.$$





Příklad 41.
Vypočtěte

$$\int_\gamma |z|\,\mathrm{d}z,$$

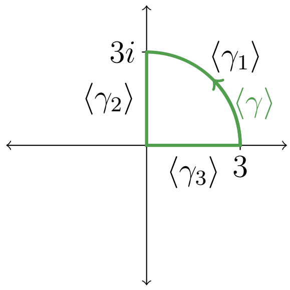
je-li

$$\gamma(t) := \begin{cases} 3e^{it}, & t \in \langle 0, \frac{\pi}{2}\rangle, \\ i\big(3 + \frac{\pi}{2} - t\big), & t \in \langle \frac{\pi}{2}, \frac{\pi}{2} + 3\rangle, \\ t - \frac{\pi}{2} - 3, & t \in \langle \frac{\pi}{2} + 3, \frac{\pi}{2} + 6\rangle. \end{cases}$$

Řešení:
Volme

$$\gamma_1(t) := 3e^{it},\ t \in \langle 0, \frac{\pi}{2}\rangle,$$
$$\gamma_2(t) := ti,\ i \in \langle 0, 3\rangle,$$
$$\gamma_3(t) := t,\ i \in \langle 0, 3\rangle.$$

Pak



a navíc

$$\gamma_1'(t) = 3ie^{it},$$
$$\gamma_2'(t) = i,$$
$$\gamma_3'(t) = 1,$$

a proto

$$\begin{aligned} \underline{\int_\gamma |z|\,\mathrm{d}z} &= \int_{\gamma_1} |z|\,\mathrm{d}z - \int_{\gamma_2} |z|\,\mathrm{d}z + \int_{\gamma_3} |z|\,\mathrm{d}z = \\ &= \int_0^{\frac{\pi}{2}} 3 \cdot 3ie^{it}\,\mathrm{d}t - \int_0^3 ti\,\mathrm{d}t + \int_0^3 t\,\mathrm{d}t = \\ &= 9i \int_0^{\frac{\pi}{2}} (\cos t + i \sin t)\,\mathrm{d}t + (1-i)\int_0^3 t\,\mathrm{d}t = \\ &= 9i[\sin t]_0^{\frac{\pi}{2}} + 9[\cos t]_0^{\frac{\pi}{2}} + (1-i)\left[\frac{t^2}{2}\right]_0^3 = \\ &= 9i - 9 + \frac{9}{2}(1-i) = \\ &= \underline{-\frac{9}{2} + \frac{9}{2}i}. \end{aligned}$$

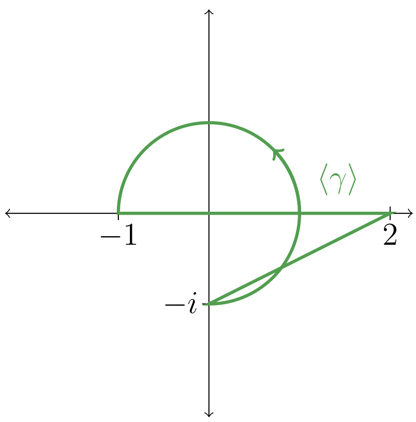
Příklad 42.
Vypočtěte

$$\int_\gamma z^3 \, dz,$$

je-li

$$\gamma(t) := \begin{cases} e^{it}, & t \in \langle -\frac{\pi}{2}, \pi \rangle, \\ \frac{3}{\pi}t - 4, & t \in \langle \pi, 2\pi \rangle, \\ -\frac{2+i}{\pi}t + 6 + 2i, & t \in \langle 2\pi, 3\pi \rangle. \end{cases}$$

Řešení:



Stačí aplikovat Cauchyho větu.

$$\underline{\int_\gamma z^3 \, dz = 0,}$$

protože $f(z) := z^3$ je holomorfní funkce na jednoduše souvislé oblasti $\mathbb{C}$ a γ je po částech hladká uzavřená křivka v $\mathbb{C}$.

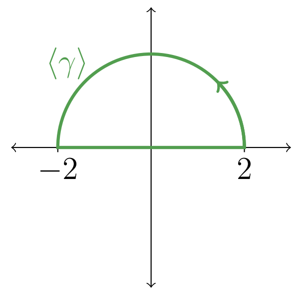
Příklad 43.
Vypočtěte

$$\int_\gamma |z| \, \overline{z} \, dz,$$

je-li γ taková jednoduchá uzavřená po částech hladká a kladně orientovaná křivka, že $\langle\gamma\rangle$ je hranicí množiny

$$\{z \in \mathbb{C}\colon \ |z| < 2 \ \land \ \operatorname{Im} z > 0\}.$$

Řešení:



Definujme křivky

$$\gamma_1(t) := 2e^{it},\ t \in \langle 0, \pi\rangle,$$
$$\gamma_2(t) := t,\ t \in \langle -2, 2\rangle.$$

Pak

$$\gamma_1'(t) = 2ie^{it},$$
$$\gamma_2'(t) = 1,$$

a proto

$$\underline{\int_\gamma |z|\,\overline{z}\,\mathrm{d}z} = \int_{\gamma_1} |z|\,\overline{z}\,\mathrm{d}z + \int_{\gamma_2} |z|\,\overline{z}\,\mathrm{d}z =$$
$$= \int_0^\pi 2\cdot 2e^{-it}\cdot 2ie^{it}\,\mathrm{d}t + \underbrace{\int_{-2}^2 |t|\,t\,\mathrm{d}t}_{=0} =$$
$$= 8i\int_0^\pi 1\,\mathrm{d}t = \underline{8\pi i}.$$

PŘÍKLAD 44.
Vypočtěte pomocí Cauchyho integrálních vzorců daný integrál[4]

a)
$$\int_k \frac{z^2+i}{z}\,\mathrm{d}z, \quad \text{kde} \quad k = \{z \in \mathbb{C}\colon\ |z-2i| = 1\};$$

b)
$$\int_k \frac{\sin z}{z+i}\,\mathrm{d}z, \quad \text{kde} \quad k = \{z \in \mathbb{C}\colon\ |z+i| = 1\};$$

[4] *Úmluva.* Symbolem $\int_k f(z)\,\mathrm{d}z$, kde $k \subset \mathbb{C}$, rozumíme $\int_\gamma f(z)\,\mathrm{d}z$, kde γ je taková jednoduchá uzavřená po částech hladká kladně orientovaná křivka, že $\langle\gamma\rangle = k$.

c)

$$\int_k \frac{\sin z}{z^2 - 7z + 10}\, \mathrm{d}z, \quad \text{kde} \quad k = \{z \in \mathbb{C} \colon \ |z| = 3\};$$

d)

$$\int_k \frac{\sin z}{(z - 2i)^3}\, \mathrm{d}z, \quad \text{kde} \quad k = \{z \in \mathbb{C} \colon \ |z| = 3\};$$

e)

$$\int_k \frac{\cos z}{z^2 - \pi^2}\, \mathrm{d}z, \quad \text{kde} \quad k = \{z \in \mathbb{C} \colon \ |z| = 4\};$$

f)

$$\int_k \frac{e^{\frac{1}{z}}}{(z^2 - 4)^2}\, \mathrm{d}z, \quad \text{kde} \quad k = \{z \in \mathbb{C} \colon \ |z - 2| = 1\};$$

g)

$$\int_\gamma \frac{e^z \cos(\pi z)}{z^2 + 2z}\, \mathrm{d}z, \quad \text{kde} \quad \gamma(t) := \frac{3}{2} e^{it}, \ t \in \langle 0, 2\pi \rangle;$$

h)

$$\int_\gamma \frac{\mathrm{d}z}{(z^2 - 1)^3}, \quad \text{kde} \quad \gamma(t) := \frac{-2 + e^{-4\pi i t}}{2}, \ t \in \langle 0, 4 \rangle;$$

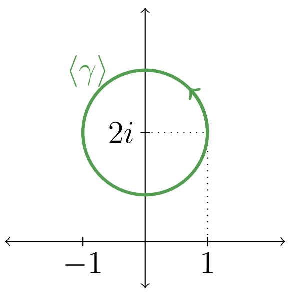
i)

$$\int_\gamma \frac{\mathrm{d}z}{(1 - z)(z + 2)(z - i)^2},$$

kde γ je taková jednoduchá uzavřená po částech hladká kladně orientovaná křivka, že $-2 \in \operatorname{int} \gamma, \ i \in \operatorname{int} \gamma, \ 1 \in \operatorname{ext} \gamma$.

Řešení:

a)



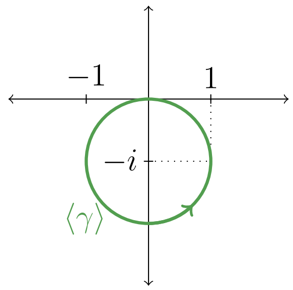
Funkce „$\frac{z^2+i}{z}$“ je holomorfní na jednoduše souvislé oblasti $\Omega := \{z \in \mathbb{C} \colon \ \operatorname{Im} z > 0\}$ a $k = \langle \gamma \rangle \subset \Omega$, a proto z Cauchyho věty vyplývá

$$\underline{\int_k \frac{z^2 + i}{z}\, \mathrm{d}z = \int_\gamma \frac{z^2 + i}{z} = 0.}$$

V zadání však máme: „Vypočtěte pomocí Cauchyho integrálních vzorců …“. Lze to udělat třeba takto:

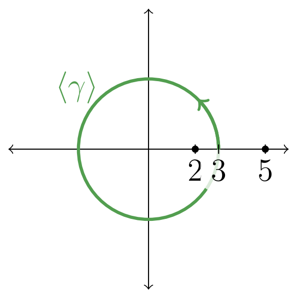
$$\underline{\int_k \frac{z^2 + i}{z}\, \mathrm{d}z} = \int_\gamma \frac{\frac{z^2+i}{z}(z - 2i)}{z - 2i}\, \mathrm{d}z = 2\pi i \left[\frac{z^2 + i}{z}(z - 2i) \right]_{z=2i} = \underline{0}.$$

b)



$$\underline{\int_k \frac{\sin z}{z+i}\,\mathrm{d}z} = \int_\gamma \frac{\sin z}{z-(-i)}\,\mathrm{d}z =$$
$$= 2\pi i\,[\sin z]_{z=-i} =$$
$$= 2\pi i\,\frac{e^{i(-i)} - e^{-i(-i)}}{2i} =$$
$$= \pi(e - e^{-1}) = \underline{2\pi \sinh 1}.$$

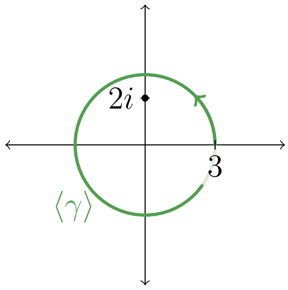
c)



$z^2 - 7z + 10 = (z-5)(z-2)$, a proto

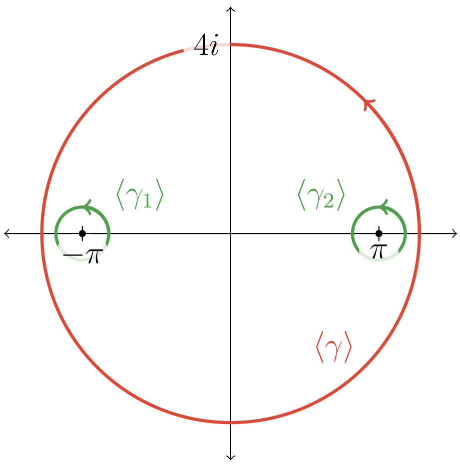
$$\underline{\int_k \frac{\sin z}{z^2-7z+10}\,\mathrm{d}z} = \int_\gamma \frac{\frac{\sin z}{z-5}}{z-2}\,\mathrm{d}z =$$
$$= 2\pi i \left[\frac{\sin z}{z-5}\right]_{z=2} =$$
$$= 2\pi i\,\frac{\sin 2}{-3} = \underline{-\left(\frac{2}{3}\pi \sin 2\right) i}.$$

d)



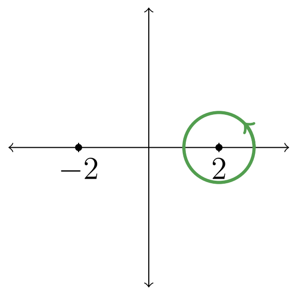
$$\underline{\int_k \frac{\sin z}{(z-2i)^3}\,\mathrm{d}z} = \frac{2\pi i}{2!}\left[(\sin z)''\right]_{z=2i} =$$
$$= \pi i\,[-\sin z]_{z=2i} =$$
$$= -\pi i\,\frac{e^{-2}-e^{2}}{2i} = \underline{\pi \sinh 2}.$$

e)



$$\underline{\int_k \frac{\cos z}{z^2-\pi^2}\,\mathrm{d}z} = \int_{\gamma_1} \frac{\frac{\cos z}{z-\pi}}{z-(-\pi)}\,\mathrm{d}z + \int_{\gamma_2} \frac{\frac{\cos z}{z+\pi}}{z-\pi}\,\mathrm{d}z =$$
$$= 2\pi i\left(\left[\frac{\cos z}{z-\pi}\right]_{z=-\pi} + \left[\frac{\cos z}{z+\pi}\right]_{z=\pi}\right) =$$
$$= 2\pi i\left(\frac{-1}{-2\pi} + \frac{-1}{2\pi}\right) = \underline{0}.$$

f)



