Optimal Shape Design in Magnetostatics

Disputation of the Ph.D. thesis

November 27, 2003

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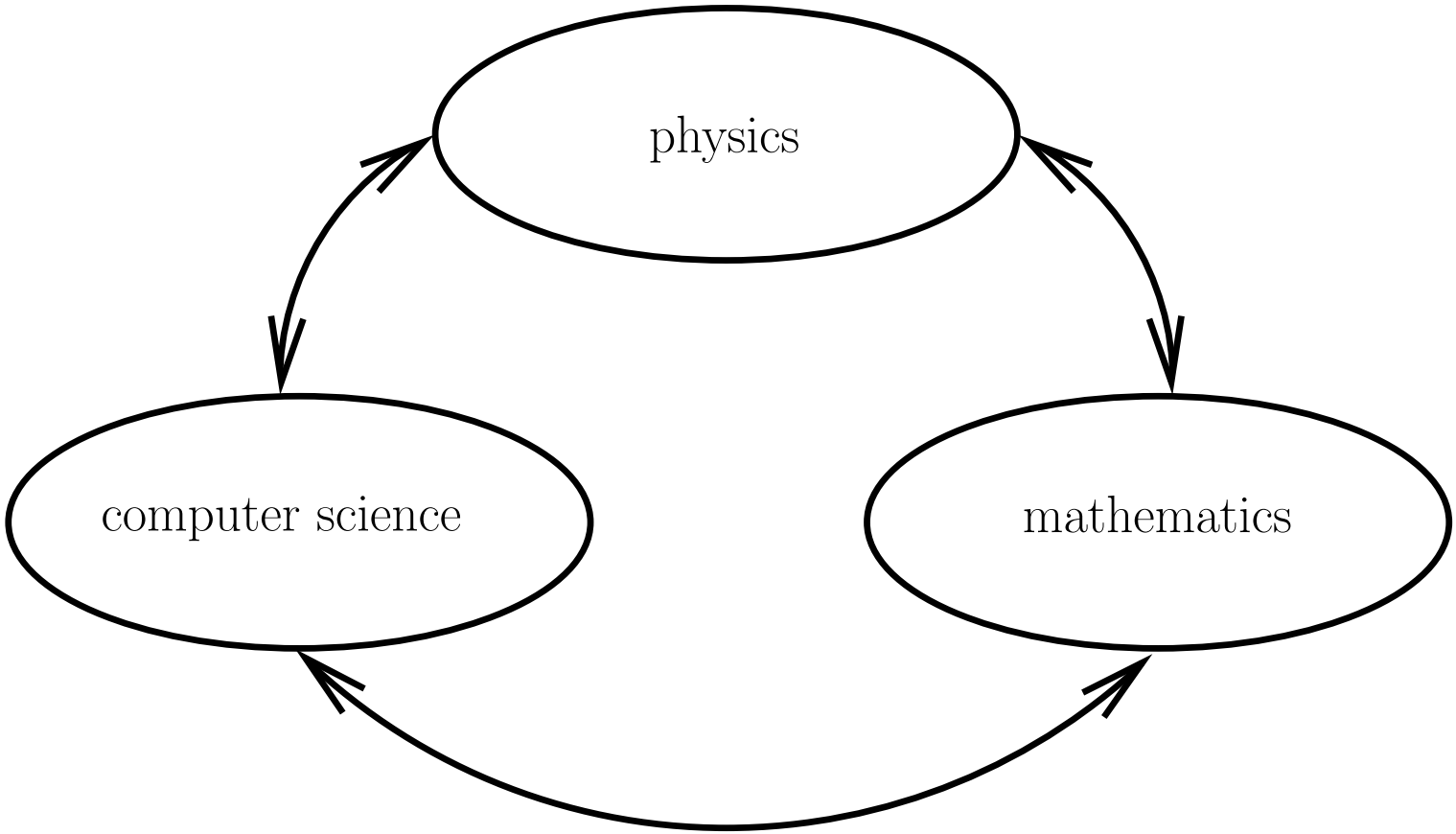
Prof. U. Langer, Inst. of Computational Mathematics, University Linz

Outline: On the road



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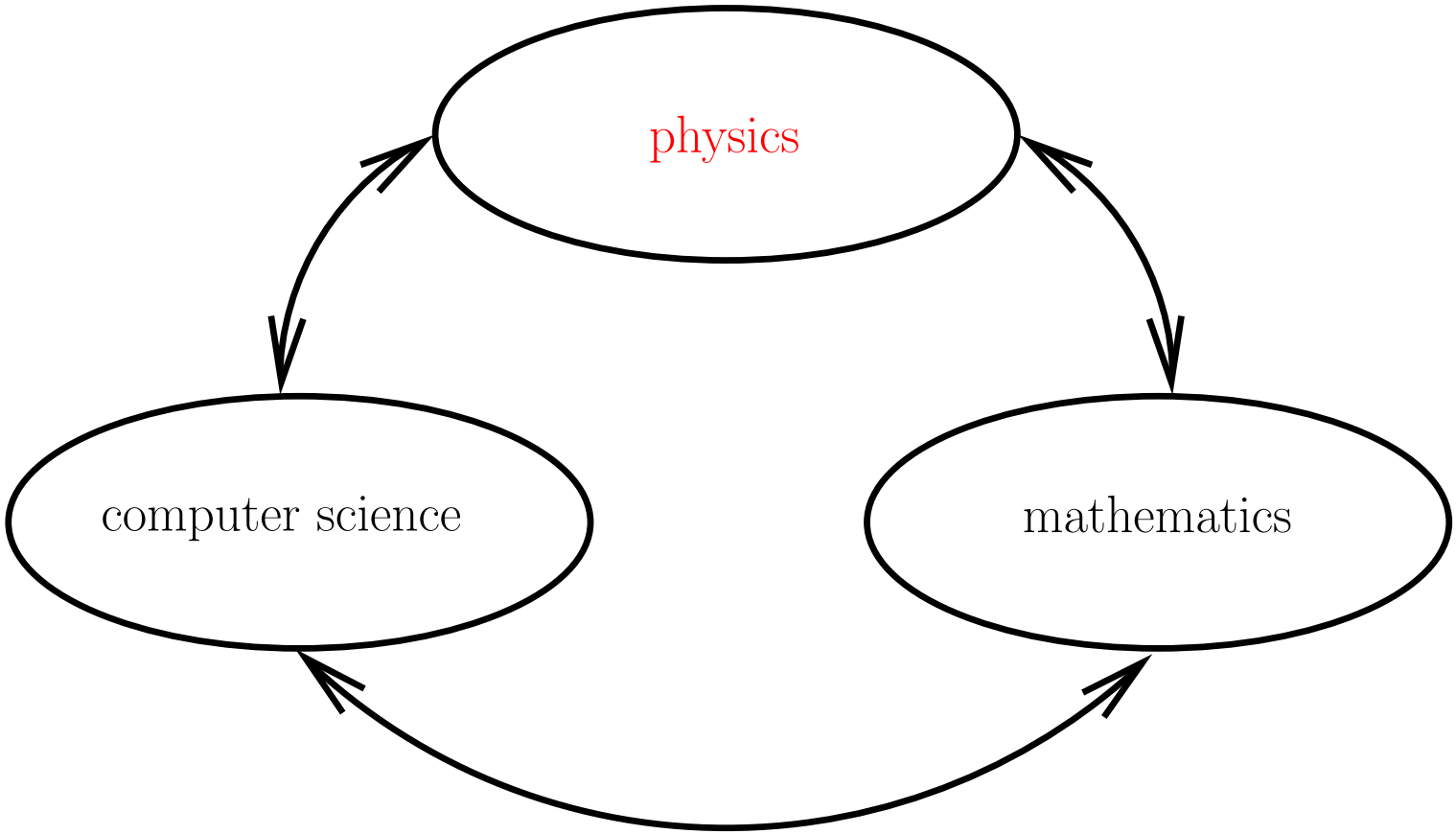
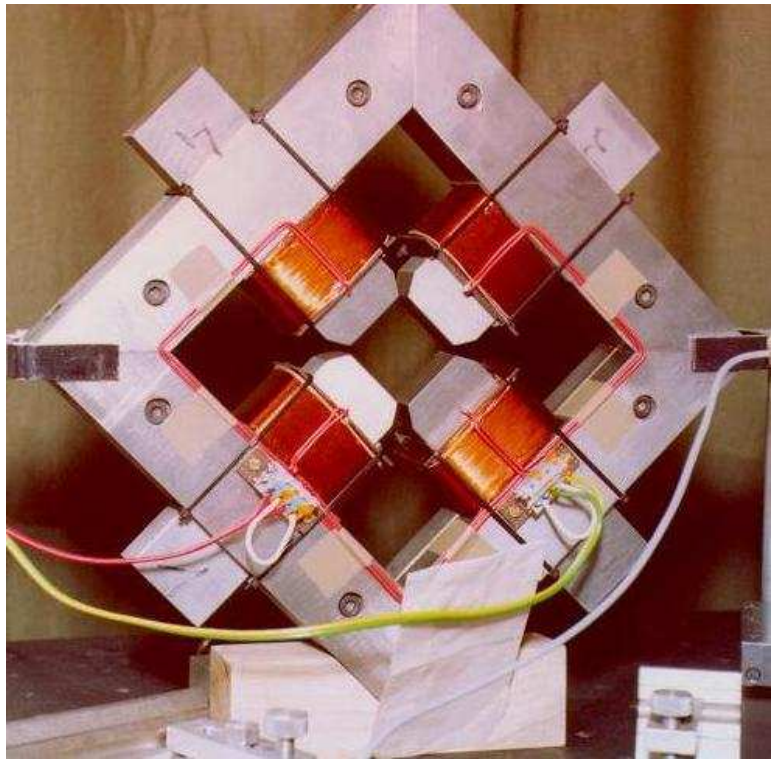


Homogeneous electromagnets

Maltese cross electromagnet



- is used for measurements of magnetooptic effects,
- produces magnetic field constant in the middle,
- is capable to rotate the magnetic field,
- is produced at Institute of Physics, VŠB–TU Ostrava,
- is also used at INSA Toulouse, University Paris VI, Simon Fraser University Vancouver, Charles University Prague

Homogeneous electromagnets

Optimization problem

Find optimal shapes of the pole heads in order to minimize inhomogeneities of the magnetic field in the middle area Ω_{m} among the pole heads.

