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ON THE ROAD – BETWEEN SOBOLEV SPACES AND A MANUFACTURE OF ELECTROMAGNETS

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Abstract: The paper describes a process of an optimal shape design of electromagnets. We begin from a real–life application, namely data recording. Then we formulate the physical problem which is behind a development of magnetooptic materials of a high data recording density. Further we illustrate an interaction among physics, mathematics and computer science during the design process and we arrive at manufacturing an electromagnet. A new numerical method and a software tool have been developed and tested on this application. In the paper a great adventure of mathematical modelling is expressed rather than a serious scientific work.

Keywords: Magnetooptic effects, magnetostatics, mathematical modelling, shape optimization, scientific computing

1 Introduction

In most scientific papers things are always introduced briefly and exactly in order not to load the reader with unimportant details. This paper is not the case. Here we present some details, which appeared half the way to the important ones, however, they attracted and motivated us to the further research. For a sophisticated description and a clear presentation of the results we refer to [8, 9, 10, 11]. The main goal of this paper is to draw how exciting – pleasant as well as worrying – mathematical modelling can be.

At this place, authors usually mention the work which has been already done in the research area and how their paper contributes. Believe us that we are by far not alone in the branch. Our main result is developing a new hierarchical optimization method and testing it on this electromagnet problem, which seems to become a benchmark. What I would like to mention instead is a brief history of our research.

1.1 On the Road between Ostrava and Linz

The work began some three years ago when Prof. Pištora from Institute of Physics, VŠB–TU Ostrava, offered me (a Ph.D. student of applied math) a cooperation. The work was fairly

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clear: Find an optimal shape of pole heads of an electromagnet in order to have magnetic field in a certain area as constant as possible. At that moment I did not have an idea what the electromagnet is used for. We set a mathematical formulation of the problem and, after a couple of weeks (and more than a couple of nights), I calculated first 2–dimensional optimized shapes. These were very rough, see Fig. 1 (left), and definitely unexpected by the colleagues, nevertheless, the calculated magnetic field was correct and more constant. It was a pleasant while. I was told to give a talk and during the presentation there were many comments. It turned out not to be enough to study only mathematics. From the discussion we learned that due to the saturation, the sharp tips on the pole head behave as the air. Therefore the linear magnetostatic model that we had used was sometimes out of validity.

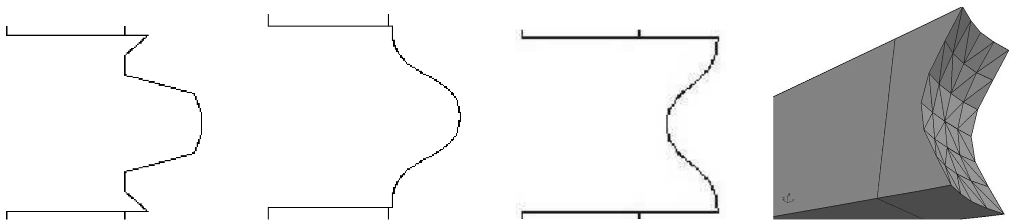


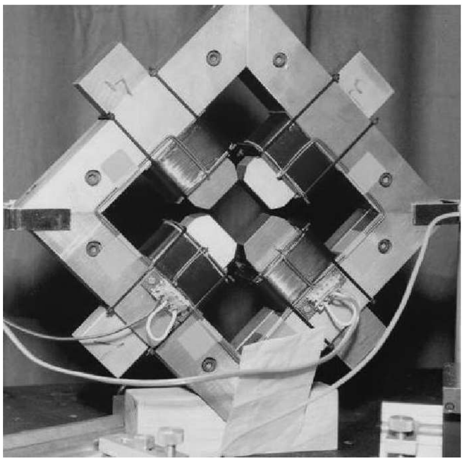
Fig. 1: An evolution of optimized pole heads

I am lucky that Prof. Dostál, who is the head of Department of Applied Mathematics, VŠB–TU Ostrava, is my Ph.D. supervisor. He arranged a cooperation with Prof. Langer from Kepler University Linz in Austria. I worked for one year in the research team in Linz. This was an exciting year. I learned a lot of mathematics and did much research work there. It was the first time I met Sobolev spaces – spaces of generalized functions, which are differentiable in some sense. It was also the first time I enjoyed proving theorems of functional analysis and applying them to my problem. The work was systematic. At any point it was clear what makes sense to do. But what I actually learned the most were courage and self-confidence. I realized that all the successful work is just a matter of clear thinking, hard working and patience. Nothing genial.