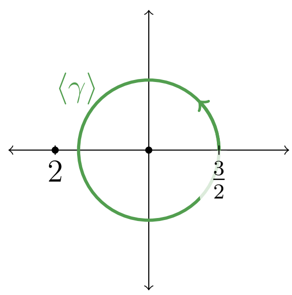
$$\underline{\int_k \frac{e^{\frac{1}{z}}}{(z^2-4)^2}\,\mathrm{d}z} = \int_k \frac{\frac{e^{\frac{1}{z}}}{(z+2)^2}}{(z-2)^2}\,\mathrm{d}z =$$

$$= 2\pi i \left[\left(\frac{e^{\frac{1}{z}}}{(z+2)^2}\right)'\right]_{z=2} =$$

$$= 2\pi i \left[\frac{-\frac{1}{z^2}e^{\frac{1}{z}}(z+2) - e^{\frac{1}{z}}2}{(z+2)^3}\right]_{z=2} =$$

$$= 2\pi i \frac{\sqrt{e}\left(-\frac{1}{4}\cdot 4 - 2\right)}{16\cdot 4} = \underline{-\frac{3\pi\sqrt{e}}{32}i}.$$

g)

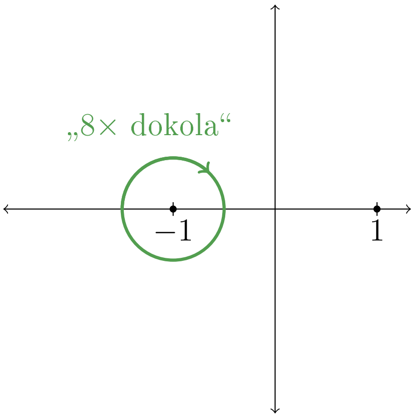


$$\underline{\int_\gamma \frac{e^z\cos(\pi z)}{z^2+2z}\,\mathrm{d}z} = \int_\gamma \frac{\frac{e^z\cos(\pi z)}{z+2}}{z-0}\,\mathrm{d}z =$$

$$= 2\pi i \left[\frac{e^z\cos(\pi z)}{z+2}\right]_{z=0} =$$

$$= 2\pi i \frac{1}{2} = \underline{\pi i}.$$

h)



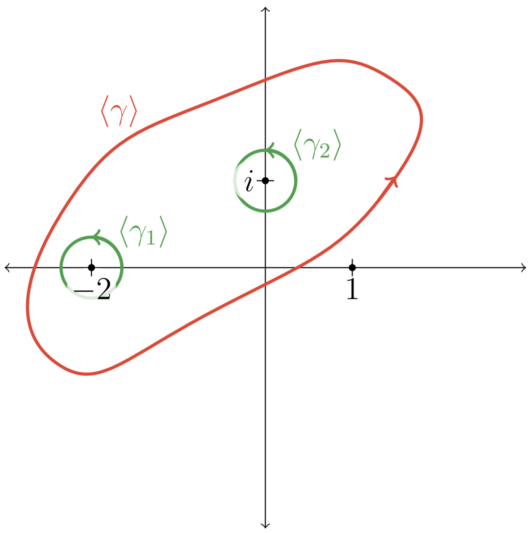
$$\underline{\int_\gamma \frac{\mathrm{d}z}{(z^2-1)^3}} = \int_\gamma \frac{\frac{1}{(z-1)^3}}{(z+1)^3}\,\mathrm{d}z =$$

$$= -8 \cdot \frac{2\pi i}{2!}\left[\left(\frac{1}{(z-1)^3}\right)''\right]_{z=-1} =$$

$$= -8\pi i\left[-3\left(\frac{1}{(z-1)^4}\right)'\right]_{z=-1} =$$

$$= -8 \cdot 12\pi i\left[\frac{1}{(z-1)^5}\right]_{z=-1} = \underline{3\pi i}.$$

i)



$$\underline{\int_\gamma \frac{\mathrm{d}z}{(1-z)(z+2)(z-i)^2}} = \int_{\gamma_1} \frac{\mathrm{d}z}{(1-z)(z+2)(z-i)^2} + \int_{\gamma_2} \frac{\mathrm{d}z}{(1-z)(z+2)(z-i)^2} =$$

$$= \int_{\gamma_1} \frac{\frac{1}{(1-z)(z-i)^2}}{z+2}\,\mathrm{d}z + \int_{\gamma_2} \frac{\frac{1}{(1-z)(z+2)}}{(z-i)^2}\,\mathrm{d}z =$$

$$= 2\pi i \left(\left[\frac{1}{(1-z)(z-i)^2} \right]_{z=-2} + \left[\left(\frac{1}{(1-z)(z+2)} \right)' \right]_{z=i} \right) =$$

$$= 2\pi i \left(\frac{1}{3(-2-i)^2} + \left[\left(\frac{1}{-z^2-z+2} \right)' \right]_{z=i} \right) =$$

$$= 2\pi i \left(\frac{1}{3(3+4i)} + \left[\frac{2z+1}{(-z^2-z+2)^2} \right]_{z=i} \right) =$$

$$= 2\pi i \left(\frac{3-4i}{75} + \frac{1+2i}{(3-i)^2} \right) = 2\pi i \left(\frac{3-4i}{75} + \frac{(1+2i)(8+6i)}{100} \right) =$$

$$= 2\pi i \left(\frac{3}{75} - \frac{4}{100} - \frac{4}{75}i + \frac{22}{100}i \right) = 2\pi i \frac{-\frac{8}{3}+11}{50} i = \underline{-\frac{1}{3}\pi}.$$

<u>PŘÍKLAD 45</u>.
Vypočtěte

a) $\int_0^{1+i} e^z\,\mathrm{d}z$;

b) $\int_0^{1+i} z^3\,\mathrm{d}z$;

c) $\int_0^{i} z^2 \sin z\,\mathrm{d}z$;

d) $\int_0^{i} z \sin z\,\mathrm{d}z$.

<u>Řešení:</u>

a)

$$\underline{\int_0^{1+i} e^z\,\mathrm{d}z} = [e^z]_0^{1+i} = e^1(\cos 1 + i \sin 1) - 1 =$$
$$= \underline{e\cos 1 - 1 + i(\sin 1)e}.$$

b)

$$\underline{\int_0^{1+i} z^3\,\mathrm{d}z} = \left[\frac{z^4}{4} \right]_0^{1+i} = \frac{1}{4}(2i)^2 = \underline{-1}.$$

c)

$$\underline{\int_0^i z^2 \sin z \, \mathrm{d}z} = \int_0^i \underbrace{z^2}_{=:u} \underbrace{\sin z}_{=:v'} \, \mathrm{d}z =$$

$$= \left[-z^2 \cos z\right]_0^i + \int_0^i \underbrace{2z}_{=:u} \underbrace{\cos z}_{=:v'} \, \mathrm{d}z =$$

$$= \cos i + [2z \sin z]_0^i - 2 \int_0^i \sin z \, \mathrm{d}z =$$

$$= \cos i + 2i \sin i + 2[\cos z]_0^i =$$

$$= \frac{e^{-1} + e^1}{2} + 2i \frac{e^{-1} - e^1}{2i} + 2\frac{e^{-1} + e^1}{2} - 2 =$$

$$= \underline{3 \cosh 1 - 2 \sinh 1 - 2}$$

(použili jsme dvakrát integraci per partes).

d)

$$\underline{\int_0^i z \sin z \, \mathrm{d}z} = [-z \cos z]_0^i + [\sin z]_0^i =$$

$$= -i \cos i + \sin i =$$

$$= -i \cosh 1 + i \sinh 1 =$$

$$= i(\sinh 1 - \cosh 1) = \underline{-\frac{1}{e} i}$$

(opět jsme integrovali pomocí per partes).

<u>PŘÍKLAD 46</u>.
Rozhodněte, zda daná řada konverguje

a) $\sum\limits_{n=1}^{\infty} \frac{i^n}{n2^n}$;

b) $\sum\limits_{n=1}^{\infty} \frac{n}{3^n}(1+i)^n$;

c) $\sum\limits_{n=1}^{\infty} \frac{(-i)^n}{3n-17}$.

<u>Řešení:</u>

a) $\underline{\sum\limits_{n=1}^{\infty} \frac{i^n}{n2^n} \text{ konverguje absolutně, neboť}}$

$$\sqrt[n]{\left|\frac{i^n}{n2^n}\right|} = \frac{1}{\sqrt[n]{n}\, 2} \to \frac{1}{2} < 1.$$

b) $\sum\limits_{n=1}^{\infty} \frac{n}{3^n}(1+i)^n$ konverguje absolutně, protože

$$\sqrt[n]{\left|\frac{n}{3^n}(1+i)^n\right|} = \frac{\sqrt[n]{n}}{3}\sqrt{2} \to \frac{\sqrt{2}}{3} < 1.$$

c) $\sum\limits_{n=1}^{\infty} \frac{(-i)^n}{3n-17}$ konverguje neabsolutně, protože řady

$$\sum_{n=1}^{\infty} \operatorname{Re}\left(\frac{(-i)^n}{3n-17}\right) \quad \text{i} \quad \sum_{n=1}^{\infty} \operatorname{Im}\left(\frac{(-i)^n}{3n-17}\right)$$

konvergují (stačí si uvědomit, že

$$\sum_{n=1}^{\infty} \frac{(-i)^n}{3n-17} = \frac{-i}{3\cdot 1-17} + \frac{-1}{3\cdot 2-17} + \frac{i}{3\cdot 3-17} + \frac{1}{3\cdot 4-17} + \frac{-i}{3\cdot 5-17} + \cdots,$$

a na příslušné řady použít Leibnizovo kritérium a snadné pozorování, že z konvergence řady $a_1 + a_2 + a_3 + \cdots$ plyne i konvergence řady $0 + a_1 + 0 + a_2 + 0 + a_3 + \cdots$), a navíc

$$\sum_{n=1}^{\infty} \left|\frac{(-i)^n}{3n-17}\right| = \sum_{n=1}^{\infty} \left|\frac{1}{3n-17}\right| = \infty$$

(viz integrální kritérium).

PŘÍKLAD 47.
Určete obor konvergence dané řady (tzn. najděte všechna $z \in \mathbb{C}$, pro která daná řada konverguje).

a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n$;

b) $\sum\limits_{n=1}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n}\right)$.

Řešení:

a) Pro $z = 1$ řada zřejmě nekonverguje. Pro $z \in \mathbb{C} \setminus \{1\}$ platí

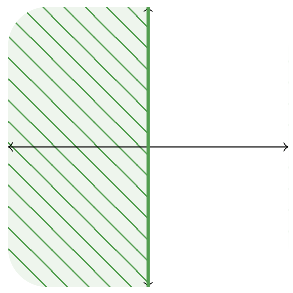
$$\sqrt[n]{\left|\frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n\right|} = \frac{1}{\left(\sqrt[n]{n}\right)^2} \cdot \left|\frac{z+1}{z-1}\right| \to \left|\frac{z+1}{z-1}\right|,$$

a proto daná řada konverguje absolutně pro každé $z \in \mathbb{C}$ takové, že $\left|\frac{z+1}{z-1}\right| < 1$, a diverguje pro každé $z \in \mathbb{C}$, pro které $\left|\frac{z+1}{z-1}\right| > 1$.

Je-li $\left|\frac{z+1}{z-1}\right| = 1$, je $\left|\frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n\right| = \frac{1}{n^2}$, a proto řada $\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n$ konverguje absolutně.

Shrnutí: <u>daná řada konverguje (absolutně) pro každé</u>

$$\underline{z \in \left\{ z \in \mathbb{C} \colon \left| \frac{z+1}{z-1} \right| \leq 1 \right\} = \{ z \in \mathbb{C} \colon \operatorname{Re} z \leq 0 \}.}$$



b) Protože pro každé $z \in \mathbb{C} \setminus \{0\}$

$$\left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \frac{|z|}{n+1} \to 0 < 1,$$

konverguje řada $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ absolutně v $\mathbb{C}$. Protože pro každé $z \in \mathbb{C} \setminus \{0\}$

$$\sqrt[n]{\left| \frac{n^2}{z^n} \right|} = \frac{\left(\sqrt[n]{n} \right)^2}{|z|} \to \frac{1}{|z|},$$

konverguje řada $\sum_{n=1}^{\infty} \frac{n^2}{z^n}$ absolutně pro $|z| > 1$ a diverguje pro $|z| < 1$. Je-li $|z| = 1$, je

$$\left| \frac{n^2}{z^n} \right| = n^2 \to \infty \neq 0,$$

a proto řada $\sum_{n=1}^{\infty} \frac{n^2}{z^n}$ diverguje.
Nyní definujme

$$s_n(z) := \sum_{k=1}^{n} \left(\frac{z^k}{k!} + \frac{k^2}{z^k} \right),$$
$$s_n^*(z) := \sum_{k=1}^{n} \frac{z^k}{k!},$$
$$s_n^{**}(z) := \sum_{k=1}^{n} \frac{k^2}{z^k}.$$

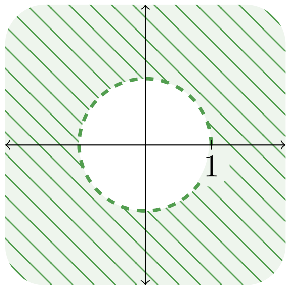
Pak pro každé $z \in \mathbb{C}$ a $n \in \mathbb{N}$ platí

$$s_n(z) = s_n^*(z) + s_n^{**}(z),$$
$$s_n^{**} = s_n(z) - s_n^*(z),$$

a navíc (už víme) $\lim s_n^*(z) \in \mathbb{C}$ pro každé $z \in \mathbb{C}$, a proto pro každé $z \in \mathbb{C}$ platí

$$\lim s_n(z) \in \mathbb{C} \iff \lim s_n^{**}(z) \in \mathbb{C}.$$

Shrnutí: daná řada konverguje (absolutně) na množině $\{z \in \mathbb{C}\colon\ |z| > 1\}$.



PŘÍKLAD 48.