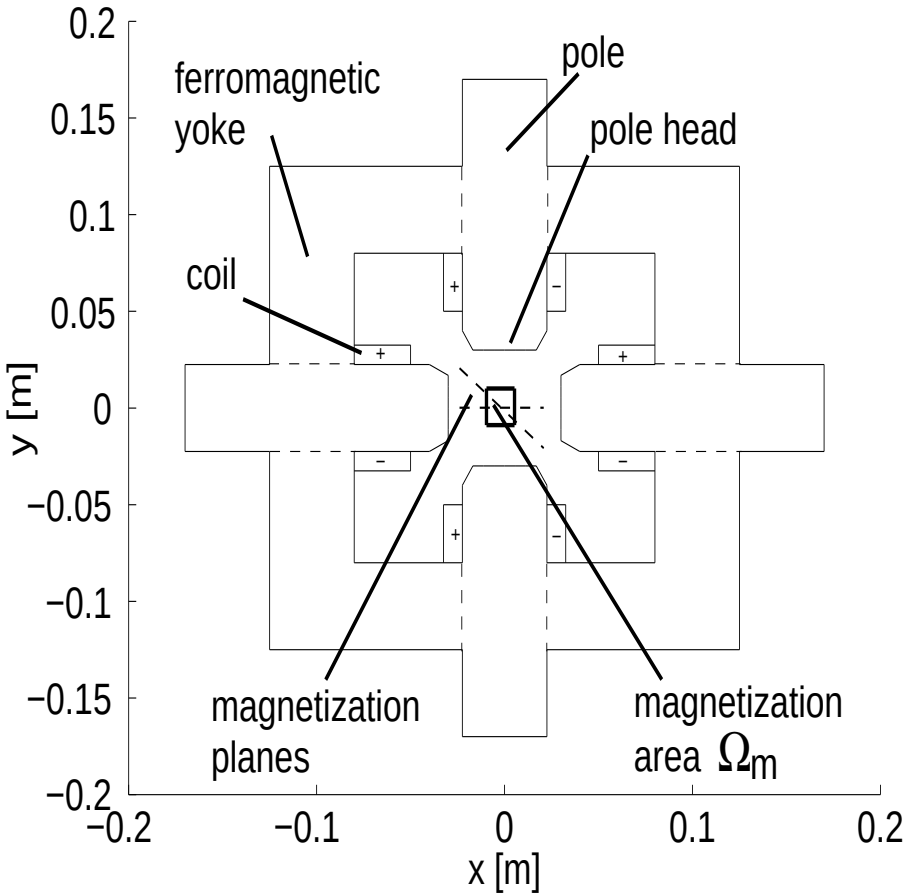
$$\min_{\alpha \in \mathcal{U}} \int_{\Omega_{\mathrm{m}}} |\mathbf{B}_\alpha(\mathbf{x}) - \mathbf{B}_\alpha^{\mathrm{avg}}|^2 \, d\mathbf{x},$$

where

$\mathbf{B}_\alpha(\mathbf{x})$ ... the magnetic flux density,

$\mathbf{B}_\alpha^{\mathrm{avg}}(\mathbf{x})$ ... the average mag. flux density over Ω_{m}, $\mathbf{B}_\alpha^{\mathrm{avg}} \geq \mathbf{B}^{\mathrm{min}}$,

$\mathcal{U}$ ... the set of admissible shapes (bounded continuous functions)



Maxwell's equations

$$\left.\begin{aligned}
\mathbf{curl}\left(\frac{1}{\mu}\mathbf{B}\right) &= \mathbf{J} + \sigma\mathbf{E} + \frac{\partial(\varepsilon\mathbf{E})}{\partial t} \\
\mathbf{curl}\,(\mathbf{E}) &= -\frac{\partial\mathbf{B}}{\partial t} \\
\mathrm{div}(\varepsilon\mathbf{E}) &= \rho \\
\mathrm{div}(\mathbf{B}) &= 0
\end{aligned}\right\} \text{ in } \Omega \times T \subset \mathbb{R}^3 \times (0, \infty)$$

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+ homogeneous boundary conditions

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \text{ and } \mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times T$$

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Maxwell's equations for linear magnetostatics

$$\left.\begin{aligned} \mathbf{curl}\left(\frac{1}{\mu(\mathbf{x})}\mathbf{B}(\mathbf{x})\right) &= \mathbf{J}(\mathbf{x}) \\ \operatorname{div}(\mathbf{B}(\mathbf{x})) &= 0 \end{aligned}\right\} \text{ in } \Omega$$

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Maxwell's equations for vector potential

$$\begin{aligned} \mathbf{curl}\left(\frac{1}{\mu(\mathbf{x})}\mathbf{curl}(\mathbf{u}(\mathbf{x}))\right) &= \mathbf{J}(\mathbf{x}) \text{ in } \Omega \\ \mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}) &= \mathbf{0} \quad \text{ on } \partial\Omega \end{aligned}$$

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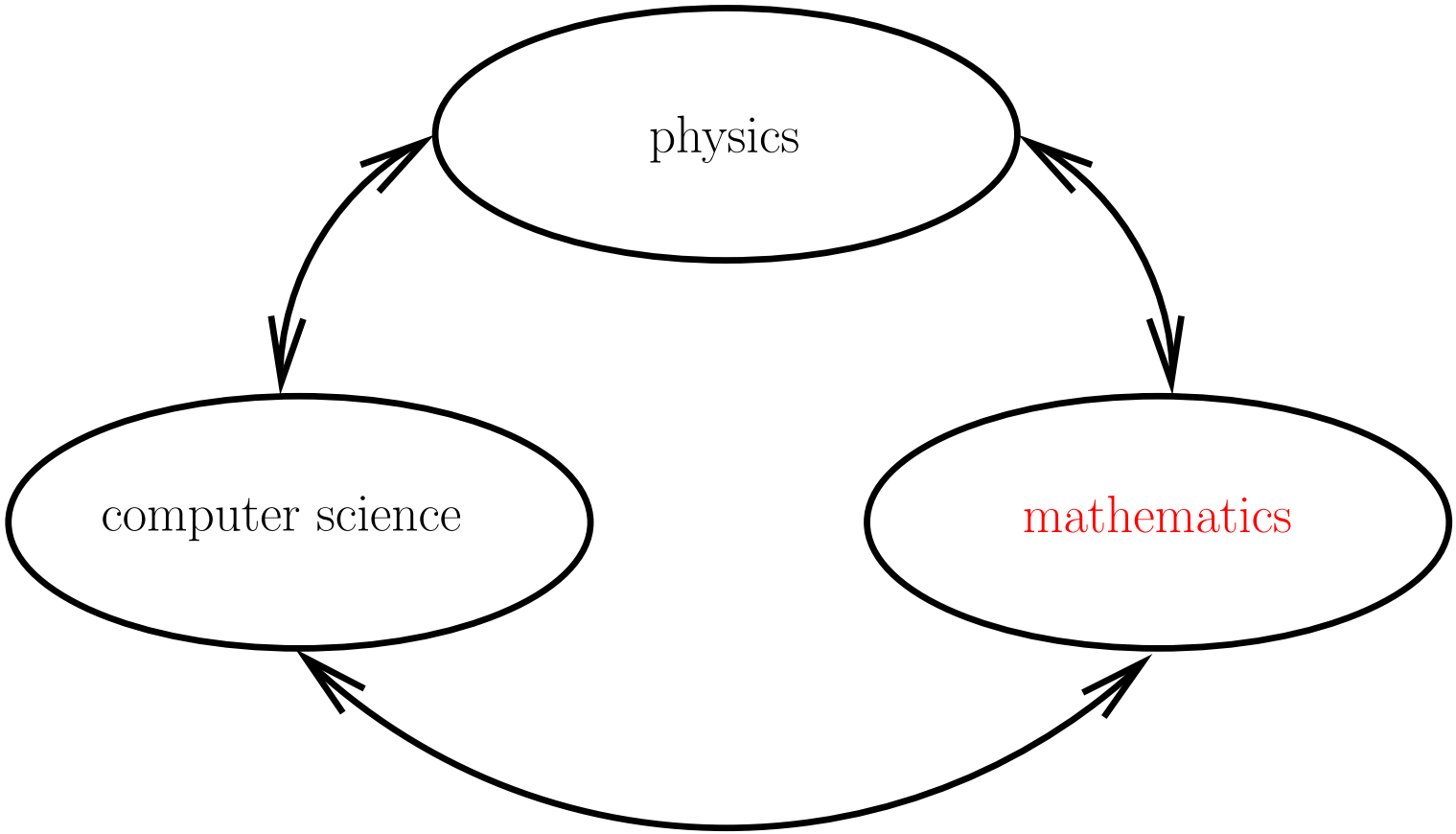