Concerning the research progress, we solved the problem in both 2 and 3 dimensions using the software, see [6], from Linz. I have contributed to this software by a special package for shape optimization, see [11]. At the same time, colleagues from Institute of Physics at VŠB–TU Ostrava managed a manufacture of the optimized pole heads and they measured the magnetic field, afterward. What a pretty surprise that the measured magnetic field had improved (in terms of the chosen optimization criterion, with respect to the initial design) even more than the calculation predicted. Nevertheless, from the very beginning we still have not found a proper criterion (an objective functional) telling us what magnetic field is optimal. The collection of various optimal shapes has been growing, see Fig. 1. We are always sure that the current result is definitely the optimal shape until we change the criterion again. Now we are optimistic, the work continues and we have been designing pole heads for an electromagnet of a different (ring) geometry. At least, we can be optimistic since more than 5 electromagnets have been already sold world–wide; to well-known laboratories at Charles University Prague, INSA Toulouse and Simon Fraser University in Vancouver.

2 Data Recording and Magnetooptic Effects

This new friend of us – the electromagnet, see Fig. 2 – is used for measurements of magnetooptic effects on thin layers, see [4, 12]. Materials of good magnetooptic properties are then used for high density data recording, i.e., magnetic or compact disc recording.



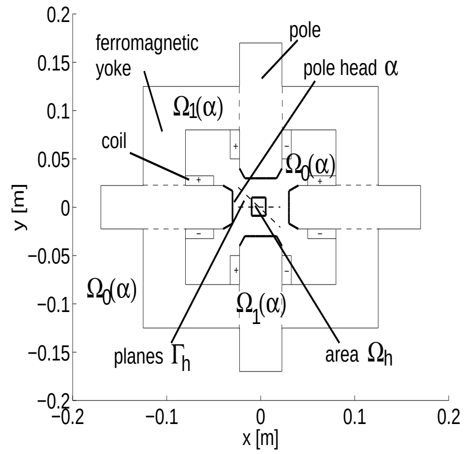


Fig. 2: The electromagnet of a so–called Maltese Cross geometry

Referring to Fig. 2, we describe the physical problem. The electromagnet consists of a ferromagnetic yoke and 4 pole heads completed by coils. A sample of some magnetooptic properties is placed into the area Ω_{h} among the pole heads. Optical beams of a given polarization reflect from the sample and their components are measured in terms of Kerr's rotation, cf. [14, p. 40]. Moreover, the magnetooptic properties are anisotropic, therefore, the measurement is done for various orientations of the magnetic field. Just by switching currents in the coils, the electromagnet is capable to generate magnetic field (step–by–step) homogeneous in the area Ω_{h} in up to 8 directions. The magnetic field should be strong enough and as homogeneous, i.e., as constant, as possible for all those 8 configurations. Unfortunately, these requirements are contradictory and we have to balance them. In fact, by choosing proper weights between the strength of the magnetic field and its homogeneity the optimization arrives at quite different shapes. It is like: Tell me what shape you would like to have and I will calculate it. From this point of view I have to be very modest and confess that the mathematics is just a tool. At the end, the experience of my colleagues gives the true solution. On the other hand, without fast computational methods the progress would definitely not be possible.

3 On the Road between Physics and Mathematics

Out of all scientific results I most appreciate the ones that are motivated by pretty and useful applications. That was exactly the case of our electromagnet problem. I liked and respected mathematics but I did not want to spend half the life closed in libraries. I rather wanted to get results as fast as possible and my motivation was (and still is) that the calculated shape of the electromagnet would be manufactured. But, working among mathematicians have influenced me and now I can see that the math is beautiful itself. Here I want to sketch practical reasons for understanding the mathematics. Namely, the used computational methods (finite element method, sequential quadratic programming, hierarchical approach) are general and powerful if the mathematical setting of the optimization problem satisfies some properties.