Určete poloměr konvergence R dané mocninné řady

a) $\sum_{n=1}^{\infty} \frac{z^n}{n^{2011}}$;

b) $\sum_{n=1}^{\infty} n^n(z-1)^n$;

c) $\sum_{n=1}^{\infty} \frac{3^n(z-1)^n}{\sqrt{(3n-2)2^n}}$;

d) $\sum_{n=0}^{\infty} \frac{(z+1+i)^n}{3^n(n-i)}$;

e) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$;

f) $\sum_{n=0}^{\infty} \big(\cos(in)\big) z^n$;

g) $\sum_{n=0}^{\infty} (n^2-n-2) z^n$;

h) $\sum_{n=0}^{\infty} \frac{z^n}{(n+8)!}$.

Řešení:

a)

$$\sqrt[n]{\frac{1}{n^{2011}}} = \frac{1}{\left(\sqrt[n]{n}\right)^{2011}} \to 1,$$

a proto

$$\underline{R = 1}.$$

b)

$$\sqrt[n]{n^n} = n \to \infty,$$

a proto

$$\underline{R = \frac{1}{\infty} = 0}.$$

c) Protože

$$\sqrt[n]{\frac{3^n}{\sqrt{(3n-2)2^n}}} = \frac{3}{\sqrt{2}} \frac{1}{\sqrt{\sqrt[n]{3n-2}}} \to \frac{3}{\sqrt{2}}$$

(stačí si uvědomit, že pro $n \geq 3$ platí $1 \leq \sqrt[n]{3n-2} \leq \sqrt[n]{n} \cdot \sqrt[n]{n} \to 1$), a proto

$$\underline{R = \frac{\sqrt{2}}{3}}.$$

d)

$$\left|\frac{\frac{1}{3^{n+1}(n+1-i)}}{\frac{1}{3^n(n-i)}}\right| = \frac{1}{3}\left|\frac{n-i}{n+1-i}\right| = \frac{1}{3}\left|\frac{1-\frac{i}{n}}{1+\frac{1-i}{n}}\right| \to \frac{1}{3},$$

a proto

$$\underline{R=3}.$$

e)

$$\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{1}{n+1}\frac{(n+1)^n(n+1)}{n^n} = \left(1+\frac{1}{n}\right)^n \to e,$$

a proto

$$\underline{R=\frac{1}{e}}.$$

f)

$$\left|\frac{\cos(i(n+1))}{\cos(in)}\right| = \left|\frac{e^{i(i(n+1))}+e^{-i(i(n+1))}}{e^{iin}+e^{-iin}}\right| =$$

$$= \frac{e^{-(n+1)}+e^{n+1}}{e^{-n}+e^n}\cdot\frac{\frac{1}{e^n}}{\frac{1}{e^n}} =$$

$$= \frac{\frac{1}{e^n e^{n+1}}+e}{\frac{1}{e^n e^n}+1} \to e,$$

a proto

$$\underline{R=\frac{1}{e}}.$$

g)

$$\left|\frac{(n+1)^2-(n+1)-2}{n^2-n-2}\right| \to 1,$$

a proto

$$\underline{R=1}.$$

h)

$$\frac{\frac{1}{(n+9)!}}{\frac{1}{(n+8)!}} = \frac{1}{n+9} \to 0,$$

a proto

$$\underline{R=\infty}.$$

<u>Příklad 49</u>.

Najděte součet dané mocninné řady v kruhu konvergence

a) $\sum\limits_{n=1}^{\infty} nz^n$;

b) $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$;

c) $\sum\limits_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$;

d) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n+1}$;

e) $\sum\limits_{n=0}^{\infty} (n^2 - n - 2) z^n$.

Řešení:

a) $\sqrt[n]{n} \to 1$, a proto poloměr konvergence dané řady je 1.

Pro každé $z \in \mathbb{C}$, $|z| < 1$, platí

$$\underline{\sum_{n=1}^{\infty} n z^n} = z \sum_{n=1}^{\infty} n z^{n-1} = z \left(\sum_{n=1}^{\infty} n \frac{z^n}{n} \right)' =$$

$$= z \left(\sum_{n=1}^{\infty} z^n \right)' = z \left(\frac{z}{1-z} \right)' =$$

$$= z \frac{1 - z + z}{(1-z)^2} = \underline{\frac{z}{(1-z)^2}}.$$

b) $\sqrt[n]{\frac{1}{n}} \to 1$, proto poloměr konvergence je 1.

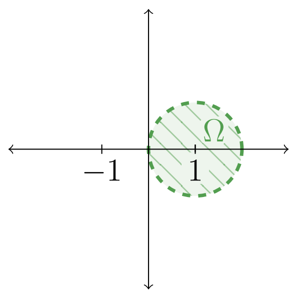
Definujme funkci $f(z) := \sum_{n=1}^{\infty} \frac{z^n}{n}$. Pak pro každé $z \in \mathbb{C}$, $|z| < 1$, platí

$$f'(z) = \sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1-z}.$$

Odtud, protože

$$|z| < 1 \ \Rightarrow \ 1 - z \in \Omega := \{w \in \mathbb{C} \colon \ |w - 1| < 1\},$$

$$\ln' w = \frac{1}{w} \text{ v } \Omega,$$



existuje $c \in \mathbb{C}$ takové, že pro každé $z \in \mathbb{C}$, $|z| < 1$, platí

$$f(z) = -\ln(1 - z) + c.$$

Navíc $f(0) = -\ln 1 + c = 0$, a proto $c = 0$.

Shrnutí: pro každé $z \in \mathbb{C}$, $|z| < 1$, je

$$\underline{\sum_{n=1}^{\infty} \frac{z^n}{n}} = f(z) = \underline{-\ln(1 - z)}.$$

c)

$$\left|\frac{\frac{1}{2n+3}}{\frac{1}{2n+1}}\right| \to 1,$$

a proto poloměr konvergence je 1.

Definujme $f(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$. Pak pro každé $z \in \mathbb{C},\ |z| < 1$, platí

$$\begin{aligned} f'(z) &= \sum_{n=0}^{\infty} z^{2n} = \frac{1}{1-z^2} = \\ &= \frac{1}{2}\frac{-1}{z-1} + \frac{1}{2}\frac{1}{z+1}. \end{aligned}$$

Odtud plyne, že existuje $c \in \mathbb{C}$ takové, že pro každé $z \in \mathbb{C}, |z| < 1$, je

$$f(z) = -\frac{1}{2}\ln(1-z) + \frac{1}{2}\ln(1+z) + c.$$

A jelikož $0 = f(0) = c$, platí pro každé $z \in \mathbb{C},\ |z| < 1$, že

$$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} = -\frac{1}{2}\ln(1-z) + \frac{1}{2}\ln(1+z).$$

d) $\sqrt[n]{\frac{1}{n+1}} \to 1$, proto poloměr konvergence je 1.

Buď $f(z) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{n+1}}{n+1}$. Pak pro $z \in \mathbb{C},\ |z| < 1$, platí

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} (-1)^{n+1} z^n = -\sum_{n=1}^{\infty} (-z)^n = \\ &= \frac{z+1-1}{1+z}. \end{aligned}$$

Odtud plyne existence $c \in \mathbb{C}$ takového, že

$$f(z) = z - \ln(1+z) + c,$$

a jelikož $0 = f(0) = c$, je

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n+1} = \begin{cases} \frac{1}{z} f(z) = 1 - \frac{\ln(1+z)}{z}, & 0 < |z| < 1, \\ 0, & z = 0. \end{cases}$$

e)

$$\left|\frac{(n+1)^2 - (n+1) - 2}{n^2 - n - 2}\right| \to 1,$$

a proto poloměr konvergence je 1.

Pro každé $z \in \mathbb{C},\ |z| < 1$, platí

$$\sum_{n=0}^{\infty} (n^2 - n - 2) z^n = \sum_{n=0}^{\infty} n^2 z^n - \sum_{n=0}^{\infty} n z^n - 2\sum_{n=0}^{\infty} z^n$$

(stačí si uvědomit, že všechny uvedené řady absolutně konvergují).

Navíc ($|z| < 1$):

- $$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

- $$\sum_{n=0}^{\infty} n z^n = \sum_{n=1}^{\infty} n z^n = z \sum_{n=1}^{\infty} n z^{n-1} =$$
$$= z \left(\sum_{n=1}^{\infty} n \frac{z^n}{n} \right)' = z \left(\frac{z}{1-z} \right)' =$$
$$= z \, \frac{1-z+z}{(1-z)^2} = \frac{z}{(1-z)^2},$$

- $$\sum_{n=0}^{\infty} n^2 z^n = \sum_{n=1}^{\infty} n^2 z^n = z \sum_{n=1}^{\infty} n^2 z^{n-1} =$$
$$= z \left(\sum_{n=1}^{\infty} n^2 \frac{z^n}{n} \right)' = z \left(\sum_{n=1}^{\infty} n z^n \right)' =$$
$$= z \left(\frac{z}{(1-z)^2} \right)' = z \frac{(1-z)^2 + z \, 2 \, (1-z)}{(1-z)^4} =$$
$$= z \, \frac{z+1}{(1-z)^3},$$

a proto pro každé $z \in \mathbb{C}, \ |z| < 1$, platí

$$\sum_{n=0}^{\infty} (n^2 - n - 2) z^n = \frac{z^2 + z - z(1-z) - 2(1-z)^2}{(1-z)^3} = \frac{2-4z}{(z-1)^3}.$$

Příklad 50.
Najděte součet dané řady

a) $\sum\limits_{n=1}^{\infty} \frac{1}{n2^n}$;

b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$.

Řešení:
Uvažujme funkci

$$f(z) := \sum_{n=1}^{\infty} \frac{z^n}{n2^n}.$$

Protože $\frac{1}{\sqrt[n]{n2^n}} \to \frac{1}{2}$, má v definici funkce f uvedená mocninná řada poloměr konvergence 2. Proto pro každé $z \in \mathbb{C},\ 0 < |z| < 2$, platí

$$f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{2^n} = \frac{1}{z} \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n = $$
$$= \frac{1}{z} \frac{\frac{z}{2}}{1 - \frac{z}{2}} = \frac{1}{2-z}.$$

Takže existuje $c \in \mathbb{C}$ takové, že $f(z) = -\ln(2-z) + c$. A jelikož $f(0) = 0 = -\ln 2 + c$, platí pro každé $z \in \mathbb{C},\ |z| < 2$,

$$f(z) = -\ln(2-z) + \ln 2.$$

a)

$$\underline{\sum_{n=1}^{\infty} \frac{1}{n2^n}} = f(1) = \underline{\ln 2},$$

b)

$$\underline{\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}} = f(-1) = \underline{-\ln 3 + \ln 2 = \ln \frac{2}{3}}.$$

Příklad 51.
Najděte Taylorovu řadu funkce f o středu z_0 a určete její poloměr konvergence, je-li

a) $f(z) := \frac{z+1}{z^2+4z-5},\ z_0 = -1$;

b) $f(z) := \frac{z}{z^2+i},\ z_0 = 0$;

c) $f(z) := \ln \frac{1+z}{1-z},\ z_0 = 0$;

d) $f(z) := e^{3z-2},\ z_0 = 1$;

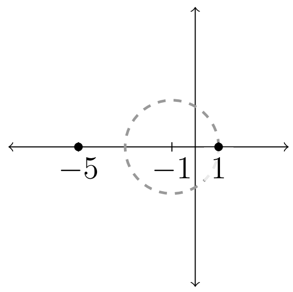
e) $f(z) := \sin(3z^2 + 2),\ z_0 = 0$;

f) $f(z) := \frac{1}{(z-1)^3},\ z_0 = 3$;

g) $f(z) := \sin^2 z,\ z_0 = 0$.

Řešení:

a) $f(z) = \frac{2}{3}\,\frac{1}{z+5} + \frac{1}{3}\,\frac{1}{z-1}$,



a proto poloměr konvergence je 2 a pro každé $z \in \mathbb{C},\ |z+1| < 2$, platí

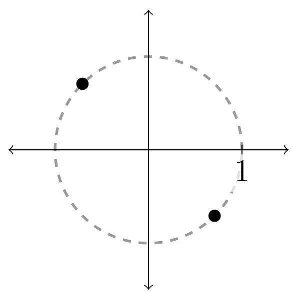
$$\underline{f(z)} = \frac{2}{3} \cdot \frac{1}{4+z+1} + \frac{1}{3} \cdot \frac{1}{-2+z+1} = \frac{2}{12} \cdot \frac{1}{1+\frac{z+1}{4}} - \frac{1}{6} \cdot \frac{1}{1-\frac{z+1}{2}} =$$

$$= \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n \frac{(z+1)^n}{4^n} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(z+1)^n}{2^n} =$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{6\cdot 4^n} - \frac{1}{6\cdot 2^n}\right)(z+1)^n =$$

$$= \underline{\sum_{n=1}^{\infty}\frac{(-1)^n - 2^n}{6\cdot 4^n}(z+1)^n}.$$

b) $z^2 + i = 0$ právě tehdy, když $z = \pm\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$,