Weak formulation of 3d linear magnetostatics

$$(W)\begin{cases} \text{Find } \mathbf{u} \in \mathbf{H_0}(\mathbf{curl};\Omega)/\mathbf{Ker_0}(\mathbf{curl};\Omega) =: \mathbf{H_{0,\perp}}(\mathbf{curl};\Omega) : \\ a(\mathbf{u},\mathbf{v}) := \displaystyle\int_\Omega \frac{1}{\mu}\mathbf{curl}(\mathbf{u})\cdot\mathbf{curl}(\mathbf{v})\,d\mathbf{x} = \int_\Omega \mathbf{J}\cdot\mathbf{v}\,d\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H_{0,\perp}}(\mathbf{curl};\Omega), \end{cases}$$

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Regularization ($\varepsilon > 0$)

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L. '03: $\exists!\mathbf{u}_\varepsilon$ solution to (W_ε) and $|\mathbf{u}_\varepsilon - \mathbf{u}|_{\mathbf{curl},\Omega} \to 0$

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Schöberl '02: Since $\mathbf{J} \in \mathbf{Ker}_0(\mathrm{div};\Omega)$, then $\mathbf{u}_\varepsilon \in \mathbf{H_{0,\perp}}(\mathbf{curl};\Omega)$ and $\|\mathbf{u}_\varepsilon - \mathbf{u}\|_{\mathbf{curl},\Omega} \to 0$

Finite element method: Nédélec's elements

- Nédélec space $\mathbf{P}^e := \left\{ \mathbf{p}(\mathbf{x}) := \mathbf{a}^e \times \mathbf{x} + \mathbf{b}^e \;\middle|\; \mathbf{a}^e, \mathbf{b}^e \in \mathbb{R}^3,\ \mathbf{x} \in \overline{K^e} \right\}$

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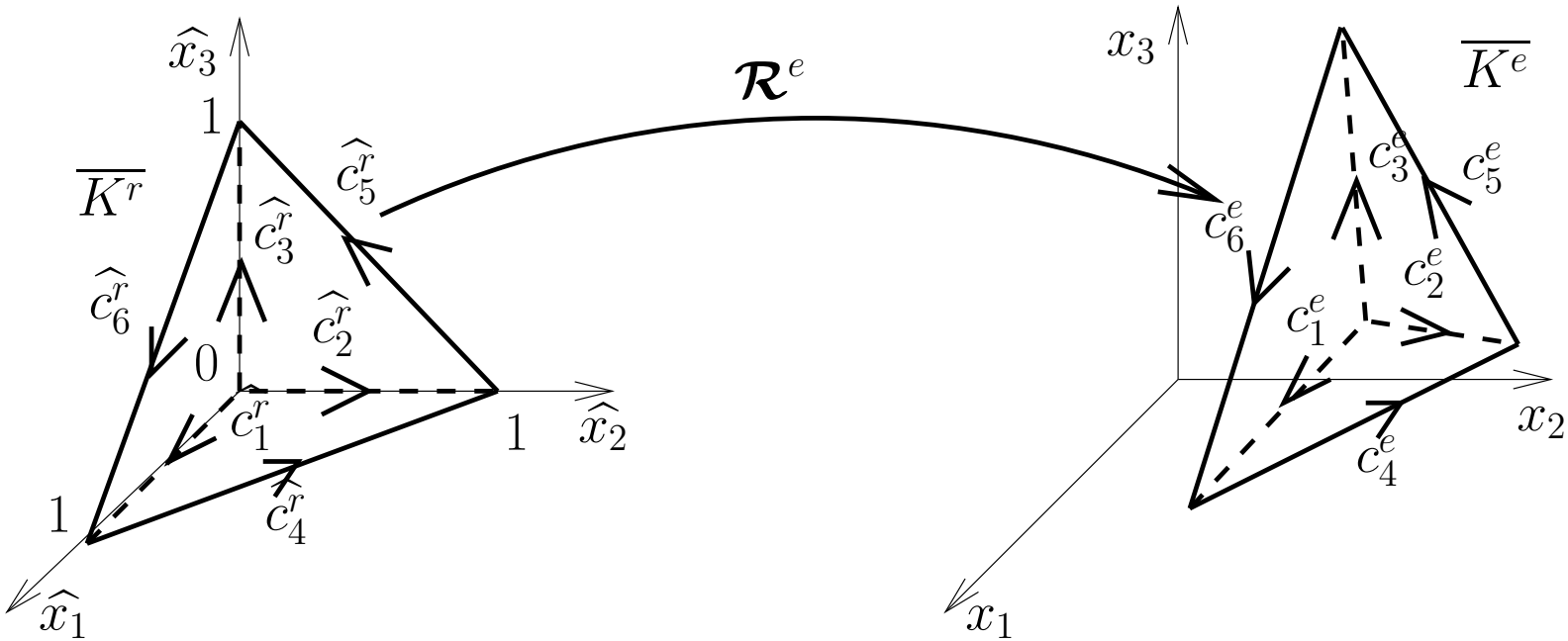
- Degrees of freedom $\sigma_i^e(\mathbf{v}) := \int_{c_i^e} \mathbf{v} \cdot \mathbf{t}_i^e \, d\mathbf{s}, \quad i = 1, \ldots, 6$



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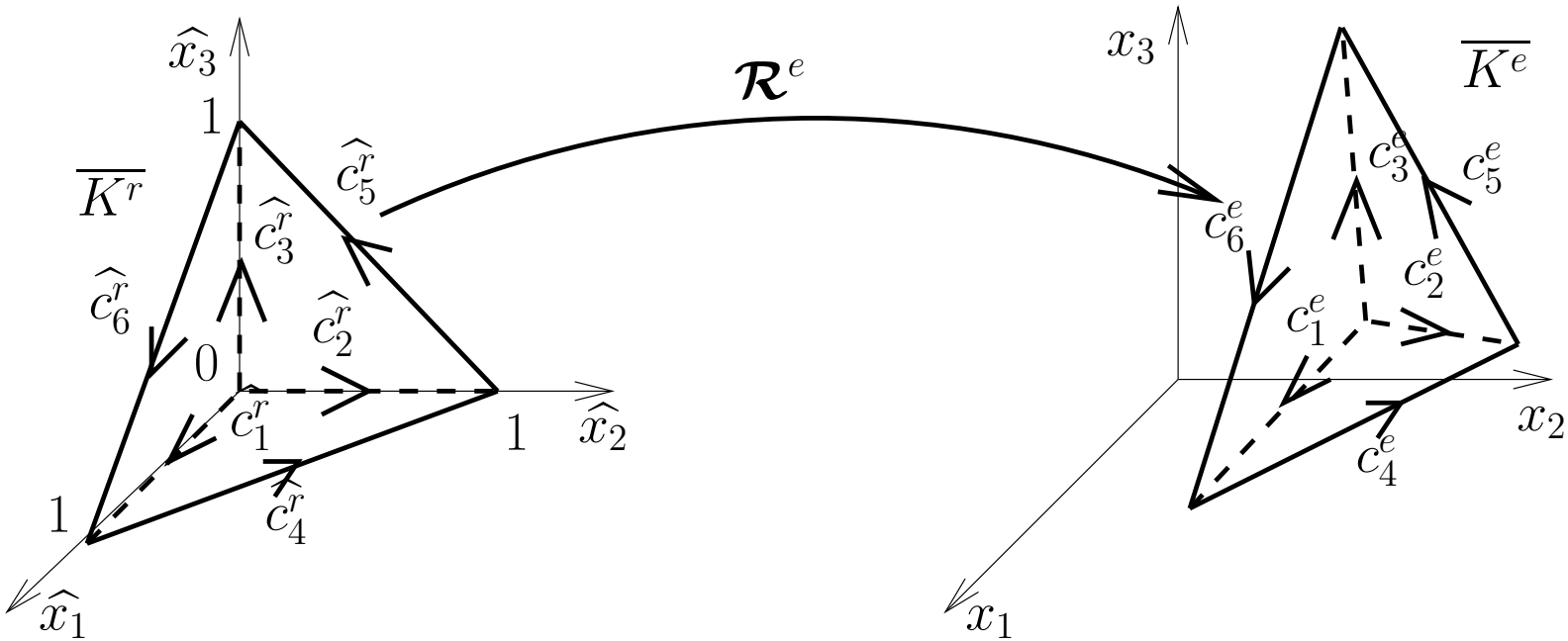

- $\mathbf{H}(\mathbf{curl})$–conformity condition: continuity of tangential components

Finite element approximation

- Inner approximation of Ω by polyhedra Ω^h

$$\forall \mathbf{x}^h \in \partial\Omega^h \; \exists \mathbf{x} \in \Omega : \|\mathbf{x}^h - \mathbf{x}\| \leq h, \text{ where } \Omega^h \subset \Omega$$

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$$\|\mathbf{u}_\varepsilon - \mathbf{X}^h(\mathbf{u}_\varepsilon^h)\|_{\mathbf{curl},\Omega} \leq \frac{1}{\min\{C, \varepsilon\}} \left(\|\mathbf{u}_\varepsilon - \mathbf{v}^h\|_{\mathbf{curl},\Omega} + \frac{\left|[a_\varepsilon - a_\varepsilon^h](\mathbf{X}^h(\mathbf{u}_\varepsilon^h) - \mathbf{v}^h, \mathbf{v}^h)\right|}{\|\mathbf{X}^h(\mathbf{u}_\varepsilon^h) - \mathbf{v}^h\|_{\mathbf{curl},\Omega}} \right)$$

- Density argument $\forall \tau > 0 \; \forall \mathbf{v} \in \mathbf{H_0}(\mathbf{curl}; \Omega) \; \exists \widetilde{\mathbf{v}} \in \mathbf{H_0^2}(\Omega) : \|\mathbf{v} - \widetilde{\mathbf{v}}\|_{\mathbf{curl},\Omega} \leq \tau$
- $\mathbf{v}^h := \mathbf{X}^h(\boldsymbol{\pi}_\mathbf{0}^h(\widetilde{\mathbf{u}}_\varepsilon|_{\overline{\Omega^h}}))$, where $\boldsymbol{\pi}_\mathbf{0}^h$ is the FE–interpolation operator
- Provided $\mu^h(\mathbf{x}) \to \mu(\mathbf{x})$ a.e. in Ω, using Lebesgue dominated convergence theorem:

$$\mathbf{X}^h(\mathbf{u}_\varepsilon^h) \to \mathbf{u}_\varepsilon \text{ in } \mathbf{H_0}(\mathbf{curl}; \Omega)$$

Finite element approximation: Prof. Langer's comment

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Can the theory be extended to non–inner domain approximations?