3.1 Linear Magnetostatic Field Problem

The optimization task includes a so–called state problem, which describes the physical field quantities. In our case it is the problem of linear magnetostatics described by the Maxwell equations, cf. [7, 13],

$$\left.\begin{aligned} \mathbf{rot}(\mathbf{H}) &= \mathbf{J} \\ \mathrm{div}(\mathbf{B}) &= 0 \end{aligned}\right\} \text{ in } \Omega \subset \mathbb{R}^3, \tag{1}$$

where $\mathbf{J}$ stands for the current density, $\mathbf{H}$ denotes the magnetic strength density and $\mathbf{B}$ denotes the magnetic flux density, which are related by the constitutive law

$$\mathbf{B} = \mu\mathbf{H}, \tag{2}$$

where $\mu > 0$ denotes the permeability. By introducing the magnetic vector potential $\mathbf{u}$

$$\mathbf{rot}(\mathbf{u}) = \mathbf{B}, \tag{3}$$

we obtain the following boundary value problem involving a partial differential equation:

$$\left.\begin{aligned} \mathbf{rot}\left(\frac{1}{\mu}\mathbf{rot}(\mathbf{u})\right) &= \mathbf{J} \quad \text{in } \Omega \\ \mathbf{n} \times \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega \end{aligned}\right\}, \tag{4}$$

where $\mathbf{n}$ denotes the unit outer normal to the boundary of Ω, which is denoted by $\partial\Omega$.

The problem (4) can be often reduced to 2 dimensions. Let us assume that

$$\mathbf{J}(\mathbf{x}) := (0, 0, J(x_1, x_2))\,,\ \mu(\mathbf{x}) := \mu(x_1, x_2) \text{ and } \mathbf{u}(\mathbf{x}) := (0, 0, u(x_1, x_2))\,, \tag{5}$$

then (4) reduces to:

$$\left.\begin{aligned} -\mathrm{div}\left(\frac{1}{\mu}\mathbf{grad}(u)\right) &= J \quad \text{in } \Omega_{2d} \\ u &= 0 \quad \text{on } \partial\Omega_{2d} \end{aligned}\right\}, \tag{6}$$

where

$$\Omega_{2d} := \left\{\mathbf{x}' = (x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2, 0) \in \Omega\right\} \tag{7}$$

represents a cross section of Ω.

3.2 Weak Formulations of Boundary Value Problems

Classical solutions to both the problems (4) and (6) need some strong requirements on the smoothness of the data μ, $\mathbf{J}$ and $\partial\Omega$. From the physics we know that the Maxwell equations hold for nonsmooth data, too. That is why we generalize the problem to a so–called weak formulation. An abstract vector boundary value problem reads as follows:

$$\left.\begin{aligned} -\mathbf{A}^*\left(\mathbf{D}\cdot\mathbf{A}(\mathbf{u})\right) = \mathbf{f} \text{ in } \Omega \\ \gamma(\mathbf{u}) = \mathbf{0} \text{ on } \partial\Omega \end{aligned}\right\}, \tag{S}$$

where $\mathbf{D}$ is a matrix involving material properties, $\mathbf{A}$, $\mathbf{A}^*$ denote linear vector differential operators of the first order and $\boldsymbol{\gamma}$ denotes a so–called trace operator. They are related by Green's formula

$$\int_\Omega \mathbf{A}(\mathbf{u})\cdot\mathbf{v}\,dx + \int_\Omega \mathbf{u}\cdot\mathbf{A}^*(\mathbf{v})\,dx = \int_{\partial\Omega} \gamma(\mathbf{u})\cdot\mathbf{v}\,ds, \tag{8}$$

where $\mathbf{u}$ and $\mathbf{v}$ are functions continuously differentiable over Ω. For example, in case of (4) $\mathbf{D} := \mu$, $\mathbf{A} := \mathbf{rot}$, $\mathbf{A}^* := -\mathbf{rot}$, $\gamma(\mathbf{u}) := \mathbf{n}\times\mathbf{u}$.