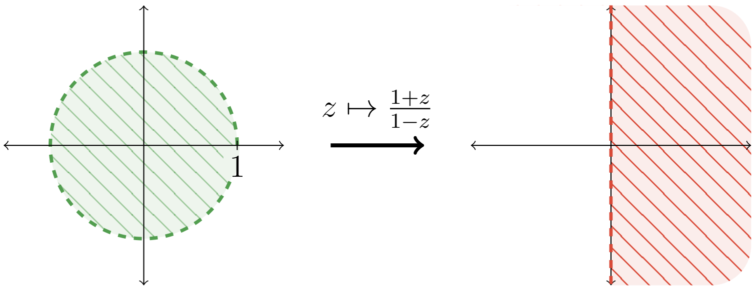
a proto má hledaná Taylorova řada poloměr konvergence 1.
Pro každé $z \in \mathbb{C},\ |z| < 1$, platí

$$\underline{f(z)} = \frac{z}{i} \cdot \frac{1}{1+\frac{z^2}{i}} = \frac{z}{i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^2}{i}\right)^n =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{i^{n+1}}z^{2n+1} =$$

$$= \underline{\sum_{n=0}^{\infty} i^{n-1} z^{2n+1}}.$$

c) Protože zřejmě



$(0 \mapsto 0,\ 1 \mapsto \infty,\ -1 \mapsto 0)$, je poloměr konvergence 1. Pro každé $z \in \mathbb{C}$, $|z| < 1$, platí

$$f'(z) = \frac{1-z}{1+z} \cdot \frac{1-z+(1+z)}{(1-z)^2} = \frac{2}{(1+z)(1-z)} =$$
$$= \frac{2}{1-z^2} = \sum_{n=0}^{\infty} 2z^{2n},$$

a proto existuje takové $c \in \mathbb{C}$, že

$$f(z) = \sum_{n=0}^{\infty} 2\,\frac{z^{2n+1}}{2n+1} + c.$$

A jelikož $f(0) = 0 = c$, platí pro každé $z \in \mathbb{C}$, $|z| < 1$, že

$$f(z) = \sum_{n=0}^{\infty} 2\,\frac{z^{2n+1}}{2n+1}.$$

d) Zřejmě poloměr konvergence je ∞. Víme, že pro každé $z \in \mathbb{C}$ je $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, a proto

$$f(z) = e^{3z-2} = e^{3(z-1)+1} = e\,e^{3(z-1)} =$$
$$= \sum_{n=0}^{\infty} \frac{e \cdot 3^n}{n!}(z-1)^n.$$

e) Poloměr konvergence je ∞ a pro každé $z \in \mathbb{C}$ je

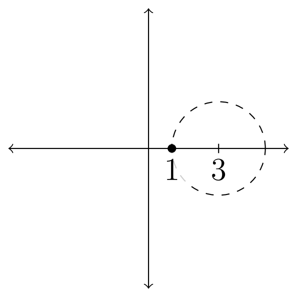
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$
$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

Odtud plyne, že pro každé $z \in \mathbb{C}$ platí

$$\underline{f(z) =} \sin(3z^2)\cos 2 + \cos(3z^2)\sin 2 =$$
$$= \sum_{n=0}^{\infty} \underbrace{\cos 2 \cdot (-1)^n \frac{3^{2n+1}}{(2n+1)!}}_{=:\alpha_n} z^{4n+2} + \sum_{n=0}^{\infty} \underbrace{\sin 2 \cdot (-1)^n \frac{3^{2n}}{(2n)!}}_{=:\beta_n} z^{4n} =$$
$$= \underline{\sum_{n=0}^{\infty} a_n z^{2n}},$$

kde pro každé $k \in \mathbb{N} \cup \{0\}$ je $a_{2k} := \beta_k$ a $a_{2k+1} := \alpha_k$.

f)



Zřejmě poloměr konvergence je 2. Pro každé $z \in U(3,2)$ platí

$$\frac{1}{z-1} = \frac{1}{2+z-3} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-3}{2}} =$$
$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{z-3}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-3)^n$$

a současně

$$\left(\frac{1}{z-1}\right)'' = \left(-\frac{1}{(z-1)^2}\right)' = 2\,\frac{1}{(z-1)^3}.$$

Odtud snadno plyne, že pro každé $z \in U(3,2)$ je

$$\underline{f(z) =} \frac{1}{2} \left(\frac{1}{z-1}\right)'' =$$
$$= \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} n (z-3)^{n-1}\right)' =$$
$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n}{2^{n+1}} n(n-1)(z-3)^{n-2} =$$
$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{2^{n+2}} n(n-1)(z-3)^{n-2} =$$
$$= \underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+4}} (n+2)(n+1)(z-3)^n}.$$

g) Poloměr konvergence je ∞ a pro každé $z \in \mathbb{C}$ platí

$$\underline{f(z) =} \sin^2 z = \frac{1-\cos(2z)}{2} = \\ = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2z)^{2n}}{(2n)!} = \\ = \underline{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n-1}}{(2n)!}z^{2n}}.$$

Příklad 52.
Určete obor konvergence dané Laurentovy řady (tzn. najděte všechna $z \in \mathbb{C}$, pro která daná řada konverguje).

a) $\sum\limits_{n=-\infty}^{\infty} 2^{-|n|} z^n$;

b) $\sum\limits_{n=-\infty}^{\infty} \frac{(z-i)^n}{n^2+1}$.

Řešení:

a)

$$\sum_{n=-\infty}^{\infty} 2^{-|n|} z^n = \sum_{n=0}^{\infty} 2^{-n} z^n + \sum_{n=1}^{\infty} 2^{-n} z^{-n} = \\ = \sum_{n=0}^{\infty} \frac{1}{2^n} z^n + \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{1}{z^n}.$$

Protože mocninná řada $\sum\limits_{n=0}^{\infty} \frac{1}{2^n} z^n$ má poloměr konvergence 2 (zřejmě $\sqrt[n]{\frac{1}{2^n}} \to \frac{1}{2}$), platí implikace

$$|z| < 2 \;\Rightarrow\; \sum_{n=0}^{\infty} \frac{1}{2^n} z^n \text{ konverguje absolutně,}$$

$$|z| > 2 \;\Rightarrow\; \sum_{n=0}^{\infty} \frac{1}{2^n} z^n \text{ diverguje.}$$

Je-li $|z| = 2$, je $\left|\frac{1}{2^n} z^n\right| = 1 \to 1 \neq 0$, a proto řada $\sum\limits_{n=0}^{\infty} \frac{1}{2^n} z^n$ diverguje.

Nyní uvažujme hlavní část dané řady, tj. řadu $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{z^n}$. Už jsme zjistili, že

$$\left|\frac{1}{z}\right| < 2 \left(\text{tj. } |z| > \frac{1}{2}\right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{z^n} \text{ konverguje absolutně,}$$

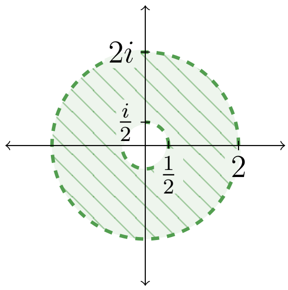
$$\left|\frac{1}{z}\right| > 2 \left(\text{tj. } |z| < \frac{1}{2}\right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{z^n} \text{ diverguje,}$$

$$\left|\frac{1}{z}\right| = 2 \left(\text{tj. } |z| = \frac{1}{2}\right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{z^n} \text{ diverguje.}$$

Shrnutí: daná řada (absolutně) konverguje pro každé

$$z \in P\left(0, \frac{1}{2}, 2\right) = \left\{z \in \mathbb{C}\colon\ \frac{1}{2} < |z| < 2\right\},$$

jinde diverguje.



b)

$$\sum_{n=-\infty}^{\infty} \frac{(z-i)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}(z-i)^n + \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{1}{(z-i)^n}.$$

Jelikož

$$\frac{\frac{1}{(n+1)^2+1}}{\frac{1}{n^2+1}} \to 1,$$

víme že

$$|z-i| < 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+1}(z-i)^n \text{ konverguje absolutně,}$$

$$|z-i| > 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+1}(z-i)^n \text{ diverguje.}$$

Je-li $|z-i| = 1$, je

$$\sum_{n=0}^{\infty} \left|\frac{(z-i)^n}{n^2+1}\right| = \sum_{n=0}^{\infty} \frac{1}{n^2+1},$$

a proto (viz integrální kritérium) řada $\sum\limits_{n=0}^{\infty} \frac{(z-i)^n}{n^2+1}$ konverguje absolutně.

Zjistili jsme (taky), že

$$\left|\frac{1}{z-i}\right| < 1 \text{ (tj. } |z-i| > 1) \ \Rightarrow\ \sum_{n=1}^{\infty} \frac{1}{n^2+1}\frac{1}{(z-i)^n} \text{ konverguje absolutně,}$$

$$\left|\frac{1}{z-i}\right| > 1 \text{ (tj. } |z-i| < 1) \ \Rightarrow\ \sum_{n=1}^{\infty} \frac{1}{n^2+1}\frac{1}{(z-i)^n} \text{ diverguje,}$$

$$\left|\frac{1}{z-i}\right| = 1 \text{ (tj. } |z-i| = 1) \ \Rightarrow\ \sum_{n=1}^{\infty} \frac{1}{n^2+1}\frac{1}{(z-i)^n} \text{ konverguje absolutně.}$$

Shrnutí: daná řada (absolutně) konverguje pro každé $z \in \{z \in \mathbb{C}\colon\ |z-i| = 1\}$. Jinde diverguje.

Příklad 53.

Najděte Laurentovu řadu funkce f na daném „mezikruží“

a) $f(z) := \frac{\cos z}{z^2},\ 0 < |z| < 1;$

b) $f(z) := \frac{1}{z^2+1},\ |z| > 1;$

c) $f(z) := \frac{z^2+1}{z(z-i)},\ \frac{1}{2} < |z-i| < 1;$

d) $f(z) := \frac{1}{2z-5},\ |z| > \frac{5}{2};$

e) $f(z) := \frac{1}{z(z-2)},\ 1 < |z-2| < 2;$

f) $f(z) := \frac{z}{(z^2+1)^2},\ 0 < |z-i| < 2;$

g) $f(z) := \frac{z-\sin z}{z^4},\ 0 < |z| < \infty;$

h) $f(z) := \frac{z+2}{z^2-4z+3},\ 2 < |z-1| < \infty;$

i) $f(z) := \frac{1}{z(z-3)^2},\ 1 < |z-1| < 2.$

Řešení:

a) Pro každé $z \in \mathbb{C},\ 0 < |z| < 1$, platí

$$f(z) = \frac{\cos z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n-2}}{(2n)!}.$$

(Tyto vztahy platí dokonce pro každé $z \in \mathbb{C}\setminus\{0\}$.)

b) Pro každé $z \in \mathbb{C},\ |z| > 1$, platí

$$f(z) = \frac{1}{z^2+1} = \frac{1}{z^2}\frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}}.$$

c) Pro každé $z \in \mathbb{C},\ \frac{1}{2} < |z-i| < 1$, platí

$$\begin{aligned} f(z) &= \frac{z^2+1}{z(z-i)} = \frac{z+i}{z} = 1 + \frac{i}{z} = \\ &= 1 + \frac{i}{i+z-i} = 1 + \frac{1}{1+\frac{z-i}{i}} = \\ &= 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{i^n}(z-i)^n = \\ &= 1 + \sum_{n=0}^{\infty} i^n (z-i)^n. \end{aligned}$$

(Tyto vztahy platí dokonce pro všechna $z \in \mathbb{C}$ taková, že $0 < |z-i| < 1$.)

d) Pro každé $z \in \mathbb{C},\ |z| > \frac{5}{2}$, platí

$$\underline{f(z) =} \frac{1}{2z-5} = \frac{1}{2z} \cdot \frac{1}{1-\frac{5}{2z}} = \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n \frac{1}{z^n} =$$

$$= \sum_{n=0}^{\infty} \frac{5^n}{2^{n+1}} \frac{1}{z^{n+1}} =$$

$$= \underline{\sum_{n=1}^{\infty} \frac{5^{n-1}}{2^n} \frac{1}{z^n}}.$$

e) Pro každé $z \in \mathbb{C},\ 1 < |z-2| < 2$, platí

$$\underline{f(z) =} \frac{1}{z(z-2)} = \frac{1}{z-2} \cdot \frac{1}{2+z-2} =$$

$$= \frac{1}{z-2} \cdot \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-1} =$$

$$= \underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-1}}.$$

(Uvedené vztahy platí dokonce pro všechna $z \in \mathbb{C}$ taková, že $0 < |z-2| < 2$.)

f) Protože pro každé $z \in \mathbb{C},\ 0 < |z-i| < 2$, platí

$$f(z) = \frac{z}{(z^2+1)^2} = \frac{1}{(z-i)^2} \frac{z+i-i}{(z+i)^2} = \frac{1}{(z-i)^2} \left(\frac{1}{z+i} - \frac{i}{(z+i)^2}\right)$$

a navíc

$$\frac{1}{z+i} = \frac{1}{2i+z-i} = \frac{1}{2i} \cdot \frac{1}{1+\frac{z-i}{2i}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2i)^{n+1}} (z-i)^n,$$

$$-\left(\frac{1}{z+i}\right)^2 = \left(\frac{1}{z+i}\right)' = \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2i)^{n+1}}\, n\, (z-i)^{n-1} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2i)^{n+2}} (n+1)(z-i)^n,$$

<u>je pro každé $z \in \mathbb{C},\ 0 < |z-i| < 2$,</u>

$$\underline{f(z) =} \sum_{n=0}^{\infty} \left((-1)^n \frac{1}{(2i)^{n+1}} + \frac{(-1)^{n+1}}{(2i)^{n+2}}\, i\, (n+1)\right) (z-i)^{n-2} =$$

$$= \underline{\sum_{n=0}^{\infty} \left(\frac{-i^{n+1}}{2^{n+1}} + \frac{i^{n+1}}{2^{n+2}} (n+1)\right) (z-i)^{n-2}}.$$

