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SETTING THE PASSBAND WIDTH IN THE VOLD-KALMAN ORDER TRACKING FILTER

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Abstract

Even though the basic principle of the Vold-Kalman (VK) order tracking filter was published many times, the issue dealing with the problem of the filter passband setting was omitted as yet. It is known that the filtration effect of the VK-filter is achieved by solving of the linear system equations. The advantage of this approach is that the system matrix is symmetric and the number of the non-zero diagonals is equal to the doubled number of the filter poles plus one. The system matrix is sparse because the pole number is not greater than three or four. The equation system solution can be based on the Cholesky factorization of the system matrix resulting in the solution of the forward and backward substitutions. Each substitution can be considered as a filtering process. This is a main idea how to evaluate the dependence of the filter bandwidth on a parameter determining the VK-filter property.

INTRODUCTION

Some particular class of signals consists of harmonic components that are all (or the most dominant of them) related in frequency to the fundamental frequency, e.g. engine rotational speed. These components are designated as super- or sub-harmonics (the so-called orders) of the fundamental frequency in RPM, which is measured. Test engineers focus their interest on the amplitude and phase of these orders as a function of time or more frequently as a function of RPM. This analysis technique is called as the order tracking filtration. The paper deals with the Vold-Kalman (VK) order-tracking filter of two generations [1]. After describing the theoretical principle, main attention is focused on the control of the absolute and relative VK-filter bandwidth.

Without loss of the generality, the analysis is dealing with the tracking of just one order. Under condition that the orders are not close or crossing, the multiple orders can be tracked individually in a step-by-step way.

THE FIRST GENERATION OF THE VK-FILTER

Similarly as for the Kalman filter, which is based on the process and measurement equations, the VK-filter is based on the structural and data equations that play the similar role in the filtration effect. Both these equations are excited by the unknown functions on its right side. It is assumed that for the Kalman filter these functions are stochastic with known covariance while for the VK-filter a user sets only the relationship between them. The structural equation for the first generation of the VK-filter takes the form

$$x(n) - 2\cos(\omega_C\,\Delta t)\,x(n+1) + x(n+2) = \varepsilon(n), \tag{1}$$

where the coefficient multiplying the delayed sample $x(n+1)$ can be designated by $c(n) = 2\cos(\omega_C\,\Delta t)$. If the right side of the structural equation is equal to zero then the solution of this second order difference equation is a harmonic function $x(n) = A\sin(\omega_C\, n\Delta t + \varphi)$ with discrete time $t_n = n\,\Delta t$, $n = 0, 1, 2, \ldots$, where $\Delta t = 1/f_S$ is a sampling time increment and ω_C is an angular frequency. As the sine wave can slightly change its amplitude and frequency over the time samples involved in the equation (1), the unknown non-homogeneity term, $\varepsilon(n)$, is incorporated on the right side of the structural equation.

The system of the structural equations (1) containing all the samples $x(1), \ldots, x(N)$ takes the following form, which can be rewritten in the matrix form

$$\begin{bmatrix} 1 & -c(1) & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & 1 & -c(N-2) & 1 \end{bmatrix} \begin{bmatrix} x(1) \\ \ldots \\ x(N) \end{bmatrix} = \begin{bmatrix} \varepsilon(3) \\ \ldots \\ \varepsilon(N) \end{bmatrix} \tag{2}$$

$$\mathbf{A\,x} = \boldsymbol{\varepsilon} \tag{3}$$

The value of the order instantaneous angular frequency, $\omega_C = 2\pi\, f_C$, cannot be usually measured at each recorded sample. Tacho pulses, generated once in a rotation of the shaft, give reduced information on the instantaneous fundamental frequency. Thus the VK order tracking filtration needs an estimation of the instantaneous frequency, which can be fitted by cubic splines.