This is the right time to meet Sobolev spaces, cf. [1, 5]. We will generalize (S) by looking for such functions $\mathbf{u}$ that (8) still makes sense. These functions form so-called Sobolev spaces

$$\mathbf{H}_0(\mathbf{A};\Omega) := \left\{\mathbf{u}\in\left(L^2(\Omega)\right)^{\nu_1} \mid \exists\mathbf{z}\in\left(L^2(\Omega)\right)^{\nu_2} : \mathbf{z} = \mathbf{A}(\mathbf{u}) \text{ and } \gamma(\mathbf{u}) = \mathbf{0}\right\}, \tag{9}$$

where ν_1, ν_2 are positive integers, $L^2(\Omega)$ is the space of functions, the squares of which are integrable in Lebesgue's sense, $\mathbf{A}(\mathbf{u})$ is understood in a weak sense and $\boldsymbol{\gamma}$ is now understood in a sense of traces. Finally, the weak formulation of (S) reads

$$\left.\begin{aligned} &\text{Find } \mathbf{u}\in\mathbf{H}_0(\mathbf{A};\Omega)\text{:} \\ &\int_\Omega \mathbf{A}(\mathbf{v})\cdot\left(\mathbf{D}\cdot\mathbf{A}(\mathbf{u})\right)\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx \quad \forall\mathbf{v}\in\mathbf{H}_0(\mathbf{A};\Omega) \end{aligned}\right\}. \tag{W}$$

Note that there are still some presumptions on the data but definitely not so strong as for (S).

The space $\mathbf{H}_0(\mathbf{A};\Omega)$ is then approximated by a finite–dimensional subspace, where the functions are piecewise polynomial. The problem (W) is discretized in the same fashion, which leads to a linear system of equations. This is the concise principle of the finite element method.

The weak formulation seems to be very natural since the physical laws are usually found in an integral form (since the measurements have averaging, i.e., integral features). Another nice property is that the just introduced abstract formulation fits all the linear mechanics, the heat conduction problem, the electrostatics, the magnetostatics, etc. The power of mathematics is now that it helps us to find a structure of the problem and it provides a method for solving the boundary value problems quite independently of the physics behind.

3.3 Shape Optimization Problem

The mathematical setting of the shape optimization problem reads as follows:

$$\left.\begin{aligned} &\text{Find } \alpha^*\in\mathcal{U}\text{:} \\ &\varphi(\alpha^*) \le \varphi(\alpha) \quad \forall\alpha\in\mathcal{U} \end{aligned}\right\}, \tag{P}$$

where α is a shape (a continuous function), α^* is the optimal shape, $\mathcal{U}$ denotes a set of admissible shapes and $\varphi : \mathcal{U} \mapsto \mathbb{R}$ denotes a so–called cost (objective) functional. The main difficulty – and the beauty – of the shape optimization problem is that evaluation of φ involves solving the boundary value problem, which describe, e.g., the magnetic field.

There exists a solution to the problem (P) if $\mathcal{U}$ is a compact set and φ is a continuous functional, see [3, 8]. In order to achieve the compactness, we, e.g., bound the slope (norm of the gradient) of the shapes by a positive constant. This may still arrive at ugly shapes, see Fig. 1 (left). It is due to some instability (so–called ill-posedness) of the shape optimization problem. Moreover, in our case the linear magnetostatic field problem is not valid for such designs. In order to avoid sharp tips, we look for smooth enough curves or patches, using, e.g., Bézier parameterization.