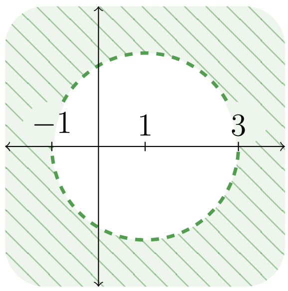
g) Pro každé $z \in \mathbb{C}$ platí, že

$$z - \sin z = z - \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)!},$$

a proto pro každé $z \in \mathbb{C},\ z \neq 0$, je

$$f(z) = \frac{z - \sin z}{z^4} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n-3}}{(2n+1)!}.$$

h) Pro každé $z \in \mathbb{C},\ 2 < |z-1| < \infty$, je



$$f(z) = \frac{z+2}{z^2 - 4z + 3} = \frac{3}{2} \cdot \frac{1}{1-z} + \frac{5}{2} \cdot \frac{1}{z-3}.$$

Navíc

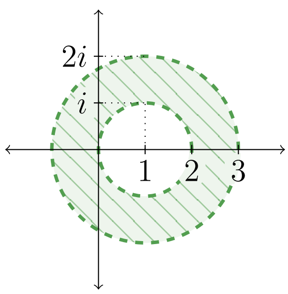
$$\frac{1}{z-3} = \frac{1}{-2+z-1} = \frac{1}{z-1} \cdot \frac{1}{1 - \frac{2}{z-1}} = \sum_{n=0}^{\infty} \frac{2^n}{(z-1)^{n+1}},$$

a proto pro každé $z \in \mathbb{C},\ 2 < |z-1|$, je

$$f(z) = -\frac{3}{2} \frac{1}{z-1} + \sum_{n=1}^{\infty} \frac{5 \cdot 2^{n-2}}{(z-1)^n} =$$

$$= \frac{1}{z-1} + \sum_{n=2}^{\infty} \frac{5 \cdot 2^{n-2}}{(z-1)^n}.$$

i) Protože pro každé $z \in \mathbb{C},\ 1 < |z-1| < 2$, je



$$f(z) = \frac{1}{z(z-3)^2} = \frac{1}{9} \cdot \frac{1}{z} + \frac{1}{9} \cdot \frac{1}{3-z} + \frac{1}{3} \cdot \frac{1}{(z-3)^2}$$

a navíc

$$\frac{1}{z} = \frac{1}{1+z-1} = \frac{1}{z-1}\cdot\frac{1}{1+\frac{1}{z-1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(z-1)^n},$$

$$\frac{1}{z-3} = \frac{1}{-2+z-1} = -\frac{1}{2}\cdot\frac{1}{1-\frac{z-1}{2}} = -\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}},$$

$$\left(\frac{1}{z-3}\right)^2 = -\left(\frac{1}{z-3}\right)' = \sum_{n=1}^{\infty}\frac{n(z-1)^{n-1}}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}(z-1)^n,$$

platí pro každé $z \in \mathbb{C}$, $1 < |z-1| < 2$, že

$$\underline{f(z)} = \sum_{n=1}^{\infty}\frac{1}{9}\frac{(-1)^{n-1}}{(z-1)^n} + \sum_{n=0}^{\infty}\left(\frac{1}{9}\cdot\frac{1}{2^{n+1}} + \frac{1}{3}\cdot\frac{n+1}{2^{n+2}}\right)(z-1)^n =$$

$$= \underline{\sum_{n=1}^{\infty}\frac{1}{9}\frac{(-1)^{n-1}}{(z-1)^n} + \sum_{n=0}^{\infty}\frac{3n+5}{9\cdot 2^{n+2}}(z-1)^n}.$$

Příklad 54.
Najděte Laurentův rozvoj funkce f na všech „maximálních mezikružích“ se středem z_0, na nichž je funkce f holomorfní, je-li

a) $f(z) := \frac{z^2-z+3}{z^3-3z+2}$, $z_0 = 0$;

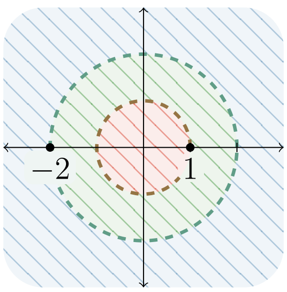
b) $f(z) := \frac{z+1}{z^2}$, $z_0 = 1+i$.

Řešení:

a)

$$f(z) = \frac{z^2-z+3}{z^3-3z+2} = \frac{z^2-z+3}{(z-1)^2(z+2)} = \frac{1}{z+2} + \frac{1}{(z-1)^2}$$

a protože f je zřejmě holomorfní na $\mathbb{C}\setminus\{-2,\ 1\}$, máme právě tři „maximální mezikruží“:



α) $P(0,0,1)$,

β) $P(0,1,2)$,

γ) $P(0,2,\infty)$.

$\alpha)$ Je-li $z \in \mathbb{C},\ |z| < 1$, je

$$\underline{f(z) =} \frac{1}{z+1} + \frac{1}{(z-1)^2} = \frac{1}{2} \cdot \frac{1}{1+\frac{z}{2}} + \left(\frac{-1}{z-1}\right)' =$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n + \left(\frac{1}{1-z}\right)' =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \left(\sum_{n=0}^{\infty} z^n\right)' =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \sum_{n=1}^{\infty} n(z^{n-1}) =$$

$$= \underline{\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2^{n+1}} + n + 1\right) z^n}.$$

$\beta)$ Pro každé $z \in \mathbb{C},\ 1 < |z| < 2$, platí

$$\underline{f(z) =} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \left(\frac{-1}{z} \frac{1}{1-\frac{1}{z}}\right)' =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n - \left(\sum_{n=0}^{\infty} z^{-n-1}\right)' =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \sum_{n=0}^{\infty} (n+1) \frac{1}{z^{n+2}} =$$

$$= \underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \sum_{n=2}^{\infty} \frac{n-1}{z^n}}.$$

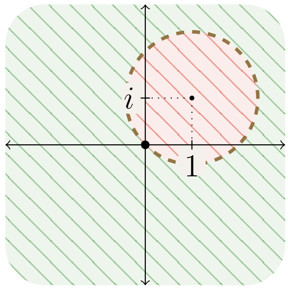
$\gamma)$ Pro každé $z \in \mathbb{C}$ takové, že $|z| > 2$, platí

$$\underline{f(z) =} \frac{1}{z} \frac{1}{1+\frac{2}{z}} + \sum_{n=2}^{\infty} \frac{n-1}{z^n} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{z^{n+1}} + \sum_{n=2}^{\infty} \frac{n-1}{z^n} =$$

$$= \underline{\frac{1}{z} + \sum_{n=2}^{\infty} \frac{(-2)^{n-1} + n - 1}{z^n}}.$$

b) Protože f je zřejmě holomorfní na $\mathbb{C}\setminus\{0\}$ a $|z_0-0|=\sqrt{2}$, máme právě dvě „maximální mezikruží“:



α) $P(1+i,0,\sqrt{2})$,

β) $P(1+i,\sqrt{2},\infty)$.

α) Pro $z\in\mathbb{C},\ |z-1-i|<\sqrt{2}$, platí

$$f(z)=\frac{z+1}{z^2}=\frac{1}{z}+\frac{1}{z^2}$$

a navíc

$$\frac{1}{z}=\frac{1}{1+i+z-1-i}=\frac{1}{1+i}\cdot\frac{1}{1+\frac{z-1-i}{1+i}}=$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^{n+1}}(z-1-i)^n,$$

$$\frac{1}{z^2}=-\left(\frac{1}{z}\right)'=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(1+i)^{n+1}}\,n\,(z-1-i)^{n-1},$$

a proto

$$f(z)=\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(1+i)^{n+1}}+\frac{(-1)^{n+2}}{(1+i)^{n+2}}(n+1)\right)(z-1-i)^n.$$

β) Pro každé $z\in\mathbb{C},\ |z-1-i|>\sqrt{2}$, platí

$$\frac{1}{z}=\frac{1}{1+i+z-1-i}=\frac{1}{z-1-i}\cdot\frac{1}{1+\frac{1+i}{z-1-i}}=\sum_{n=0}^{\infty}\frac{(-1)^n(1+i)^n}{(z-1-i)^{n+1}},$$

$$\frac{1}{z^2}=-\left(\frac{1}{z}\right)'=\sum_{n=0}^{\infty}\frac{(-1)^n(1+i)^n(n+1)}{(z-1-i)^{n+2}},$$

a proto

$$f(z)=\frac{1}{z-1-i}+\sum_{n=1}^{\infty}\frac{(-1)^n(1+i)^n+(-1)^{n-1}(1+i)^{n-1}\,n}{(z-1-i)^{n+1}}.$$

PŘÍKLAD 55.
Určete typ každé z izolovaných singularit funkce f, je-li

a) $f(z) := z^5 + 4z^3 - 2 + \frac{2}{z} + \frac{3}{z^2}$;

b) $f(z) := \frac{z^2-4}{z-2}$;

c) $f(z) := \frac{1}{z-z^3}$;

d) $f(z) := \frac{z^4}{z^4+1}$;

e) $f(z) := \frac{e^z}{z^2+4}$;

f) $f(z) := \frac{z^2+4}{e^z}$;

g) $f(z) := \frac{1-e^z}{2+e^z}$;

h) $f(z) := e^{\frac{1}{z^2}}$;

i) $f(z) := \frac{1}{(z-3)^2(2-\cos z)}$;

j) $f(z) := \frac{z}{\sin z}$;

k) $f(z) := z^2 \sin \frac{z}{z+1}$;

l) $f(z) := \frac{1-\cos z}{\sin^2 z}$.

Řešení:

a) Funkce $f(z) = z^5 + 4z^3 - 2 + \frac{2}{z} + \frac{3}{z^2}$ má dvě izolované singularity: 0 a ∞.
Zřejmě platí

- 0 je 2-násobný pól f,
- ∞ je 5-násobný pól f.

b) Funkce $f(z) = \frac{z^2-4}{z-2}$ má dvě izolované singularity: 2 a ∞.

- Protože
$$\lim_{z\to 2} \frac{z^2-4}{z-2} = \lim_{z\to 2} (z+2) = 4,$$
je 2 odstranitelná singularita f.
- $$\lim_{z\to\infty} \frac{f(z)}{z} = 1 \neq 0,$$
a proto ∞ je jednoduchý pól f.

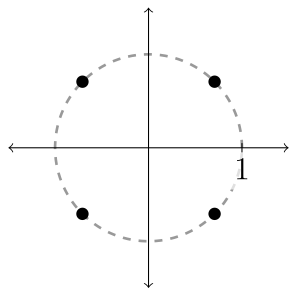
c) Funkce $f(z) = \frac{1}{z-z^3} = \dfrac{1}{z(1-z)(1+z)}$ má čtyři izolované singularity: $0,\ 1,\ -1$ a ∞.

- $0, 1$ a -1 jsou jednoduché póly f.
- Protože
$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1}{z^3\left(\frac{1}{z^2}-1\right)} = \frac{1}{-\infty} = 0,$$
je ∞ odstranitelná singularita f.

d) $f(z) = \frac{z^4}{z^4+1}$ a protože

$$z^4 + 1 = 0 \;\Leftrightarrow\; z \in \left\{ \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}} \right\},$$

má funkce f pět izolovaných singularit:



- $\frac{1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$, $\frac{-1-i}{\sqrt{2}}$ a $\frac{-1+i}{\sqrt{2}}$ jsou jednoduché póly f
- a, protože $\lim\limits_{z\to\infty} f(z) = 1$, je ∞ odstranitelná singularita f.

e) Funkce $f(z) = \frac{e^z}{z^2+4}$ má tři izolované singularity: $2i$, $-2i$ a ∞.

- $2i$ a $-2i$ jsou jednoduché póly f.
- Protože

$$\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{e^x}{x^2+4} \overset{\text{l'H.}}{=} \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{e^x}{2x} \overset{\text{l'H.}}{=} \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{e^x}{2} = \infty,$$

$$f(2n\pi i) = \frac{e^{2n\pi i}}{(2n\pi i)^2 + 4} = \frac{1}{-4n^2\pi^2 + 4} \to 0,$$

$\lim\limits_{z\to\infty} f(z)$ neexistuje, a proto ∞ je podstatná singularita f.

f) Funkce $f(z) = \frac{z^2+4}{e^z}$ má pouze jednu izolovanou singularitu, a to ∞.

- Protože

$$\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = 0,$$

$$\lim_{\substack{x\to-\infty \\ x\in\mathbb{R}}} f(x) = \infty \cdot \infty = \infty,$$

neexistuje $\lim\limits_{z\to\infty} f(z)$. Odtud plyne, že ∞ je podstatná singularita f.

g) Protože $f(z) = \frac{1-e^z}{2+e^z}$ a současně

$$2 + e^z = 0 \;\Leftrightarrow\; z = \mathrm{Ln}(-2) = \ln 2 + (2k+1)\pi i =: z_k,\;\; k \in \mathbb{Z},$$

má f izolované singularity právě v bodech z_k.

- Navíc

$$\begin{aligned} [(2+e^z)']_{z=z_k} = [e^z]_{z=z_k} &= -2 \neq 0, \\ [1-e^z]_{z=z_k} &= \;\;\, 3 \neq 0, \end{aligned}$$

a proto $z_k = \ln 2 + (2k+1)\pi i$, $k \in \mathbb{Z}$, jsou jednoduché póly f.

Pozor: ∞ není izolovaná singularita f.

h) $f(z) = e^{\frac{1}{z^2}}$ a pro všechna $z \in \mathbb{C} \setminus \{0\}$ platí, že

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^{2n}}.$$

Odtud plyne, že

- 0 je podstatná singularita f,
- ∞ je odstranitelná singularita f.

i) $f(z) = \frac{1}{(z-3)^2(2-\cos z)}$ a protože

$$\begin{aligned} 2 = \cos z = \frac{e^{iz}+e^{-iz}}{2} \;&\Leftrightarrow\; 4 = e^{iz} + e^{-iz} \;\Leftrightarrow \\ \Leftrightarrow\; e^{2iz} - 4e^{iz} + 1 = 0 \;&\Leftrightarrow\; e^{iz} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3} > 0 \;\Leftrightarrow \\ \Leftrightarrow\; iz = \mathrm{Ln}(2 \pm \sqrt{3}) &= \ln(2 \pm \sqrt{3}) + 2k\pi i,\;\; k \in \mathbb{Z} \;\Leftrightarrow \\ \Leftrightarrow\; z = z_k &:= 2k\pi - i\ln(2 \pm \sqrt{3}),\;\; k \in \mathbb{Z}, \end{aligned}$$

má funkce f izolované singularity v bodech 3 a z_k, $k \in \mathbb{Z}$.