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$$\forall \mathbf{x}^h \in \partial\Omega^h \; \exists \mathbf{x} \in \Omega : \|\mathbf{x}^h - \mathbf{x}\| \leq h, \text{ where } \Omega, \Omega^h \subset \widehat{\Omega}$$

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- Using 2nd Strang's lemma, for $\mathbf{v}^h := \mathbf{X}^h(\boldsymbol{\pi}_{\mathbf{0}}^h(\widetilde{\mathbf{u}}_\varepsilon|_{\overline{\Omega^h}}))$:

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- Provided $\mu^h \to \mu$ a.e. in Ω, it yields $\mathbf{X}^h(\mathbf{u}_\varepsilon^h) \to \mathbf{X}(\mathbf{u}_\varepsilon)$ in $\mathbf{H_0}(\mathbf{curl}; \widehat{\Omega})$

Optimal shape design: Continuous setting

Set of admissible shapes

$$\mathcal{U} := \{\alpha \in C(\overline{\omega}) \mid \alpha_{\mathrm{l}} \leq \alpha(\mathbf{x}) \leq \alpha_{\mathrm{u}} \text{ and } |\alpha(\mathbf{x}) - \alpha(\mathbf{y})| \leq C_{\mathrm{L}} \|\mathbf{x} - \mathbf{y}\|\}$$

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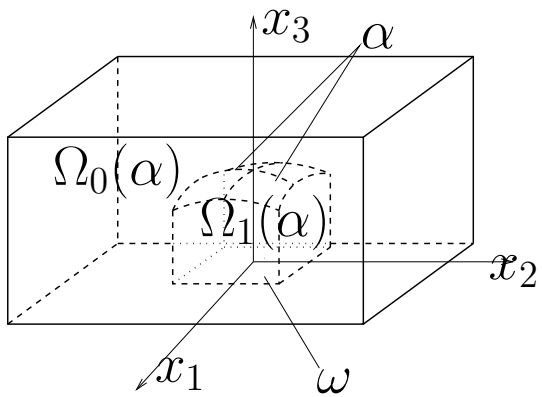

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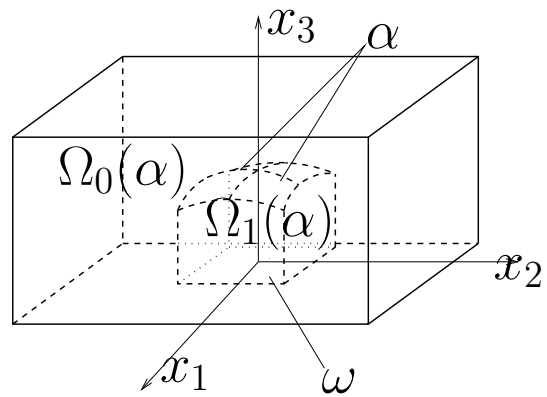
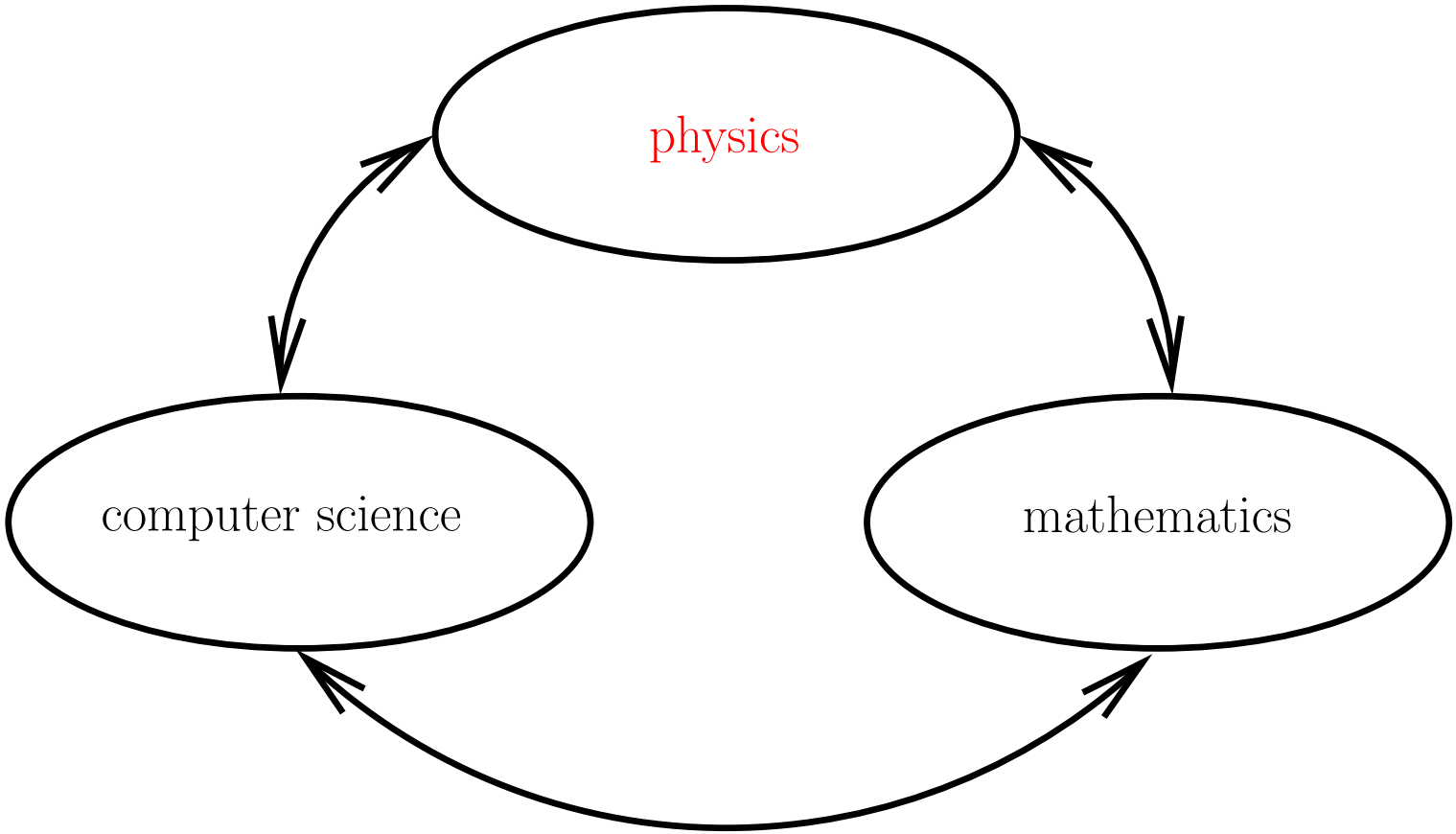


Multistate problem

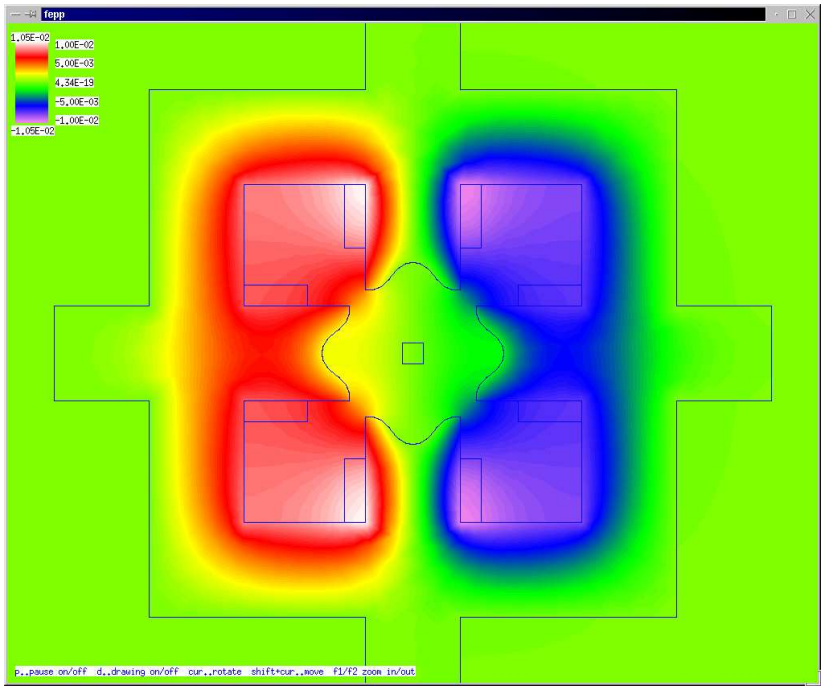
$$(W^v(\alpha)) \begin{cases} \text{Find } \mathbf{u}^v(\alpha) \in \mathbf{H}_{\mathbf{0},\perp}(\mathbf{curl};\Omega) : \\ \displaystyle\int_{\Omega_0(\alpha)} \frac{1}{\mu_0}\mathbf{curl}(\mathbf{u}^v(\alpha)) \cdot \mathbf{curl}(\mathbf{v})\, d\mathbf{x} + \int_{\Omega_1(\alpha)} \frac{1}{\mu_1}\mathbf{curl}(\mathbf{u}^v(\alpha)) \cdot \mathbf{curl}(\mathbf{v})\, d\mathbf{x} = \\ \displaystyle = \int_{\Omega} \mathbf{J}^v \cdot \mathbf{v}\, d\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H}_{\mathbf{0},\perp}(\mathbf{curl};\Omega) \end{cases}$$