Concerning the cost functional, during first calculations we prescribed the following one:

$$\varphi(\alpha) := \frac{1}{|\Gamma_{\mathrm{h}}|\,\|\mathbf{B}_\alpha^{\mathrm{avg}}\|^2} \int_{\Gamma_{\mathrm{h}}} \|\mathbf{B}_\alpha(x) - \mathbf{B}_\alpha^{\mathrm{avg}}\|^2 \, ds, \tag{10}$$

where Γ_{h} is a part of a plane (or line segment), see Fig. 2, which is orthogonal to the magnetic flux, $|\Gamma_{\mathrm{h}}|$ denotes its length, $\mathbf{B}_\alpha^{\mathrm{avg}}$ stands for the average magnetic flux density along the plane Γ_{h}, $\mathbf{B}_\alpha := \mathbf{rot}(\mathbf{u}_\alpha)$ is the magnetic flux density and $\mathbf{u}_\alpha$ denotes a weak solution to the linear magnetostatic problem (4) (or (6)) in case of $\Omega \subset \mathbb{R}^3$ (or $\Omega_{2d} \subset \mathbb{R}^2$, respectively). However, the functional (10) is not mathematically correct since Γ_{h} is of zero Lebesgue measure with respect to the computational domain Ω (or Ω_{2d}). Even from the physical point of view, the magnetostatic field can not be measured at a point, along a line or a surface, since the Maxwell equations describe macroscopic properties. In other words, every magnetic probe has a small nonzero volume and it does not measure pointwise but rather averaged magnetic field over the volume.

The next, mathematically correct, cost functional that we defined was

$$\varphi(\alpha) := \frac{1}{|\Omega_{\mathrm{h}}|} \int_{\Omega_{\mathrm{h}}} \|\mathbf{B}_\alpha(x) - \mathbf{B}_\alpha^{\mathrm{avg}}\|^2 \, dx, \tag{11}$$

where $\Omega_{\mathrm{h}} \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) denotes the area where the magnetic field should be homogeneous, see Fig. 2, $|\Omega_{\mathrm{h}}|$ denotes its volume (or area), $\mathbf{B}_\alpha^{\mathrm{avg}}$ stands for the average magnetic flux density over the area Ω_{h} and $\mathbf{B}_\alpha := \mathbf{rot}(\mathbf{u}_\alpha)$ is the magnetic flux density, where $\mathbf{u}_\alpha$ is a weak solution to the problem (4) (or (6)). The functional (11) is well–defined and continuous with respect to a proper metric defined over the set of admissible shapes $\mathcal{U}$.

In both the cases we consider the following state dependent constraint on a minimal magnetic flux density

$$\forall \alpha \in \mathcal{U} : g(\alpha) \le 0, \text{ where } g(\alpha) := B_{\min}^{\mathrm{avg}} - \|\mathbf{B}_\alpha^{\mathrm{avg}}\|, \; B_{\min}^{\mathrm{avg}} > 0. \tag{12}$$

The optimization with the incorrect functional (10) arrived at the 2–dimensional shape in Fig. 1 (2nd from the left). The optimization with the functional (11) arrived at the next 2–dimensional shape, see Fig. 1 (2nd from the right), and at the 3–dimensional shape, see Fig. 1 (right).

4 On the Road between Mathematics and Computers

There is hardly any mathematician who does not use a computer. The rather small gap between the math and the computer science is mainly due to the fact that the computer guys do not like time–consuming plays with expressions on a sheet of paper. On the other hand, the math guys will never admit that a computer can be more intelligent and they do not let it influence their lives. Am I exaggerating? I know.

4.1 Shape Optimization Software Library

What I want to express here is that the systematic and well–structured programming work, the object–oriented technology and a little software engineering knowledge helped me in building a software package for the optimal shape design. Namely, in the Newton–like optimization algorithms, e.g., the sequential quadratic programming, the computational time is proportional to the number of evaluations of the cost functional and its gradient. Making use of the structure of the shape optimization problem, we can implement the gradient of the cost functional very efficiently in an object–oriented library, the data flow diagram of which is in Fig. 3. It is enough to realize that the cost functional is a compound of the following mappings:

- shape parameterization $p \mapsto \alpha$, using, e.g., Bézier curves or patches,
- deformation of the rest of the discretized grid $\alpha \mapsto x$,
- assembling the matrix and the right–hand side vector of the linear system which is given by a discretization of the boundary value problem (the finite element method) $x \mapsto \mathbf{K}, \mathbf{f}$,
- solving the linear system $\mathbf{K}, \mathbf{f} \mapsto \mathbf{u}$,
- evaluating the cost functional and the state dependent constraints $\alpha, \mathbf{u} \mapsto \varphi, g$.