- Snadno spočteme, že

$$[(2-\cos z)']_{z=z_k} = [\sin z]_{z=z_k} \neq 0,$$

a proto f má v bodech $z_k = 2k\pi - i\ln(2 \pm \sqrt{3})$, kde $k \in \mathbb{Z}$, jednoduché póly.

- Je zřejmé, že 3 je 2-násobný pól funkce f.

(∞ není izolovanou singularitou f.)

j) Funkce $f(z) = \frac{z}{\sin z}$ má zřejmě izolované singularity v kořenech funkce sinus. Rozmyslete si, že

- 0 je odstranitelná singularita f,
- $k\pi$, kde $k \in \mathbb{Z} \setminus \{0\}$, jsou jednoduché póly f.

(∞ není izolovanou singularitou f.)

k) Funkce $f(z) = z^2 \sin \frac{z}{z+1}$ má právě dvě izolované singularity: -1 a ∞.

- -1 je podstatná singularita f (neboť $\lim\limits_{z\to -1} f(z)$ neexistuje),
- ∞ je dvojnásobný pól f (neboť $\lim\limits_{z\to\infty} \frac{f(z)}{z^2} = \lim\limits_{z\to\infty} \sin\left(\frac{z}{z+1}\right) = \sin 1 \neq 0$).

l) Funkce $f(z) = \frac{1-\cos z}{\sin^2 z}$ má zřejmě izolované singularity právě v kořenech funkce sinus.

- Protože[5]
$$\lim_{z\to 2k\pi} \frac{1-\cos z}{\sin^2 z} \overset{\text{l'H.}}{=} \lim_{z\to 2k\pi} \frac{\sin z}{2\sin z\cos z} = \frac{1}{2},$$
platí, že body $2k\pi$, kde $k \in \mathbb{Z}$, jsou odstranitelné singularity f.
- Protože
$$\lim_{z\to(2k+1)\pi} \big(z-(2k+1)\pi\big)^2 \cdot \frac{1-\cos z}{\sin^2 z} \overset{\text{l'H.}}{=} 2 \lim_{z\to(2k+1)\pi} \frac{2\big(z-(2k+1)\pi\big)}{2\sin z\cos z} \overset{\text{l'H.}}{=}$$
$$\overset{\text{l'H.}}{=} -2 \lim_{z\to(2k+1)\pi} \frac{1}{\cos z} = 2 \neq 0,$$
jsou body $(2k+1)\pi$, kde $k \in \mathbb{Z}$, 2-násobné póly f.

(∞ není izolovanou singularitou f.)

Příklad 56.

Dokažte l'Hospitalovo pravidlo:

Nechť funkce f a g jsou holomorfní a nekonstantní na nějakém prstencovém okolí bodu $z_0 \in \mathbb{C}$ a nechť $\lim\limits_{z\to z_0} f(z) = \lim\limits_{z\to z_0} g(z) = 0$.
Potom platí
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}.$$

Řešení:

Z předpokladů plyne, že existují čísla $p, q \in \mathbb{N}$, okolí $U(z_0)$ bodu z_0 a funkce f_1 a g_1, které jsou holomorfní a `nenulové` na $U(z_0)$ a takové, že pro každé $z \in U(z_0) \setminus \{z_0\}$ platí

$$f(z) = (z-z_0)^p f_1(z),$$
$$g(z) = (z-z_0)^q g_1(z).$$

[5]Používáme l'Hospitalovo pravidlo dokázané v následujícím příkladu.

Odtud

$$\underline{\lim_{z\to z_0}\frac{f(z)}{g(z)}} = \lim_{z\to z_0}(z-z_0)^{p-q}\frac{f_1(z)}{g_1(z)} = \underline{\begin{cases}\infty, & p<q,\\ 0, & p>q,\\ \frac{f_1(z_0)}{g_1(z_0)}, & p=q,\end{cases}}$$

a současně

$$\underline{\lim_{z\to z_0}\frac{f'(z)}{g'(z)}} = \lim_{z\to z_0}\frac{p(z-z_0)^{p-1}f_1(z)+(z-z_0)^p f_1'(z)}{q(z-z_0)^{q-1}g_1(z)+(z-z_0)^q g_1'(z)} =$$

$$= \lim_{z\to z_0}(z-z_0)^{p-q}\frac{pf_1(z)+(z-z_0)f_1'(z)}{qg_1(z)+(z-z_0)g_1'(z)} =$$

$$\underline{= \begin{cases}\infty, & p<q,\\ 0, & p>q,\\ \frac{f_1(z_0)}{g_1(z_0)}, & p=q.\end{cases}}$$

Věta je dokázána.

<u>Příklad 57</u>.

Vypočtěte reziduum funkce f ve všech jejích izolovaných singularitách, je-li

a) $f(z) := \frac{1}{z+z^3}$;

b) $f(z) := \frac{z^2}{(1+z)^3}$;

c) $f(z) := \frac{1}{(z^2+1)^3}$;

d) $f(z) := \frac{z^3+1}{z-2}$;

e) $f(z) := \frac{1}{z^6(z^2+1)^2}$;

f) $f(z) := \operatorname{tg} z$;

g) $f(z) := \frac{1}{\sin z}$;

h) $f(z) := \operatorname{cotg}^3 z$;

i) $f(z) := \sin z \cdot \sin\frac{1}{z}$;

j) $f(z) := \frac{\sin(\pi z)}{(z-1)^3}$.

<u>Řešení:</u>

a) Funkce $f(z) = \dfrac{1}{z+z^3} = \dfrac{1}{z(z-i)(z+i)}$ má zřejmě 4 izolované singularity: $0,\ i,\ -i$ a ∞.
Nyní použijeme (stejně jako v řadě dalších příkladů) větu 9.3, část (iii) – viz [1]:

- $\underline{\operatorname{res} f(0)} = \left[\frac{1}{1+3z^2}\right]_{z=0} = \underline{1}$,
- $\underline{\operatorname{res} f(i)} = \left[\frac{1}{1+3z^2}\right]_{z=i} = \frac{1}{1-3} = \underline{-\frac{1}{2}}$,
- $\underline{\operatorname{res} f(-i)} = \left[\frac{1}{1+3z^2}\right]_{z=-i} = \frac{1}{1-3} = \underline{-\frac{1}{2}}$

a větu 9.3, část (v) – opět viz [1]:

- $\underline{\operatorname{res} f(\infty)} = -\left(1-\frac{1}{2}-\frac{1}{2}\right) = \underline{0}$.

b) $f(z) = \frac{z^2}{(1+z)^3}$ má zřejmě dvě izolované singularity.

- -1 je trojnásobný pól funkce f, a proto

$$\underline{\operatorname{res} f(-1) =} \frac{1}{2}\left[(z^2)''\right]_{z=-1} = \underline{1}.$$

- $\underline{\operatorname{res} f(\infty) = -1}$.

c) Funkce $f(z) := \frac{1}{(z^2+1)^3}$ má tři izolované singularity: trojnásobné póly v bodech i a $-i$ a odstranitelnou singularitu v ∞.

- $$\underline{\operatorname{res} f(\pm i) =} \frac{1}{2}\left[\left(\frac{1}{(z\pm i)^3}\right)''\right]_{z=\pm i} = \frac{1}{2}\left[3\cdot 4\,\frac{1}{(z\pm i)^5}\right]_{z=\pm i} = \\ = 6\,\frac{1}{(\pm 2i)^5} = \underline{\mp\frac{3}{16}}i,$$

- $\underline{\operatorname{res} f(\infty) = 0}$.

d) Funkce $f(z) = \frac{z^3+1}{z-2}$ má dvě izolované singularity: 2 (jednoduchý pól) a ∞.

- $$\underline{\operatorname{res} f(2) =} \left[\frac{z^3+1}{1}\right]_{z=2} = \underline{9},$$

- $\underline{\operatorname{res} f(\infty) = -9}$.

e) Protože $f(z) = \dfrac{1}{z^6(z^2+1)^2} = \dfrac{1}{z^6(z+i)^2(z-i)^2}$, jsou čísla $\pm i$ dvojnásobné póly f, 0 je 6-tinásobný pól f a ∞ je odstranitelná singularita f.

- $$\underline{\operatorname{res} f(\pm i) =} \left[\left(\frac{1}{z^6(z\pm i)^2}\right)'\right]_{z=\pm i} = \\ = -\left[\frac{6z^5(z\pm i)^2 + z^6\,2(z\pm i)}{z^{12}(z\pm i)^4}\right]_{z=\pm i} = \\ = \underline{\pm\frac{7}{4}}i.$$

- Protože $f(z) = 1 : (z^6 + 2z^8 + z^{10}) = \frac{1}{z^6} + \cdots,$[6] je $\underline{\operatorname{res} f(\infty) = 0}$,
- $\underline{\operatorname{res} f(0) =} -\frac{7}{4}i + \frac{7}{4}i - 0 = \underline{0}$.

f) Funkce $f(z) = \operatorname{tg} z = \frac{\sin z}{\cos z}$ má jednoduché póly v bodech $\frac{\pi}{2} + k\pi$, kde $k \in \mathbb{Z}$, a platí

$$\underline{\operatorname{res} f\left(\frac{\pi}{2} + k\pi\right) =} \left[\frac{\sin z}{-\sin z}\right]_{z=\frac{\pi}{2}+k\pi} = \underline{-1}.$$

(∞ není izolovanou singularitou f.)

[6] V příslušném Laurentově rozvoji funkce f je koeficient u $\frac{1}{z}$ roven 0.

g) $f(z)=\frac{1}{\sin z}$ má jednoduché póly v bodech $k\pi$, kde $k\in\mathbb{Z}$, v nichž platí

$$\underline{\operatorname{res} f(k\pi)} = \left[\frac{1}{\cos z}\right]_{z=k\pi} = \underline{(-1)^k}.$$

h) $f(z)=\operatorname{cotg}^3 z=\frac{\cos^3 z}{\sin^3 z}$ má trojnásobné póly v bodech $k\pi$, kde $k\in\mathbb{Z}$, navíc

$$\frac{\cos z}{\sin z}=\left(1-\frac{z^2}{2}+\frac{z^4}{24}-\cdots\right):\left(z-\frac{z^3}{6}+\frac{z^5}{120}-\cdots\right)=\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\cdots,$$

a proto

$$\left(\frac{\cos z}{\sin z}\right)^3=\left(\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\cdots\right)\left(\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\cdots\right)\left(\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\cdots\right).$$

$\operatorname{res} f(0)$ je „koeficient u $\frac{1}{z}$“, z čehož plyne

$$\operatorname{res} f(0)=3\left(-\frac{1}{3}\right)=-1.$$

Protože funkce f má periodu π, tzn. $f(z)=f(z-k\pi)$, je

$$\underline{\operatorname{res} f(k\pi)}=\operatorname{res} f(0)=\underline{-1 \text{ pro každé } k\in\mathbb{Z}}.$$

i) $f(z)=\sin z\cdot\sin\frac{1}{z}$ má izolované singularity v 0 a v ∞.
Protože pro každé $z\in\mathbb{C}\setminus\{0\}$ je

$$f(z)=\left(\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\right)\left(\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)!}\cdot\frac{1}{z^{2k+1}}\right)=\cdots,$$

platí[7]

$$\underline{\operatorname{res} f(0)=\operatorname{res} f(\infty)=0}.$$

j) Funkce $f(z)=\frac{\sin(\pi z)}{(z-1)^3}$ má dvě izolované singularity: 1 a ∞.

- Protože 1 je dvojnásobným pólem f, je

$$\underline{\operatorname{res} f(1)}=\lim_{z\to1}\left(\frac{\sin(\pi z)}{(z-1)^3}(z-1)^2\right)'=\lim_{z\to1}\left(\frac{\sin(\pi z)}{z-1}\right)'=$$

$$=\lim_{z\to1}\frac{\pi\cos(\pi z)(z-1)-\sin(\pi z)}{(z-1)^2}\overset{\text{l'H}}{=}$$

$$\overset{\text{l'H}}{=}\lim_{z\to1}\frac{-\pi^2\sin(\pi z)(z-1)+\pi\cos(\pi z)-\pi\cos(\pi z)}{2(z-1)}=$$

$$=\lim_{z\to1}\left(-\frac{\pi^2}{2}\sin(\pi z)\right)=\underline{0}.$$

[7] V příslušné Laurentově řadě funkce f jsou nenulové koeficienty pouze u „sudých mocnin“ z.

Jinak:

$$f(z) = \frac{\sin(\pi z)}{(z-1)^3} = -\frac{\sin(\pi(z-1))}{(z-1)^3} =$$
$$= -\frac{1}{(z-1)^3}\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{(2n+1)!}(z-1)^{2n+1} =$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\pi^{2n+1}}{(2n+1)!}(z-1)^{2n-2}.$$

V právě spočtené Laurentově řadě funkce f jsou nenulové koeficienty pouze u „sudých mocnin“ $(z-1)$, a proto $\underline{\operatorname{res} f(1) = 0}$.

- $\underline{\operatorname{res} f(\infty) = 0}$.