Instead of observing the sampled sinusoidal signal $x(n)$, samples $y(n)$ are recorded. The signal $y(n)$ is combined from both the signals satisfying the structural equation (1) as well as random noise and other sinusoidal components differing in the frequency with the sinusoidal signal $x(n)$. The random noise and other sinusoidal are combined into the signal $\eta(n)$. Formally, it can be written as N equations or in the matrix form

$$y(n) = x(n) + \eta(n), \quad n = 1, 2, .., N , \tag{4}$$

$$\mathbf{y} - \mathbf{x} = \boldsymbol{\eta} . \tag{5}$$

THE SECOND GENERATION OF THE VK-FILTER

The algorithm of the first generation gives the time signal $x(t) = A(n)\sin(\omega t + \varphi)$ as a component of the recorded signal $y(t)$ while the second-generation algorithm results in the direct evaluation of the harmonic component envelope. Assuming that the filter is focused at only one harmonic component the data equation can be written as

$$y(n) = x(n)\exp(j\Theta(n)) + \eta(n), \tag{6}$$

where the signal phase is a result of the angular frequency integration

$$\Theta(n) = \sum_{i=0}^{n} \omega(i)\Delta t . \tag{7}$$

The complex envelope $x(n)$ is the low frequency modulation of the complex carrier wave $\exp(j\Theta(n))$. Low frequency modulation causes envelope smoothness. In other words envelope is locally approximated by a low order polynomial and it is filtered in some specific way. This condition can be expressed by a structural equation with the non-homogeneity term $\varepsilon(n)$. The polynomial order designates the number of the filter poles. The equations for 1-, 2- and 3-pole filter are given by

$$\nabla x(n) = x(n) - x(n+1) = \varepsilon(n). \tag{8}$$

$$\nabla^2 x(n) = x(n) - 2x(n+1) + x(n+2) = \varepsilon(n). \tag{9}$$

$$\nabla^3 x(n) = x(n) - 3x(n+1) + 3x(n+2) - x(n+3) = \varepsilon(n). \tag{10}$$

Note that the difference operator of a given order annihilates all polynomial of one order less.

The system of equations (8, 9, 10) for all the samples $x(1),..., x(N)$ takes the same form of the structural equations $\mathbf{A\,x} = \boldsymbol{\varepsilon}$ as the matrix and vector equation (3). The matrix **A** for the 2-pole filter is of the same structure as the matrix **A** for the first generation VK-filter with substitution $c(n) = 2$.

The system of data equations for the second generation of the VK-filter takes the following form

$$\mathbf{y} - \mathbf{C\,x} = \boldsymbol{\eta}, \tag{11}$$

where $\eta(n)$ is of the same meaning as in the equation (4) and C is a diagonal matrix ($\mathbf{C} = \mathbf{C}^T$), which diagonal elements are the signal phase samples

$$\mathbf{C} = diag\{\exp(j\Theta(1)), \exp(j\Theta(2)), ..., \exp(j\Theta(N))\}. \tag{12}$$

GLOBAL SOLUTION OF BOTH THE GENERATION VK-FILTERS

The system of the data equations (5) or (11) and the structural equations (3) is an underdetermined system for the unknown waveform $x(n)$. The additional condition for the equation solution is that the variances of the non-homogeneity terms, $\varepsilon(n)$ and the other sinusoidal components and background random noise $\eta(n)$, have to be minimal while maintaining the given relationship between them. The global solution can be found using the standard least square technique.

The sum of the squares of all the unknown non-homogeneity terms for the first and second-generation algorithm can be expressed as a scalar product

$$\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = \mathbf{x}\mathbf{A}^T \mathbf{A}\,\mathbf{x}\,, \tag{13}$$

where a row vector $\boldsymbol{\varepsilon}^T$ is a transpose of the column vector $\boldsymbol{\varepsilon}$. The sum of the squares of the signal $\eta(n)$ in both the VK-filter generations can be written in the form

First generation VK-filter: $\boldsymbol{\eta}^T \boldsymbol{\eta} = (\mathbf{y}^T - \mathbf{x}^T)(\mathbf{y} - \mathbf{x})$,

Second generation VK-filter: $\boldsymbol{\eta}^H \boldsymbol{\eta} = (\mathbf{y}^T - \mathbf{x}^H \mathbf{C}^H)(\mathbf{y} - \mathbf{C}\mathbf{x})$, (14)

where the upper index *H* designates the complex conjugate quantities. As the matrix **C** is complex both the vectors $\mathbf{x}, \boldsymbol{\eta}$ are complex as well.

The weighted sum of the particular sums (13) and (14) gives the loss function

$$J = r^2 \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} + \boldsymbol{\eta}^T \boldsymbol{\eta}. \tag{15}$$

where *r* is a weighting factor [3, 4]. The choice of a large value for the weighting factor *r* leads to the highly selective filtration in the frequency domain that takes a long time to converge in amplitude. In contrast, fast convergence with low frequency resolution is achieved by choosing *r* small.