These mappings as well as their gradients are evaluated independently and the gradient of the cost functional is then given by their product. For more details on the design of the library we refer to [11].

4.2 Hierarchical Optimization Strategy

This is the most challenging part of our mathematical research. It is originally the idea of Prof. Langer from Linz. It involves usage of multigrid strategies, cf. [2], in solving shape optimization problems.

We distinguish between a classical and a hierarchical approach. By the classical approach we understand optimization with a fixed number of design variables and a fixed discretization grid. In our hierarchical approach we solve several optimization problems such that the first one is discretized on a very coarse grid and the optimized coarse design is used as an initial guess for the next, finer discretized problem. We refine the problem up to the required level. The hierarchical approach turns out to be very efficient, see Fig. 4. The strategy succeeds if the coarse optimized design approximates the finest one well. Some results have been already published in [9].

5 Back to the Electromagnet

Home, sweet home. After long travelling among different research areas we arrived at some optimized shapes and it is the time to return and discuss the designs with my dear colleagues from Institute of Physics. As for me, the discussion means doing some work on a computer – a lot of simulations, a lot of graphs. However, I like this work very much. We discuss what really happens, what simplifications we have supposed and what design will be manufactured. Then I am just looking forward to seeing the real electromagnet and I am pretty nervous as for the measurements.

At last! The 2–dimensional optimized pole heads, see Fig. 1 (2nd on the right), were manufactured, the magnetic field was measured and the real value of (11) decreased 4.5–times