Příklad 58.
Vypočtěte pomocí reziduové věty daný integrál

a)
$$\int_\gamma \frac{\cos z}{z^3}\,\mathrm{d}z, \quad \text{kde} \quad \gamma(t) := 3e^{it},\ t \in \langle 0, 2\pi\rangle;$$

b)
$$\int_\gamma \frac{1}{z+2}\cos\frac{1}{z}\,\mathrm{d}z, \quad \text{kde} \quad \gamma(t) := 18e^{it},\ t \in \langle 0, 2\pi\rangle;$$

c)
$$\int_k \frac{z^3}{z^4-1}\,\mathrm{d}z, \quad \text{kde} \quad k = \{z \in \mathbb{C}\colon\ |z| = 2\};$$

d)
$$\int_k \frac{z^3}{z+1}e^{\frac{1}{z}}\,\mathrm{d}z, \quad \text{kde} \quad k = \{z \in \mathbb{C}\colon\ |z| = 2\};$$

e)
$$\int_\gamma z\sin\frac{z+1}{z-1}\,\mathrm{d}z, \quad \text{kde} \quad \gamma(t) := 2e^{-it},\ t \in \langle 0, 6\pi\rangle;$$

f)
$$\int_\gamma \frac{e^{\pi z}}{2z^2-i}\,\mathrm{d}z,$$

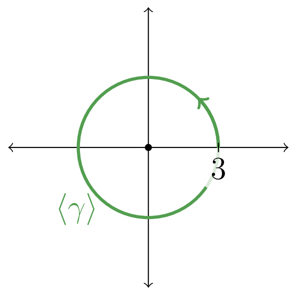
kde γ je taková jednoduchá uzavřená po částech hladká kladně orientovaná křivka, že

$$\operatorname{int}\gamma = \{z \in \mathbb{C}\colon\ |z| < 1\ \wedge\ 0 < \arg z < \frac{\pi}{2}\};$$

g)
$$\int_k \frac{\mathrm{d}z}{z^5(z^{10}-2)}, \quad \text{kde} \quad k = \{z \in \mathbb{C}\colon\ |z| = 2\}.$$

<u>Řešení:</u>

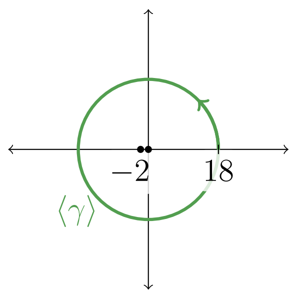
a)



Zřejmě $z=0$ je trojnásobným pólem funkce $\dfrac{\cos z}{z^3}$, a proto

$$\underline{\int_\gamma \frac{\cos z}{z^3}\,\mathrm{d}z} = 2\pi i \operatorname*{res}_{z=0} \frac{\cos z}{z^3} = \\ = 2\pi i \,\frac{1}{2}\,\left[(\cos z)''\right]_{z=0} = \\ = \pi i \left[(-\sin z)'\right]_{z=0} = \\ = \pi i \left[-\cos z\right]_{z=0} = \underline{-\pi i}.$$

b)



Zřejmě

$$\int_\gamma \underbrace{\frac{1}{z+2}\cos\frac{1}{z}}_{=:f(z)}\,\mathrm{d}z = 2\pi i\Big(\operatorname{res} f(-2) + \operatorname{res} f(0)\Big) = 2\pi i\big(-\operatorname{res} f(\infty)\big).$$

Pro každé $z \in \mathbb{C}$, $|z|>2$, platí, že

$$\frac{1}{z+2} = \frac{1}{z}\,\frac{1}{1+\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\,\frac{2^n}{z^n} = \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}},$$

a proto taky

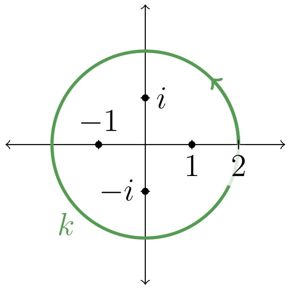
$$f(z) = \left(\sum_{n=1}^{\infty}\frac{(-2)^{n-1}}{z^n}\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\,\frac{1}{z^{2n}}\right).$$

Odtud plyne, že

$$\operatorname{res} f(\infty) = -(-2)^0\cdot(-1)^0\cdot\frac{1}{0!} = -1,$$

$$\underline{\int_\gamma \frac{1}{z+2}\cos\frac{1}{z}\,\mathrm{d}z} = -2\pi i \operatorname{res} f(\infty) = \underline{2\pi i}.$$

c)



$$\int_k \underbrace{\frac{z^3}{z^4-1}}_{=:f(z)}\,\mathrm{d}z = 2\pi i\Big(\operatorname{res} f(1) + \operatorname{res} f(-1) + \operatorname{res} f(i) + \operatorname{res} f(-i)\Big) = 2\pi i\big(-\operatorname{res} f(\infty)\big).$$

Pro každé $z \in \mathbb{C},\ |z| > 1$, platí, že

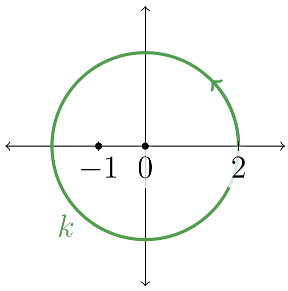
$$f(z) = \frac{z^3}{z^4-1} = z^3\frac{1}{z^4}\frac{1}{1-\frac{1}{z^4}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z^4}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{4n+1}},$$

a proto

$$\operatorname{res} f(\infty) = -1,$$

$$\underline{\int_k \frac{z^5}{z^4-1}\,\mathrm{d}z} = 2\pi i\big(-\operatorname{res} f(\infty)\big) = \underline{2\pi i}.$$

d)



Zřejmě

$$\int_k \underbrace{\frac{z^3}{z+1}e^{\frac{1}{z}}}_{=:f(z)}\,\mathrm{d}z = 2\pi i\Big(\operatorname{res} f(-1) + \operatorname{res} f(0)\Big) = 2\pi i\big(-\operatorname{res} f(\infty)\big).$$

Protože pro každé $z \in \mathbb{C},\ |z| > 1$, platí, že

$$\frac{z^3}{z+1} = z^2\frac{1}{1+\frac{1}{z}} = z^2\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n-2}},$$

$$f(z) = \left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n-2}}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n}\right),$$

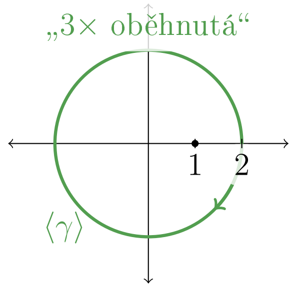
je

$$\operatorname{res} f(\infty) = -\left((-1)^0\cdot\frac{1}{3!} + (-1)^1\cdot\frac{1}{2!} + (-1)^2\cdot\frac{1}{1!} + (-1)^3\cdot\frac{1}{0!}\right) = \frac{1}{3}.$$

Odtud

$$\underline{\int_k \frac{z^3}{z+1} e^{\frac{1}{z}}\,\mathrm{d}z} = 2\pi i\big(-\operatorname{res} f(\infty)\big) = \underline{-\frac{2\pi i}{3}}.$$

e)



Zřejmě

$$\int_\gamma \underbrace{z \sin\frac{z+1}{z-1}}_{=:f(z)}\,\mathrm{d}z = -3\cdot 2\pi i \operatorname{res} f(1).$$

Protože

$$\sin\frac{z+1}{z-1} = \sin\left(\frac{z-1}{z-1} + \frac{2}{z-1}\right) = \sin\left(1 + \frac{2}{z-1}\right) =$$
$$= \sin 1\, \cos\frac{2}{z-1} + \cos 1\, \sin\frac{2}{z-1},$$

je pro každé $z \in \mathbb{C},\ z \neq 1$,

$$f(z) = (z-1+1)\left(\sin 1 \cos\frac{2}{z-1} + \cos 1 \sin\frac{2}{z-1}\right) =$$
$$= \sin 1\left[\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{(2n)!}\frac{1}{(z-1)^{2n-1}} + \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{(2n)!}\frac{1}{(z-1)^{2n}}\right] +$$
$$+ \cos 1\left[\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+1)!}\frac{1}{(z-1)^{2n}} + \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+1)!}\frac{1}{(z-1)^{2n+1}}\right].$$

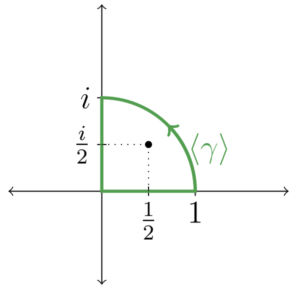
Odtud plyne, že

$$\operatorname{res} f(1) = \left((-1)\frac{2^2}{2!}\right)\sin 1 + \cos 1\left((-1)^0\,\frac{2^1}{1!}\right) = -2\sin 1 + 2\cos 1,$$

$$\underline{\int_\gamma z\sin\frac{z+1}{z-1}\,\mathrm{d}z} = -3\cdot 2\pi i \operatorname{res} f(1) = \underline{12\pi i\big(\sin 1 - \cos 1\big)}.$$

f) Protože

$$2z^2 - i = 0 \;\Leftrightarrow\; z^2 = \frac{i}{2} \;\Leftrightarrow\; z = \pm\frac{1+i}{2},$$

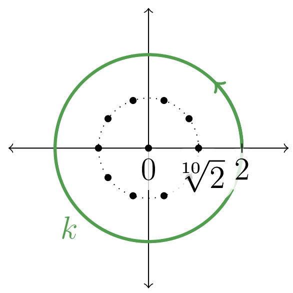


je

$$\underline{\int_\gamma \frac{e^{\pi z}}{2z^2 - i}\,\mathrm{d}z} = 2\pi i \operatorname*{res}_{z=\frac{1+i}{2}} \left(\frac{e^{\pi z}}{2z^2 - i}\right) =$$

$$= 2\pi i \frac{e^{\pi\left(\frac{1+i}{2}\right)}}{4\left(\frac{1+i}{2}\right)} = \underline{\frac{\pi}{2}(i-1)e^{\frac{\pi}{2}}}.$$

g) Protože funkce $f(z) := \frac{1}{z^5(z^{10}-2)}$ má zřejmě 12 izolovaných singularit, z nichž 11 (0 a kořeny rovnice $z^{10} = 2$) leží „uvnitř“ k a tím dvanáctým je ∞,



je

$$\int_k \frac{\mathrm{d}z}{z^5(z^{10}-2)} = 2\pi i\big(-\operatorname{res} f(\infty)\big).$$

Pro každé $z \in \mathbb{C}$, $\sqrt[10]{2} < |z|$, je

$$f(z) = \frac{1}{z^{15}}\,\frac{1}{1-\frac{2}{z^{10}}} = \frac{1}{z^{15}}\sum_{n=0}^{\infty}\frac{2^n}{z^{10n}} = \sum_{n=0}^{\infty}\frac{2^n}{z^{10n+15}},$$

a proto

$$\underline{\int_k \frac{\mathrm{d}z}{z^5(z^{10}-2)}} = 2\pi i\big(-\operatorname{res} f(\infty)\big) = \underline{0}.$$

Příklad 59.
Vypočtěte pomocí reziduové věty daný integrál[8]

a)
$$\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{5+3\cos x};$$

b)
$$\int_{-\infty}^{\infty} \frac{x^2\ \mathrm{d}x}{x^4+6x^2+25};$$

c)
$$\int_{0}^{\infty} \frac{x^4+1}{x^6+1}\ \mathrm{d}x;$$

d)
$$\int_{0}^{\infty} \frac{x^2}{(x^2+1)^3}\ \mathrm{d}x;$$

e)
$$\int_{-\pi}^{\pi} \frac{\cos x}{3+2\sin x}\ \mathrm{d}x;$$

f)
$$\int_{0}^{2\pi} \frac{\cos^2(2x)}{5-4\cos x}\ \mathrm{d}x;$$

g)
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^6};$$

h)
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+x+1}.$$

Řešení:

a) Definujme křivku $\gamma(t) := e^{it}$, $t \in \langle 0, 2\pi\rangle$. Pak pomocí substituce[9] $e^{ix} = z$, pro niž

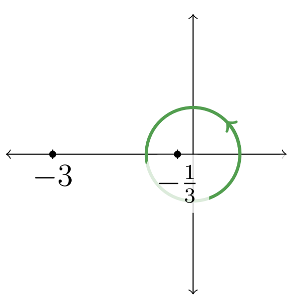
$$\cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{z+\frac{1}{z}}{2},$$
$$ie^{ix}\,\mathrm{d}x = \mathrm{d}z, \text{ tj. } \mathrm{d}x = \frac{1}{iz}\,\mathrm{d}z,$$

dostaneme

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{5+3\cos x} = \int_\gamma \frac{1}{\left(5+3\frac{z+\frac{1}{z}}{2}\right)}\,\frac{1}{iz}\,\mathrm{d}z = \int_\gamma \frac{2\,\mathrm{d}z}{i(10z+3z^2+3)}.$$

Protože

$$3z^2+10z+3=0 \;\Leftrightarrow\; z = \frac{-10\pm\sqrt{100-36}}{6} \;\Leftrightarrow\; z\in\left\{-\frac{1}{3}, -3\right\},$$



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[8] Uvedené integrály je třeba chápat jako „*reálné*“ integrály z funkce *reálné* proměnné.
[9] Prohlédněte si kapitolu 9.3, část a) – viz [1].

je

$$\underline{\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{5+3\cos x}} = \int_\gamma \frac{2\,\mathrm{d}z}{i(10z+3z^2+3)} =$$

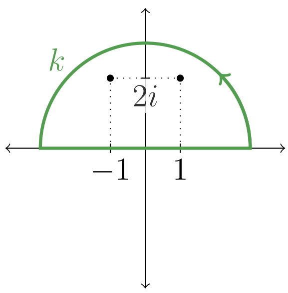
$$= 2\pi i \operatorname*{res}_{z=-\frac{1}{3}} \left(\frac{2}{i(3z^2+10z+3)} \right) =$$

$$= 4\pi \left[\frac{1}{6z+10} \right]_{z=-\frac{1}{3}} = \frac{4\pi}{8} = \underline{\frac{\pi}{2}}.$$

b) Nejdříve si všimněme, že

$$z^4 + 6z^2 + 25 = 0 \ \Leftrightarrow \ z^2 = -3 \pm 4i \ \Leftrightarrow \ z \in \{1+2i,\ -1-2i,\ -1+2i,\ 1-2i\}\,.$$

Buď nyní $k \subset \mathbb{C}$ hranicí množiny $\{z \in \mathbb{C}\colon\ |z| < 3 \ \wedge \ \operatorname{Im} z > 0\}$.