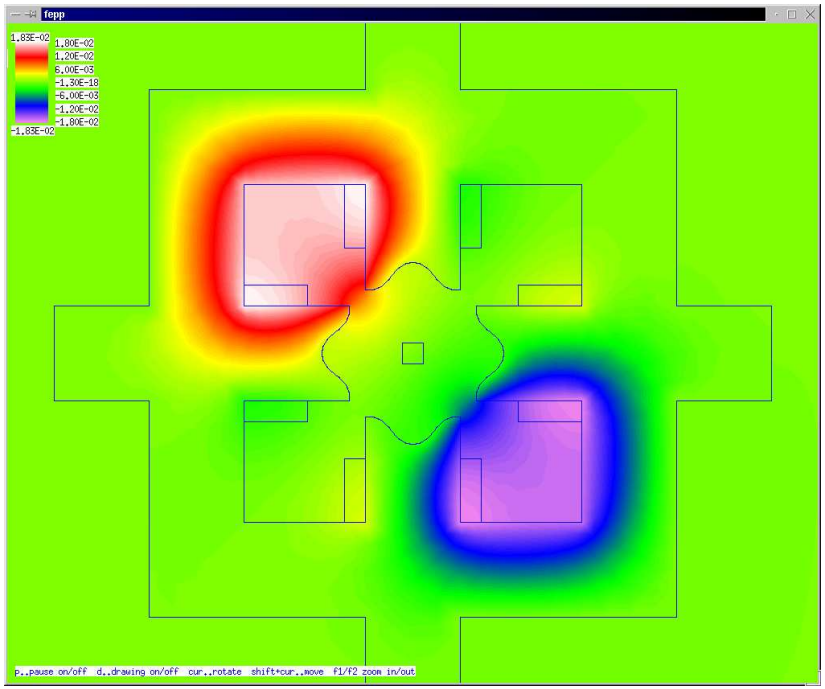
Outline: On the road



Maltese cross electromagnet: Multistate problem

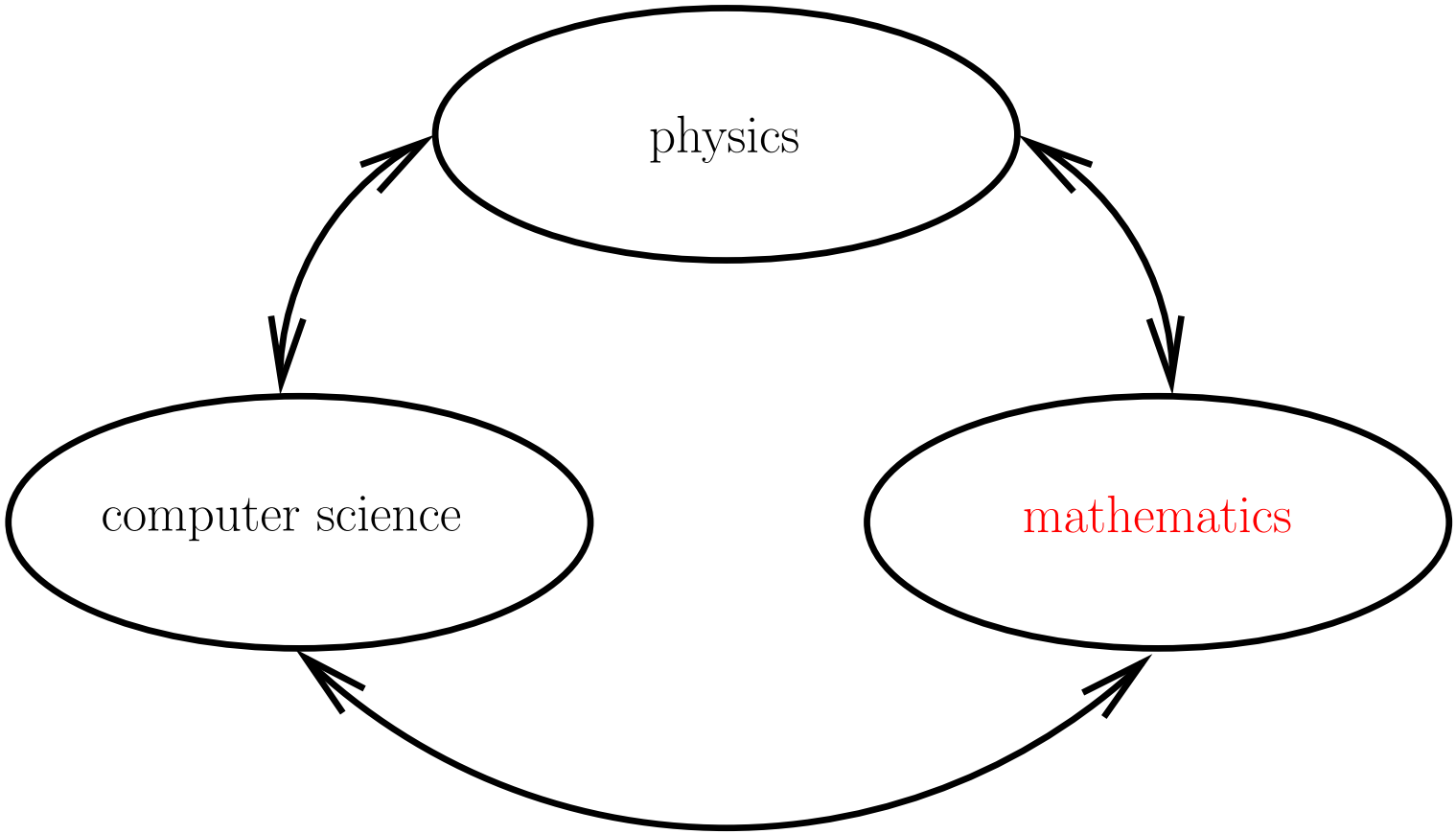
Vertical and diagonal current excitations





5 Amperes, 600 turns, relative permeability of AREMA = 5100

Outline: On the road



Optimal shape design: Continuous setting

Continuous cost functional

$$\mathcal{J}(\alpha) := \mathcal{I}(\alpha, \mathbf{curl}(u^1(\alpha)), \ldots, \mathbf{curl}(\mathbf{u}^{n_{\mathrm{v}}}(\alpha))), \text{ where } \mathbf{u}^v(\alpha) \text{ is the solution to } (W^v(\alpha))$$

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Optimization problem

$$(P) \begin{cases} \text{Find } \alpha^* \in \mathcal{U} : \\ J(\alpha^*) \leq J(\alpha) \quad \forall \alpha \in \mathcal{U} \end{cases}, \quad \text{there exists } \alpha^* \text{ a solution to } (P)$$

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Shape parameterization

- $\boldsymbol{\Upsilon} \subset \mathbb{R}^{n_\Upsilon}$ a compact subset
- $F : \boldsymbol{\Upsilon} \mapsto \mathcal{U}$ a continuous mapping, e.g., Beziér parameterization

$$(\widetilde{P}) \begin{cases} \text{Find } \mathbf{p}^* \in \boldsymbol{\Upsilon} : \\ [J \circ F](\mathbf{p}^*) \leq [J \circ F](\mathbf{p}) \quad \forall \mathbf{p} \in \boldsymbol{\Upsilon} \end{cases}, \quad \text{there exists } \mathbf{p}^* \text{ a solution to } (\widetilde{P})$$

Optimal shape design: Existence and convergence analysis

Existence

$$\left.\begin{array}{l} \mathcal{U} \text{ compact} \\ \mathbf{u}^v : \mathcal{U} \mapsto \mathbf{H}_{\mathbf{0},\perp}(\mathbf{curl};\Omega) \text{ continuous} \\ \mathcal{I} : \mathcal{U} \times \left[\mathbf{L}^2(\Omega)\right]^{n_{\mathrm{v}}} \mapsto \mathbb{R} \text{ continuous} \end{array}\right\} \Rightarrow \exists \alpha^* \in \mathcal{U} \text{ a solution to } (P)$$

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Convergence

$$\left.\begin{array}{l}\exists \alpha^*, {\alpha_{\varepsilon_n}^{h_n}}^* \text{ solutions to } (P), (P_{\varepsilon_n}^{h_n})\\ \forall \alpha \in \mathcal{U} : \pi_\omega^{h_n}(\alpha) \rightrightarrows \alpha\\ \mathbf{u}_{\varepsilon_n}^{v,h_n} : \mathcal{U}^{h_n} \mapsto \mathbf{H}_{\mathbf{0},\perp}(\mathbf{curl};\Omega^{h_n})^{h_n} \text{ continuous}\\ \mathcal{I} : \mathcal{U} \times \left[\mathbf{L}^2(\Omega)\right]^{n_{\mathrm{v}}} \mapsto \mathbb{R} \text{ continuous}\end{array}\right\} \Rightarrow \exists \{n_k\}_{k=1}^{\infty} \subset \{n\}_{n=1}^{\infty} : {\alpha_{\varepsilon_{n_k}}^{h_{n_k}}}^* \rightrightarrows \alpha^*$$

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Bottleneck

For fine discretizations it is hard to find a continuous shape–to–mesh mapping!