The first derivative of the loss function (15) with respect to the vector **x** gives a condition for the minimum of this function, which is called a normal equation.

First generation VK-filter: $\frac{\partial J}{\partial \mathbf{x}} = 2 r^2 \mathbf{A}^T \mathbf{A}\,\mathbf{x} + 2(\mathbf{x} - \mathbf{y}) = \mathbf{0}$

Second generation VK-filter: $\frac{\partial J}{\partial x} = 2 r^2 \mathbf{A}^T \mathbf{A}\,\mathbf{x} + 2(\mathbf{x} - \mathbf{C}^H \mathbf{y}) = \mathbf{0}$ (16)

The matrix equations (16) are of the same form but for an exception. The vector **y** is multiplied by $\mathbf{C}^H$, which shifts the frequency of the tracked components toward to zero. The pass band filter becomes the low pass filter.

The unknown waveform in the case of the first generation VK-filter and the unknown envelope in the case of the second-generation VK-filter result from the equations (16)

First generation VK-filter $\mathbf{x} = \left(r^2 \mathbf{A}^T \mathbf{A} + \mathbf{E}\right)^{-1} \mathbf{y}$, Second generation VK-filter $\mathbf{x} = \left(r^2 \mathbf{A}^T \mathbf{A} + \mathbf{E}\right)^{-1} \mathbf{C}^H \mathbf{y}$, (17)

The product of the matrixes, $\mathbf{A}^T$ and $\mathbf{A}$, gives a symmetric positive semi definite matrix. The matrix $r^2 \mathbf{A}^T \mathbf{A}$ becomes the symmetric positive definite matrix $\mathbf{B} = r^2 \mathbf{A}^T \mathbf{A} + \mathbf{E}$ by adding the unity matrix **E.** The matrix **B** consists of the limit number of the non-zero diagonals. Therefore, it is easy to invert it. The number of the non-zero diagonals of the matrix **B** for the VK-filter of the second-generation is equal to $2p+1$, where p is the number of the filter poles. This number of the non-zero diagonals of the matrix **B** for the VK-filter of the first-generation can be designated as well by $2p+1$, where $p = 2$.

Employing the Cholesky factorization of the matrix **B** into the matrix product $\mathbf{B} = \mathbf{L}\,\mathbf{U}$, where **L** is a lower-triangular matrix and $\mathbf{U} = \mathbf{L}^T$ is an upper-triangular matrix, is the easiest way how to solve the equation system (16). The only condition for the Cholesky factorization is that all the main minor determinants are equal to a positive value what can be easily proved. The main advantage of the Cholesky factorization algorithm is that it saves the number of the non-zero diagonals in the triangular matrices at the value $p+1$. The solution of the system (16) is broken down into two linear equation systems, the forward reduction and backward substitution.

Forward reduction (first system)	Backward substitution (second system)	
$z_1 = y_1 / u_{1,1}$	$x_N = z_1 / u_{N,N}$	(18)
$z_2 = \left(y_2 - u_{1,2} z_1\right) / u_{2,2}$	$x_{N-2} = \left(z_{N-2} - u_{N-1,N} x_N\right) / u_{N-1,N-1}$	(19)
….	….	
$j = p+1,\ldots,N$	$j = N - (p+1),\ldots,1$	(20)
$z_j = \left(y_j - u_{j-1,j}\, z_{j-1} \ldots - u_{j-p,j}\, z_{j-p}\right) / u_{j,j}$	$x_j = \left(z_j - u_{j,j+1} x_{j+1} \ldots - u_{j,j+p} x_{j+p}\right) / u_{j,j}$	(21)

In the forward reduction, the linear equation system $\mathbf{L}\,\mathbf{z} = \mathbf{y}$ ($\mathbf{U}^T \mathbf{z} = \mathbf{y}$) for an unknown vector **z** is solved while in the backward substitution the unknown vector **x** of the equation system $\mathbf{U}\,\mathbf{x} = \mathbf{z}$ is evaluated.

The value of weighting factor r has to be limited not to lose the effect of adding unity to main matrix diagonal on the positive definiteness by rounding the diagonal elements due to the limit bit number for saving quantities in a computer memory. Taking into consideration the maximal value of the diagonal components of the matrix $r^2 \mathbf{A}^T \mathbf{A}$ and 14 decimal places for double-precision computer-arithmetic, the limit value r_{MAX} for the weighting factor is shown in the next table. This table gives also the lower limit for the relative VK-filter bandwidth, which is introduced in the next chapter.