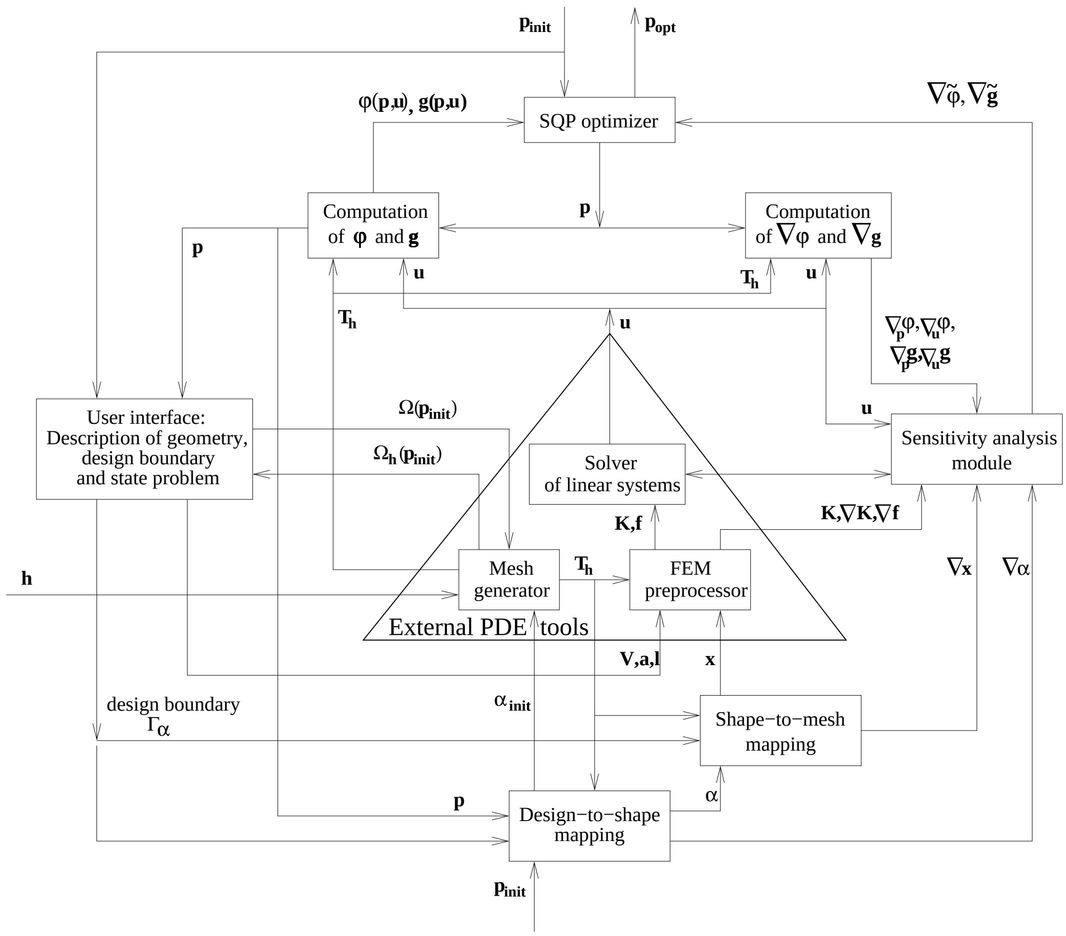


Fig. 3: Object–oriented library for optimal shape design

with respect to the initial design. It was even better than the calculation had predicted (the calculated value of (11) decreased only twice). A relative difference between the calculated and the measured normal component of the magnetic field over the area of homogeneity Ω_h is presented in Fig. 5. However, the difference is of about 30%, which is probably caused by employing the linear magnetostatic model.

6 Conclusion

In the paper I presented some issues on mathematical modelling and shape optimization of an electromagnet. Let me briefly summarize what we have done so far:

- We introduced a mathematical setting (in both 2 and 3 dimensions) of the shape optimization problem and proved the existence of an optimal shape.
- We calculated both the 2 and 3–dimensional optimized pole heads.

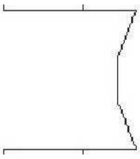
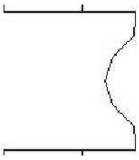
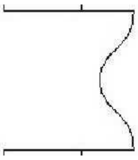
optimized designs	# of des. variables	# of unknowns	SQP iters.	CPU time [s]
	2	1386	8	55.75
	4	4705	47	1650.75
		4970	53	1812.43
	7	12272	72	7134.71
		12324	125	24239.63

Fig. 4: Hierarchical vers. classical optimization approach

– We manufactured the 2–dimensional pole heads and compared the calculated magnetic field to the measured one.
– We have implemented an object–oriented library for optimal shape design.
– We designed a hierarchical approach in the shape optimization and we tested it on the electromagnet problem.

In our paper, we visited three different worlds - the world of mathematics, physics and computer science. We were talking about working on a computer, in a laboratory, we were in Ostrava as well as visited our colleagues in Linz. In total, I have cooperated with, at least, ten colleagues and I have been enjoying this work. It is mainly an exciting game.

Well, this is the beauty of the mathematical modelling world. I kindly invite you to come in.

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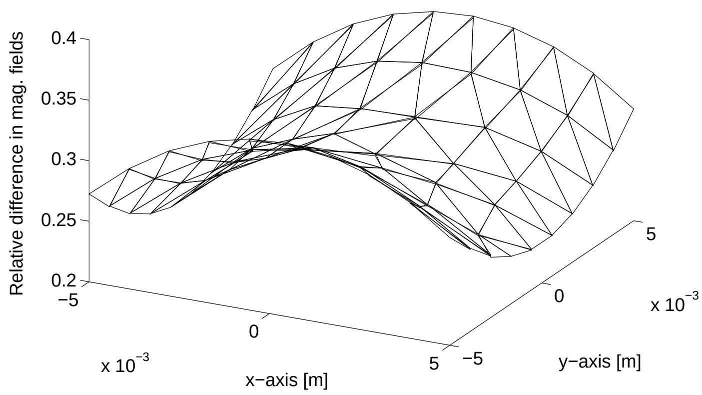

Fig. 5: Relative difference between the computed and measured fields

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