Finite–dimensional optimization

$$(\widetilde{P}) \begin{cases} \min\limits_{\mathbf{p}\in\mathbb{R}^{n_\Upsilon}} \widetilde{\mathcal{J}}(\mathbf{p}) \\ \text{subject to } \boldsymbol{v}(\mathbf{p}) \leq \mathbf{0} \end{cases}$$

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Structure of $\widetilde{\mathcal{J}}$

$\mathbf{p}$

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$$\mathbf{p} \xrightarrow{\pi_\omega^h \circ F} \boldsymbol{\alpha}^h$$

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Structure of $\widetilde{\mathcal{J}}$

$$\mathbf{p} \xrightarrow{\pi_\omega^h \circ F} \boldsymbol{\alpha}^h \xrightarrow{\text{linear elasticity}} \mathbf{x}^h$$

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Structure of $\widetilde{\mathcal{J}}$

$$\mathbf{p} \xrightarrow{\pi_\omega^h \circ F} \boldsymbol{\alpha}^h \xrightarrow{\text{linear elasticity}} \mathbf{x}^h \xrightarrow{\text{FEM}} A_\varepsilon^n, f^{v,n}$$

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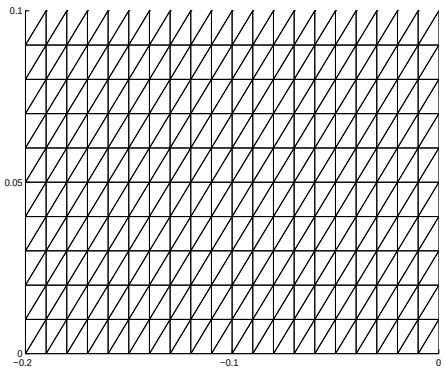
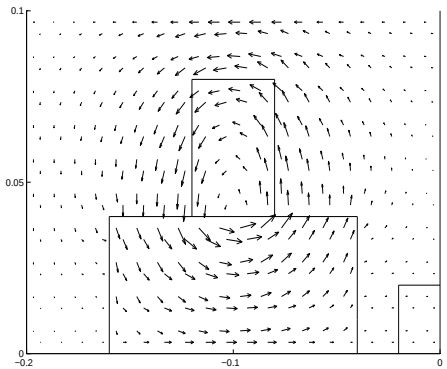
Shape–to–mesh elasticity mapping: initial design





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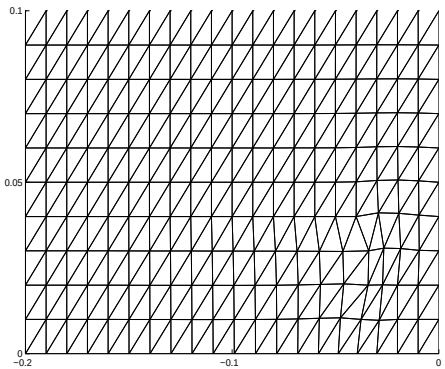
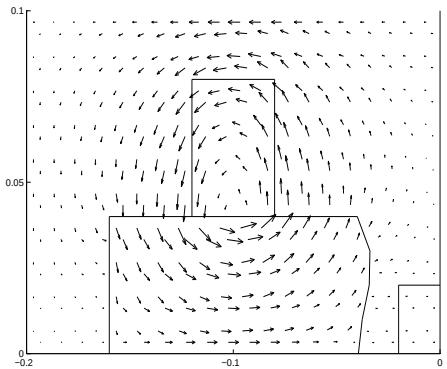
Structure of $\widetilde{\mathcal{J}}$

$$\mathbf{p} \xrightarrow{\pi_\omega^h \circ F} \boldsymbol{\alpha}^h \xrightarrow{\text{linear elasticity}} \mathbf{x}^h \xrightarrow{\text{FEM}} A_\varepsilon^n, f^{v,n} \xrightarrow{A_\varepsilon^n \cdot u_\varepsilon^{v,n} = f^{v,n}} u_\varepsilon^{v,n} \xrightarrow{\mathbf{curl}}$$

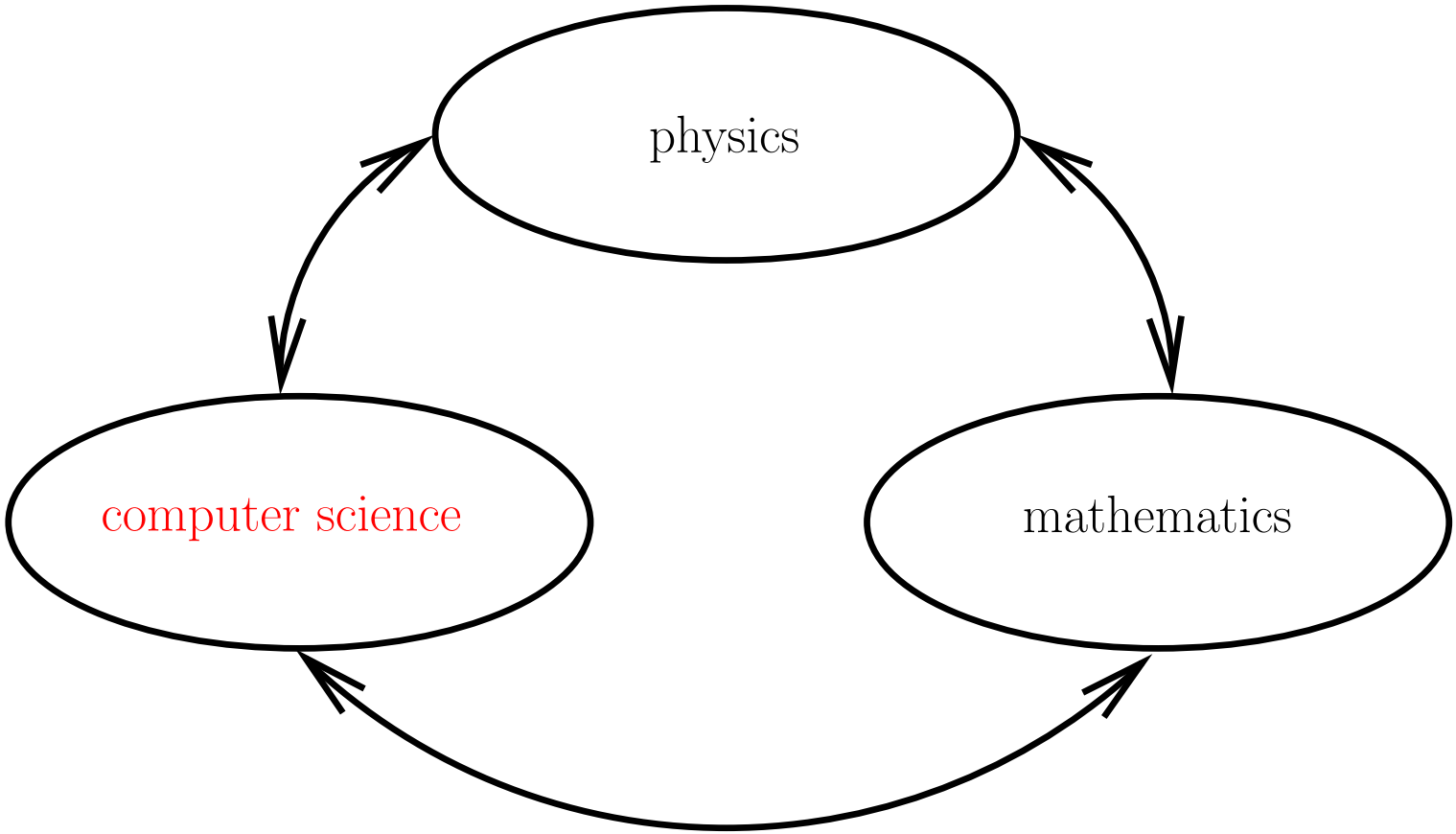
$$\xrightarrow{\mathbf{curl}} B_\varepsilon^{v,n} \xrightarrow{\mathcal{I}^h\left(\boldsymbol{\alpha}^h, \mathbf{x}^h, B_\varepsilon^{1,n}, \ldots, B_\varepsilon^{n_{\mathrm{v}},n}\right)} \widetilde{\mathcal{J}}_\varepsilon^h(\mathbf{p})$$