Pole number	$p=1$	$p=2$	$p=3$	$p=4$
$\left(r^2\mathbf{A}^T\mathbf{A}\right)_{i,i}$	$2r^2$	$6r^2$	$20r^2$	$70r^2$
$r_{MAX} \approx$	7x10^6	4x10^6	2x10^6	1.1x10^6
$100\Delta f >$	5x10^{-6} %	0.025 %	0.5 %	2 %

BANDWIDTH OF THE VK FILTER

Except for the first and last *p* rows of the matrix **B**, all the other diagonal elements are identical. Of course, the VK-filter of the first generation requires the constant angular frequency, $c(n)=c$. Except for the first and last *p* rows the same property has the matrix **U**. Therefore, it is possible to simplify the index of these matrix elements

$$u_0 = \lim_{j\to\infty} u_{j,j}, \quad u_1 = \lim_{j\to\infty} u_{j,j+1}, \quad \dots \quad u_p = \lim_{j\to\infty} u_{j,j+p}, \tag{22}$$

The equation (21) in the forward reduction is equivalent to the linear digital filter of the order *p*. The transfer function in Z-transform is as follows

$$H(z) = \frac{Z(z)}{Y(z)} = \frac{1}{u_0 + u_1 z^{-1} + \dots + u_p z^{-p}} \tag{23}$$

Taking into account the reverse order of the samples $x(N),\dots,x(1)$ in the backward substitution, the filtration process is based on the same transfer function as for the forward reduction. Altogether the forward reduction and backward substitution results in zero-phase digital filtering analogous to the *filtfilt* function in Matlab. The VK-filter transfer function is equal to the square of the transfer function (23). The roll-off of the second generation filter is equal to $-40p$ dB per decade. Substituting the complex quantity *z* by the term $\exp(j\Omega)$, where $\Omega = \omega\Delta t$ for $\Omega \in (-\pi, +\pi)$, the equation for the -3 dB cutoff frequencies Ω_L (low pass) and Ω_H (high pass) is obtained. Thus

$$\left|H\left(e^{j\Omega}\right)\right|^2 = \left|\frac{1}{u_0 + u_1 e^{-j\Omega} + u_2 e^{-j2\Omega} + \dots + u_p e^{-jp\Omega}}\right|^2 = \frac{1}{\sqrt{2}} \tag{24}$$

The relative bandwidth Δf with respect to the Nyquist frequency is given by the formulas

First generation VK-filter

$$\Delta f = \frac{f_H - f_L}{f_S/2},$$

Second generation VK-filter $\left(f_L = 0\right)$

$$\Delta f = \frac{f_H}{f_S/2}, \tag{25}$$

The difference in the formulas results from the fact that the VK-filter of the first generation is a passband filter while the second generation is a lowpass filter.

For the first generation VK-filter, the bandwidth as the exact function of the weighting factor results from the equation (24), in which $p = 2$, Thus

$$\Delta f = \frac{1}{\pi}\left(\arccos\left(\cos(\omega\Delta t) - \sqrt{\sqrt{2}-1}\Big/2r\right) - \arccos\left(\cos(\omega\Delta t) + \sqrt{\sqrt{2}-1}\Big/2r\right)\right). \quad (26)$$

The approximation of the mentioned exact function can be performed by using substitution $\Omega = \omega_C\,\Delta t + \Delta\Omega$ followed by simplification the formula by using substitution $\cos(\Delta\Omega) \approx 1$ and $\sin(\Delta\Omega) \approx \Delta\Omega$. The approximation formula is given by

$$r \approx \frac{1}{\pi\,\Delta f}\frac{\sqrt{\sqrt{2}-1}}{\sqrt{1-(\cos(\omega_C\,\Delta t))^2}}. \quad (27)$$

For the second generation VK-filter, the weighting factor as the exact function of the bandwidth and the approximation of this function, using the estimating formula $\cos(\pi\Delta f) \approx \left(1-(\pi\Delta f)^2/2\right)$, are summarized in the following table