Pak platí[10]

$$\underline{\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{x^4+6x^2+25}} = \int_k \frac{z^2\,\mathrm{d}z}{z^4+6z^2+25} =$$

$$= 2\pi i \left(\operatorname*{res}_{z=1+2i} \frac{z^2}{z^4+6z^2+25} + \operatorname*{res}_{z=-1+2i} \frac{z^2}{z^4+6z^2+25} \right) =$$

$$= 2\pi i \left(\left[\frac{z^2}{4z^3+12z} \right]_{z=1+2i} + \left[\frac{z^2}{4z^3+12z} \right]_{z=-1+2i} \right) =$$

$$= 2\pi i \left(\left[\frac{z}{4z^2+12} \right]_{z=1+2i} + \left[\frac{z}{4z^2+12} \right]_{z=-1+2i} \right) =$$

$$= 2\pi i \left(\frac{1+2i}{4(-3+4i)+12} + \frac{-1+2i}{4(-3-4i)+12} \right) =$$

$$= 2\pi i \left(\frac{1+2i}{16i} - \frac{-1+2i}{16i} \right) = \frac{\pi}{8} \Big(1+2i+1-2i \Big) = \underline{\frac{\pi}{4}}.$$

[10] Viz kapitolu 9.3, část b) v [1].

c) Protože problém

$$z^6 + 1 = 0 \ \wedge \ \operatorname{Im} z \geq 0$$

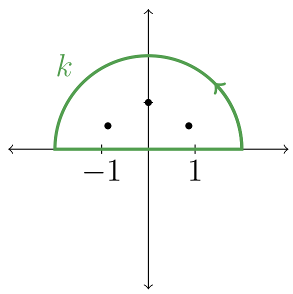
má právě tři řešení:

$$\begin{aligned} z_1 &:= e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \\ z_2 &:= e^{i\frac{\pi}{2}} = i, \\ z_3 &:= e^{\frac{5}{6}\pi i} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \end{aligned}$$

a funkce $\frac{x^4+1}{x^6+1}$ je sudá, platí

$$\begin{aligned} \int_0^\infty \frac{x^4+1}{x^6+1}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{x^4+1}{x^6+1}\,\mathrm{d}x &= \\ = \frac{1}{2}\int_k \frac{z^4+1}{z^6+1}\,\mathrm{d}z &= \\ = \frac{1}{2}\,2\pi i \sum_{j=1}^{3} \operatorname*{res}_{z=z_j} \frac{z^4+1}{z^6+1}&, \end{aligned}$$

kde $k \subset \mathbb{C}$ je hranicí množiny $\{z \in \mathbb{C}\colon\ |z| < 2 \ \wedge \ \operatorname{Im} z > 0\}$.



Odtud, protože

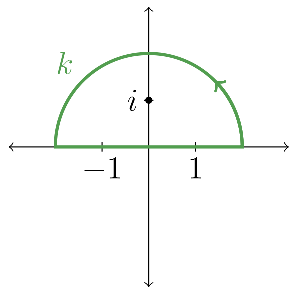
$$\begin{aligned} \operatorname*{res}_{z=z_1} \frac{z^4+1}{z^6+1} &= \left[\frac{z^4+1}{6z^5}\right]_{z=\frac{\sqrt{3}}{2}+\frac{1}{2}i} &&= \frac{1}{6}(-i), \\ \operatorname*{res}_{z=z_2} \frac{z^4+1}{z^6+1} &= \left[\frac{z^4+1}{6z^5}\right]_{z=i} &&= \frac{2}{6}(-i), \\ \operatorname*{res}_{z=z_3} \frac{z^4+1}{z^6+1} &= \left[\frac{z^4+1}{6z^5}\right]_{z=\frac{-\sqrt{3}}{2}+\frac{1}{2}i} &&= \frac{1}{6}(-i), \end{aligned}$$

plyne

$$\underline{\int_0^\infty \frac{x^4+1}{x^6+1}\,\mathrm{d}x = \frac{2}{3}\pi.}$$

d) Funkce $\frac{x^2}{(x^2+1)^3}$ je sudá, a proto pro $k \subset \mathbb{C}$, které je hranicí množiny

$$\{z \in \mathbb{C}\colon\ |z| < 2 \ \wedge\ \operatorname{Im} z > 0\}\,,$$



platí

$$\underline{\int_0^{\infty} \frac{x^2\,\mathrm{d}x}{(x^2+1)^3}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{(x^2+1)^3} = \frac{1}{2}\int_k \frac{z^2\,\mathrm{d}z}{(z^2+1)^3} =$$

$$= \frac{1}{2} 2\pi i \operatorname*{res}_{z=i}\left(\frac{z^2}{(z^2+1)^3}\right) = \pi i \frac{1}{2}\left[\left(\frac{z^2}{(z+i)^3}\right)''\right]_{z=i} =$$

$$= \frac{\pi i}{2}\left[\left(\frac{2z(z+i)^3 - z^2\,3(z+i)^2}{(z+i)^6}\right)'\right]_{z=i} =$$

$$= \frac{\pi i}{2}\left[\left(\frac{-z^2+2zi}{(z+i)^4}\right)'\right]_{z=i} =$$

$$= \frac{\pi i}{2}\left[\frac{(-2z+2i)(z+i)^4 - (-z^2+2zi)4(z+i)^3}{(z+i)^8}\right]_{z=i} =$$

$$= \frac{\pi i}{2}\left(\frac{4(2i)^3}{(2i)^8}\right) = \frac{2\pi i}{2^5 i^5} = \underline{\frac{\pi}{16}}.$$

e) Buď $\gamma(t) := e^{it}$, kde $t \in \langle 0, 2\pi\rangle$. Pak

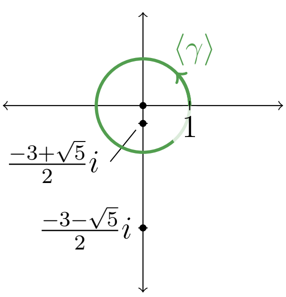
$$\int_{-\pi}^{\pi} \frac{\cos x}{3 + 2\sin x}\,\mathrm{d}x = \int_\gamma \frac{z+\frac{1}{z}}{2}\,\frac{1}{3 + 2\,\frac{z-\frac{1}{z}}{2i}}\,\frac{1}{iz}\,\mathrm{d}z =$$

$$= \frac{1}{2}\int_\gamma \frac{z^2+1}{z}\,\frac{1}{z^2+3iz-1}\,\mathrm{d}z$$

(volili jsme substituci $e^{ix} = z$ – viz kapitolu 9.3, část a) v [1]).

Protože

$$z^2 + 3iz - 1 = 0 \ \Leftrightarrow\ z = \frac{-3 \pm \sqrt{5}}{2} i,$$



je

$$\underline{\int_{-\pi}^{\pi} \frac{\cos x}{3+2\sin x}\,\mathrm{d}x} = \frac{1}{2}\int_\gamma \underbrace{\frac{z^2+1}{z}\,\frac{1}{z^2+3iz-1}}_{=:f(z)}\,\mathrm{d}z =$$

$$= \frac{1}{2}2\pi i\left(\operatorname{res} f(0) + \operatorname{res} f\Big(\frac{-3+\sqrt{5}}{2}i\Big)\right) =$$

$$= \pi i\left(\left[\frac{z^2+1}{z^2+3iz-1}\right]_{z=0} + \left[\frac{z^2+1}{z(2z+3i)}\right]_{z=\frac{-3+\sqrt{5}}{2}i}\right) =$$

$$= \pi i\left(-1 + \left[\frac{2-3iz}{2-6iz+3iz}\right]_{z=\frac{-3+\sqrt{5}}{2}i}\right) = \pi i(-1+1) = \underline{0}.$$

f) Provedeme substituci $e^{ix} = z$, pro niž

$$\cos x = \frac{z+\frac{1}{z}}{2},$$
$$\cos 2x = \frac{z^2+\frac{1}{z^2}}{2},$$
$$\mathrm{d}x = \frac{1}{iz}\,\mathrm{d}z.$$

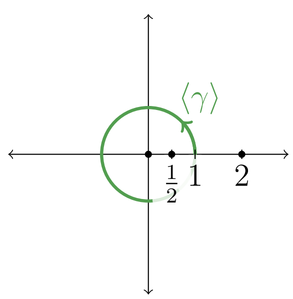
Pro $\gamma(t) := e^{it}$, kde $t \in \langle 0, 2\pi\rangle$, pak platí, že

$$\int_0^{2\pi} \frac{\cos^2 2x}{5-4\cos x}\,\mathrm{d}x = \int_\gamma \frac{1}{4}\left(\frac{z^4+1}{z^2}\right)^2 \frac{1}{5-2\frac{z^2+1}{z}}\,\frac{1}{iz}\,\mathrm{d}z =$$

$$= \int_\gamma \frac{1}{4i}\,\frac{(z^4+1)^2}{z^4}\,\frac{1}{5z-2z^2-2}\,\mathrm{d}z =$$

$$= \int_\gamma \frac{1}{4i}\underbrace{\frac{(z^4+1)^2}{z^4}\,\frac{1}{-2(z-2)\left(z-\frac{1}{2}\right)}}_{=:f(z)}\,\mathrm{d}z =$$

$$= \frac{2\pi i}{4i}\left(\operatorname{res} f(0) + \operatorname{res} f\left(\frac{1}{2}\right)\right).$$



A protože

$$\operatorname{res} f(0) = \frac{1}{3!}\left[\left(\frac{(z^4+1)^2}{5z-2z^2-2}\right)'''\right]_{z=0} = \frac{1}{6}\left(-\frac{255}{8}\right) = -\frac{255}{48},$$

$$\operatorname{res} f\left(\frac{1}{2}\right) = \left[\frac{(z^4+1)^2}{z^4}\,\frac{1}{5-4z}\right]_{z=\frac{1}{2}} = \frac{289}{48},$$

je

$$\underline{\int_0^{2\pi} \frac{\cos^2 2x}{5-4\cos x}\,\mathrm{d}x = \frac{17}{48}\pi.}$$

g) Protože rovnice $z^6+1=0$ má za podmínky $\operatorname{Im} z \geq 0$ právě tři řešení:

$$z_1 = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i,$$
$$z_2 = e^{i\frac{\pi}{2}} = i,$$
$$z_3 = e^{\frac{5}{6}\pi i} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i,$$

platí pro funkci $f(z) := \frac{1}{1+z^6}$, že

$$\underline{\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^6}} = 2\pi i\Big(\operatorname{res} f(z_1) + \operatorname{res} f(z_2) + \operatorname{res} f(z_3)\Big) =$$

$$= 2\pi i \sum_{k=1}^{3} \frac{1}{6z_k^5} = 2\pi i \sum_{k=1}^{3} \frac{z_k}{6z_k^6} =$$

$$= -\frac{2\pi i}{6}\big(z_1+z_2+z_3\big) =$$

$$= -\frac{\pi}{3}\,i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i + i - \frac{\sqrt{3}}{2} + \frac{1}{2}i\right) =$$

$$= -\frac{\pi}{3}\,i\,2i = \underline{\frac{2\pi}{3}}.$$

h) Protože

$$z^2+z+1=0 \;\Leftrightarrow\; z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i,$$

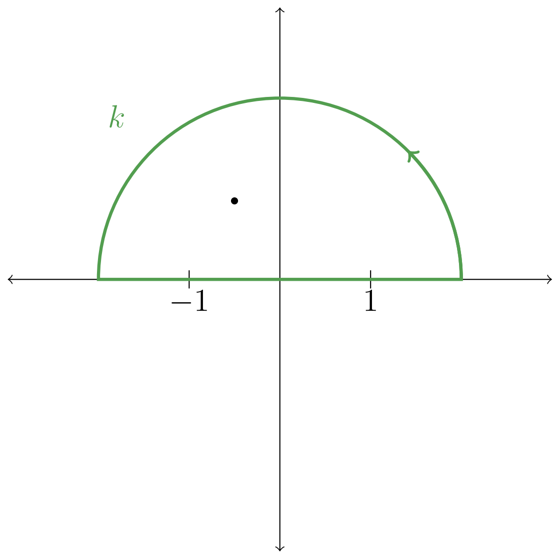
platí pro $k\subset\mathbb{C}$ definované jako hranice množiny $\{z\in\mathbb{C}\colon\ |z|<2\ \wedge \operatorname{Im} z>0\}$, že

$$\underline{\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+x+1}}=\int_k\frac{\mathrm{d}z}{z^2+z+1}=$$

$$=2\pi i\operatorname*{res}_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i}\left(\frac{1}{z^2+z+1}\right)=$$

$$=2\pi i\left[\frac{1}{2z+1}\right]_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i}=$$

$$=2\pi i\,\frac{1}{-1+\sqrt{3}i+1}=\underline{\frac{2\pi}{\sqrt{3}}}.$$



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