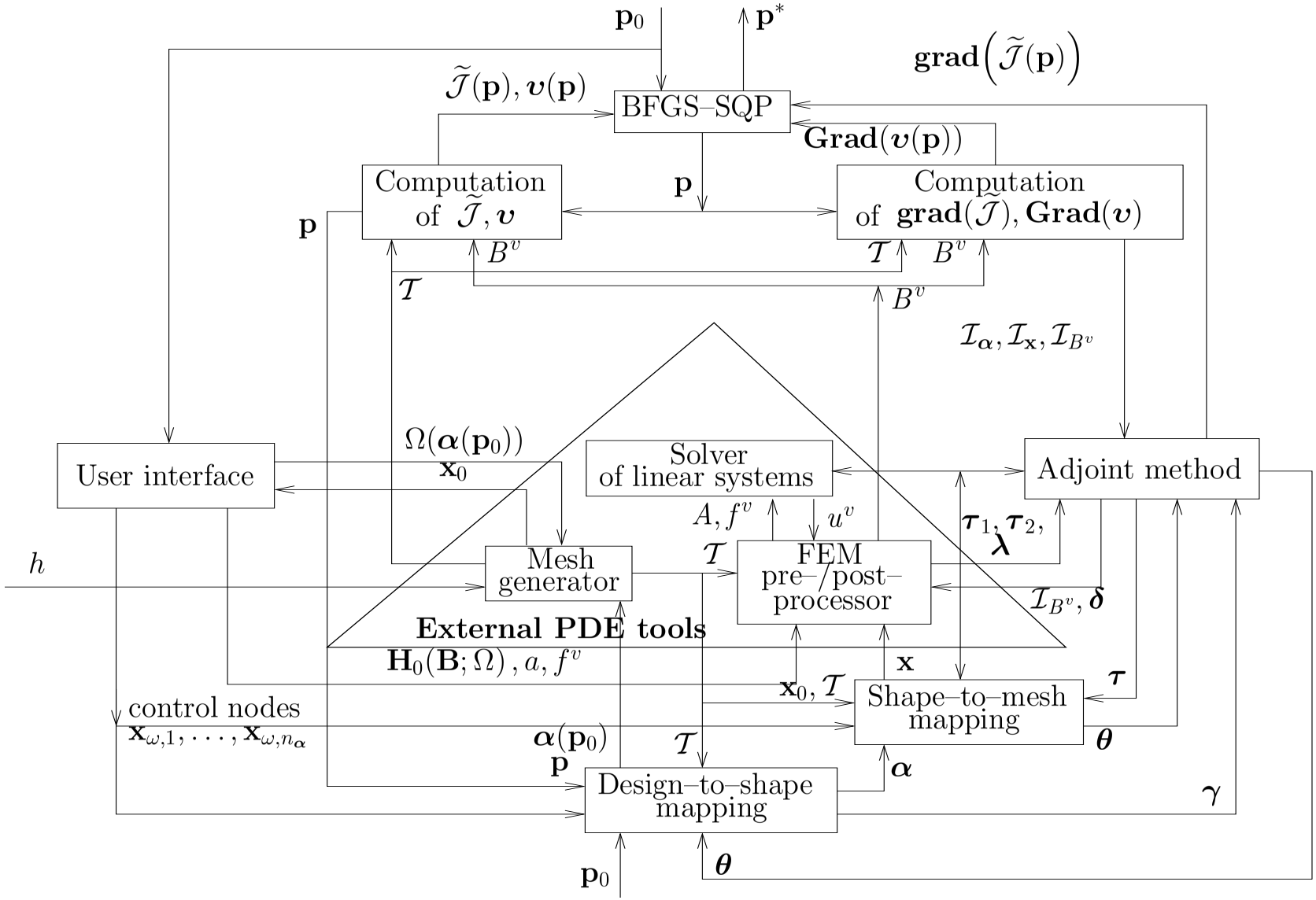
Shape–to–mesh elasticity mapping: deformed design





Outline: On the road




$\mathbf{p}_0$
$\mathbf{p}^*$
$\mathbf{grad}\left(\widetilde{\mathcal{J}}(\mathbf{p})\right)$
$\widetilde{\mathcal{J}}(\mathbf{p}), \boldsymbol{v}(\mathbf{p})$
BFGS–SQP
$\mathbf{Grad}(\boldsymbol{v}(\mathbf{p}))$
Computation of $\widetilde{\mathcal{J}}, \boldsymbol{v}$
$\mathbf{p}$
Computation of $\mathbf{grad}(\widetilde{\mathcal{J}}), \mathbf{Grad}(\boldsymbol{v})$
B^v
$\mathcal{T}$
$\mathcal{I}_{\boldsymbol{\alpha}}, \mathcal{I}_{\mathbf{x}}, \mathcal{I}_{B^v}$
$\Omega(\boldsymbol{\alpha}(\mathbf{p}_0))$
$\mathbf{x}_0$
User interface
Solver of linear systems
Adjoint method
A, f^v
u^v
$\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\lambda}$
h
Mesh generator
FEM pre–/post–processor
$\mathcal{I}_{B^v}, \boldsymbol{\delta}$
External PDE tools
$\mathbf{H}_0(\mathbf{B};\Omega), a, f^v$
$\mathbf{x}$
$\mathbf{x}_0, \mathcal{T}$
$\boldsymbol{\tau}$
Shape–to–mesh mapping
control nodes
$\mathbf{x}_{\omega,1}, \ldots, \mathbf{x}_{\omega,n_{\boldsymbol{\alpha}}}$
$\boldsymbol{\alpha}(\mathbf{p}_0)$
$\boldsymbol{\theta}$
$\boldsymbol{\alpha}$
Design–to–shape mapping
γ
$\boldsymbol{\theta}$
$\mathbf{p}_0$

Scientific software development

Numerical methods

- mesh generation

Software (SFB F013)

- NETGEN (by J. Schöberl)

Scientific software development

Numerical methods

- mesh generation
- FEM approximation using edge elements

Software (SFB F013)

- NETGEN (by J. Schöberl)
- FEPP (by J. Schöberl et al.)

Scientific software development

Numerical methods

- mesh generation
- FEM approximation using edge elements
- multigrid PCG solver

Software (SFB F013)

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Scientific software development

Numerical methods

- mesh generation
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- SQP with BFGS update of the Hessian

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Scientific software development

Numerical methods	Software (SFB F013)
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- gradients by finite differences	
or by the adjoint variable method	- FEPP sensitivity analysis module (by D.L.)

Scientific software development

Numerical methods	**Software (SFB F013)**
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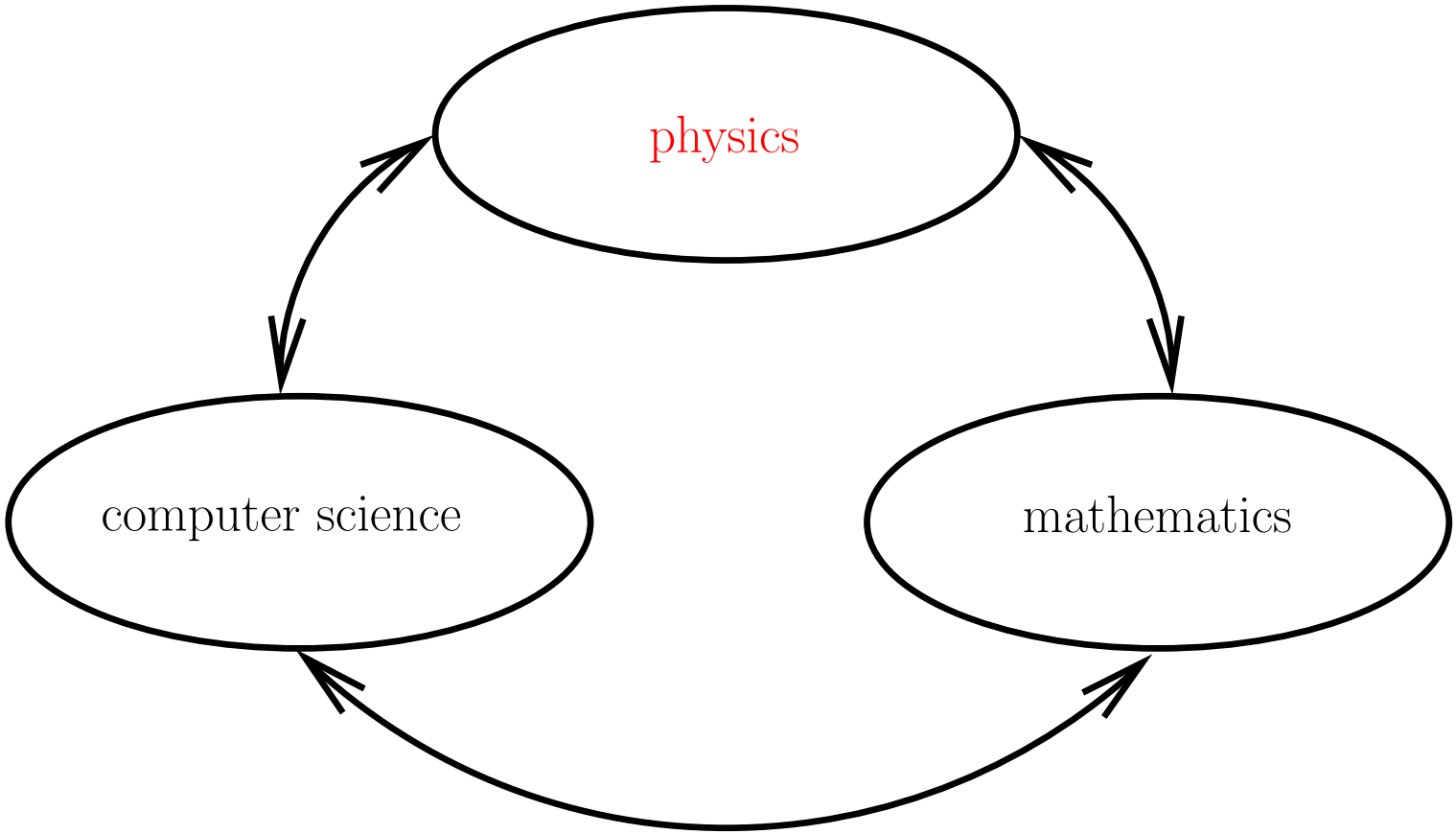
NETGEN + FEPP + PEBBLES = NgSolve