Number of poles	Solution of the equation (24)	Approximation
1	$r = \sqrt{\dfrac{\sqrt{2}-1}{2(1-\cos(\pi\Delta f))}}$	$r \approx \dfrac{0.2048624}{\Delta f}$
2	$r = \sqrt{\dfrac{\sqrt{2}-1}{6-8\cos(\pi\Delta f)+2\cos(2\pi\Delta f)}}$	$r \approx \dfrac{0.0652097315}{\Delta f^2}$
3	$r = \sqrt{\dfrac{\sqrt{2}-1}{20-30\cos(\pi\Delta f)+12\cos(2\pi\Delta f)-2\cos(3\pi\Delta f)}}$	$r \approx \dfrac{0.020756902}{\Delta f^3}$

In the case when the frequency of the tracked order is not equal to a constant value and it is not possible to evaluate the appropriate weighting factor, the loss function (15) is transferred to the form $J = \boldsymbol{\varepsilon}^T \mathbf{R}^T \mathbf{R}\,\boldsymbol{\varepsilon} + \boldsymbol{\eta}^T \boldsymbol{\eta}$, where $\mathbf{R}$ is a square diagonal matrix with diagonal elements, which are equal to the weighting factors determined for the instantaneous order frequency and the filter bandwidth, e.g. $r_{i,i} = r(f_C, \Delta f)$. The VK-filtration can work with both the absolute bandwidth Δf in Hz and the relative bandwidth $\Delta f / f_C$ in percentage to remain a constant value.

CROSSING ORDER SOLUTION

If the VK-filter is simultaneously focused at the P harmonic components the data equation can be written in the general form

$$y(n)=\sum_{k=1}^{P} x_k(n)\exp(j\Theta_k(n))+\eta(n). \quad (28)$$

The loss function takes the form

$$J=\sum_{k=1}^{P} r^2 \boldsymbol{\varepsilon}_k^T \boldsymbol{\varepsilon}_k+\boldsymbol{\eta}^T\boldsymbol{\eta}=\sum_{k=1}^{P} r^2 \mathbf{x}_k^H \mathbf{A}^T\mathbf{A}\,\mathbf{x}_k+\left(\mathbf{y}^T-\sum_{k=1}^{P}\mathbf{x}_k^H\mathbf{C}_k^H\right)\left(\mathbf{y}-\sum_{k=1}^{P}\mathbf{C}_k\mathbf{x}_k\right). \quad (29)$$

The unknown envelopes $\mathbf{x}_i^H, i=1,...,P$ are minimizing the loss function (29). To determine the loss function minimum the first derivative with respect to the unknown vectors has to be evaluated. After putting the derivative to the zero the unknown envelopes are obtained as a solution of the following equation system

$$\frac{\partial J}{\partial \mathbf{x}_i^H}=\mathbf{B}_i\mathbf{x}_i+\mathbf{C}_i^H\sum_{\substack{k=1\\k\neq i}}^{P}\mathbf{C}_k\mathbf{x}_k-\mathbf{C}_i^H\mathbf{y}=\mathbf{0},\quad i=1,...,P, \quad (30)$$

where substitution $\mathbf{B}_k=r^2\mathbf{A}^T\mathbf{A}+\mathbf{E}$ and $\mathbf{C}_i^H\mathbf{C}_i=\mathbf{E}$ are employed. The solution of the equation system is not based on the direct evaluation as in the case of the formula (17) but an iterative method is suitable. Preconditioned Conjugate Gradient (PCG) algorithm is the only effective method that can be employed [2].

SUMMARY

The main results of the paper are formulas describing the dependence of the weighting factor on the VK-filter bandwidth, which is defined by (25). These formulas play a key role in the software for the VK-filter and allow the setting of the filter bandwidth in either absolute value in Hz or relative value in percentage.

REFERENCES

[1] Vold, H. & Leuridan, J. Order Tracking at Extreme Slew Rates, Using Kalman Tracking Filters. SAE Paper Number 931288.

[2] Feldbauer, Ch. & Holdrich, R. Realisation of a Vold-Kalman Tracking Filter – A Least Square Problem, Proceedings of the COST G-6 Conference on Digital Audio Effects (DAFX-000, Verona Italy, December 7-9, 2000

[3] Tuma, J. Vold-Kalman filtration in MATLABu (in Czech). In *Proceedings of Eleventh MATLAB Conference*. Praha : Humusoft Praha, 25. 11. 2003, s. 575-586

[4] Tuma, J. Dedopplerisation in Vehicle External Noise Measurements. In *Proceedings of Eleventh International Congress on Sound and Vibration. St. Petersburg* : IIAV, 5.-8..7. 2004, P 151-158.

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