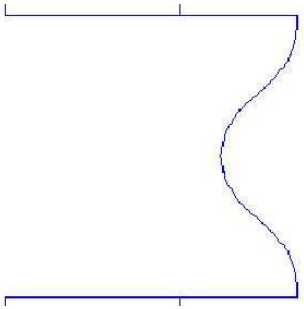
see http://www.hpfem.jku.at

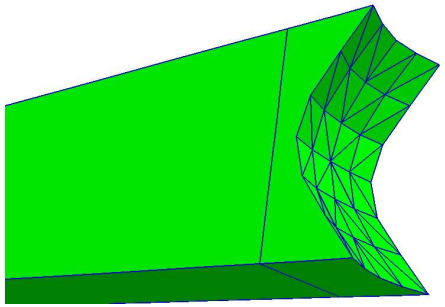
Outline: On the road



Maltese cross electromagnet

Optimized pole heads





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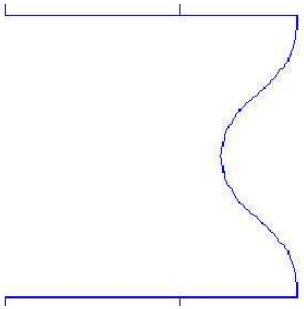
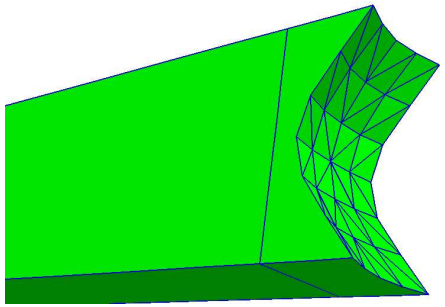




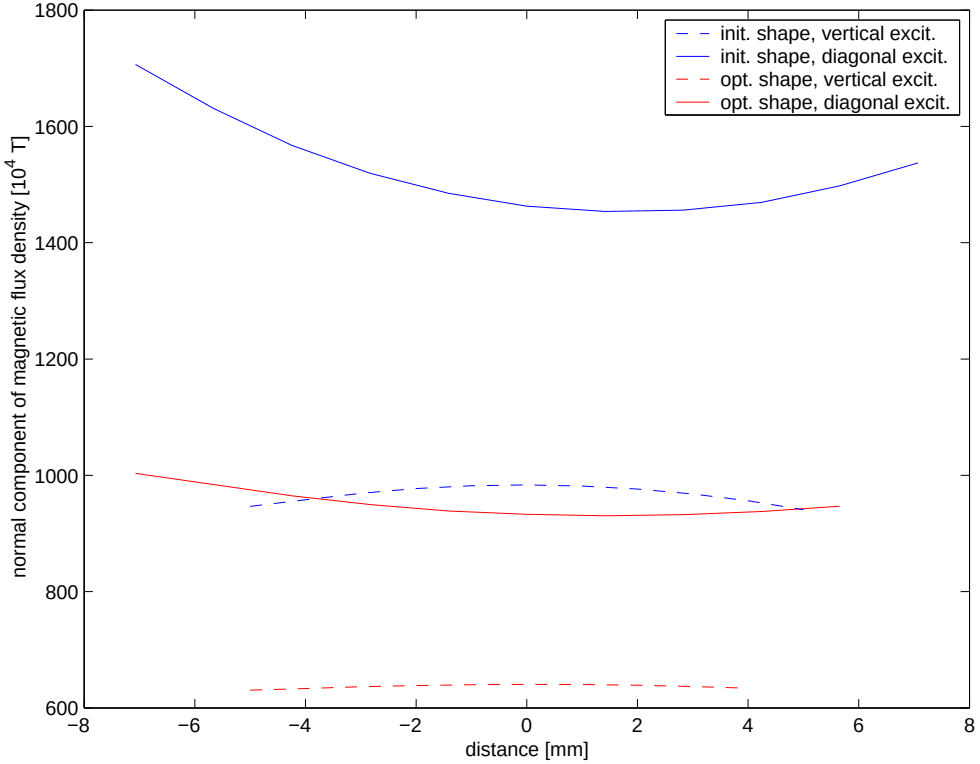
Parameters

design variables	7	12
deg. of freedom	12272	29541
SQP iterations	72	93
cost func. decrease	$1.97.10^{-6}$ to $1.49.10^{-6}$	$2.57.10^{-6}$ to $7.32.10^{-7}$
comput. time	2 hours	30 hours

Maltese cross electromagnet

Manufacture and measurements

The calculated cost functional has improved twice and the measured cost functional has improved even 4.5–times.



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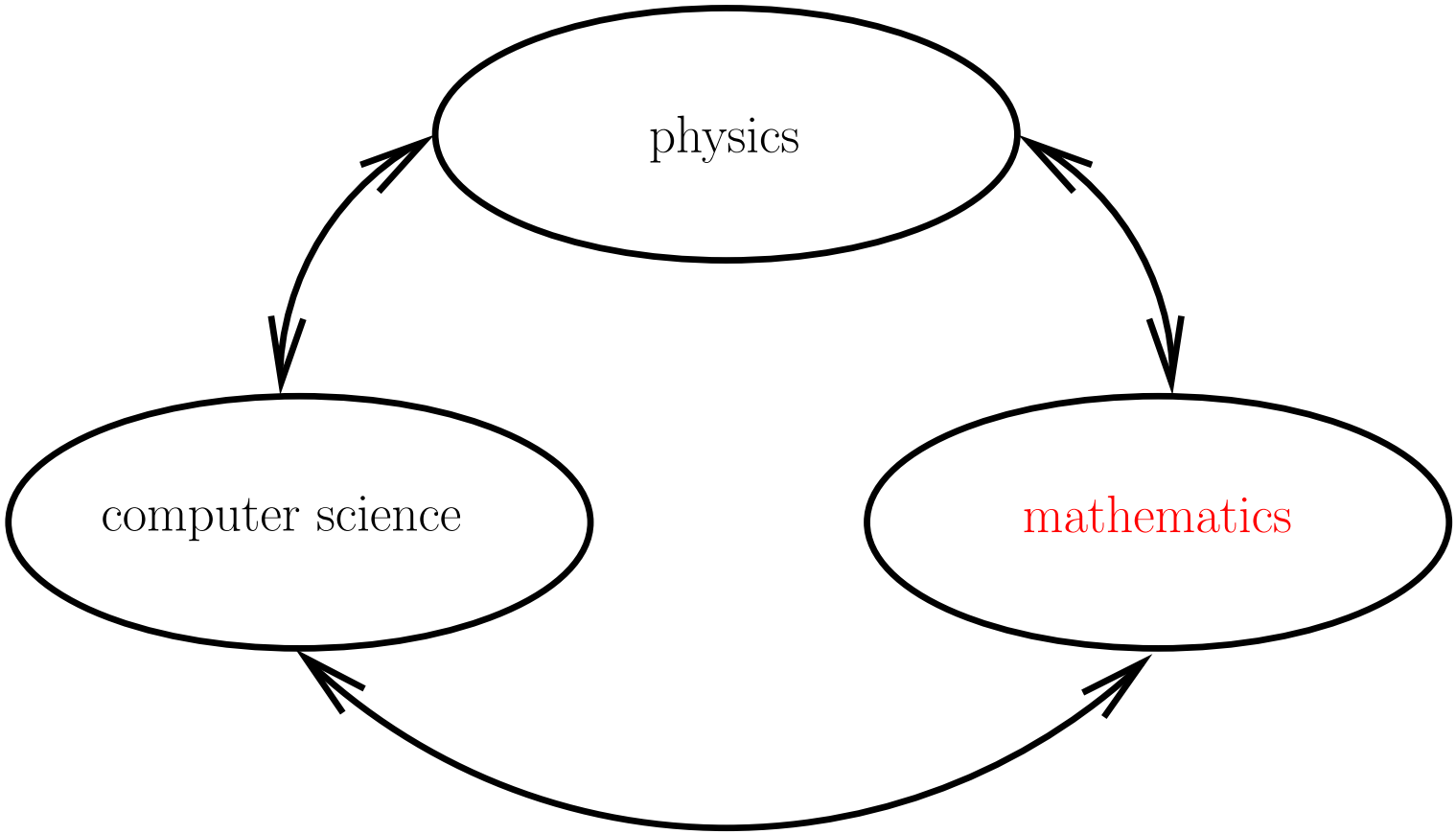
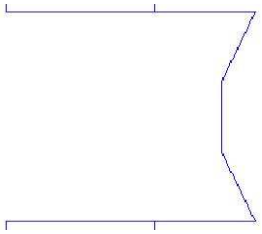


General multilevel approach

optimized designs	# of des. variables	# of unknowns	SQP iters.	CPU time
	2	2777	6	44s

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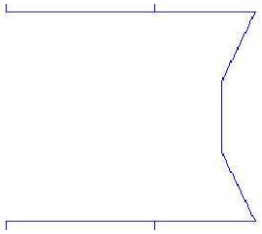
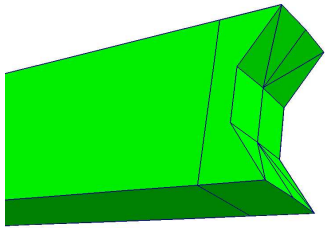
	2	2777	6	44s
	4	12086	56	7h 27m

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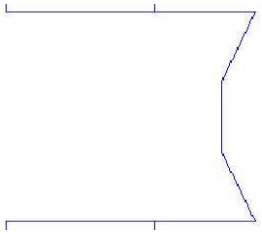
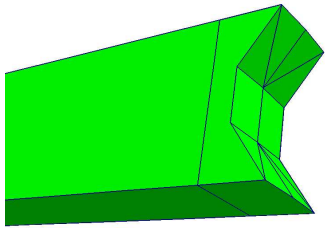
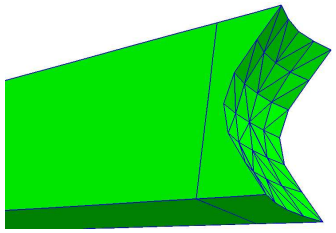
	2	2777	6	44s
	4	12086	56	7h 27m
	12	29541	93	29h 46m

Contributions of the thesis

Physics/electrical engineering

- development and manufacture of an electromagnet

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- analysis of optimal shape design problems governed by 3d linear magnetostatics
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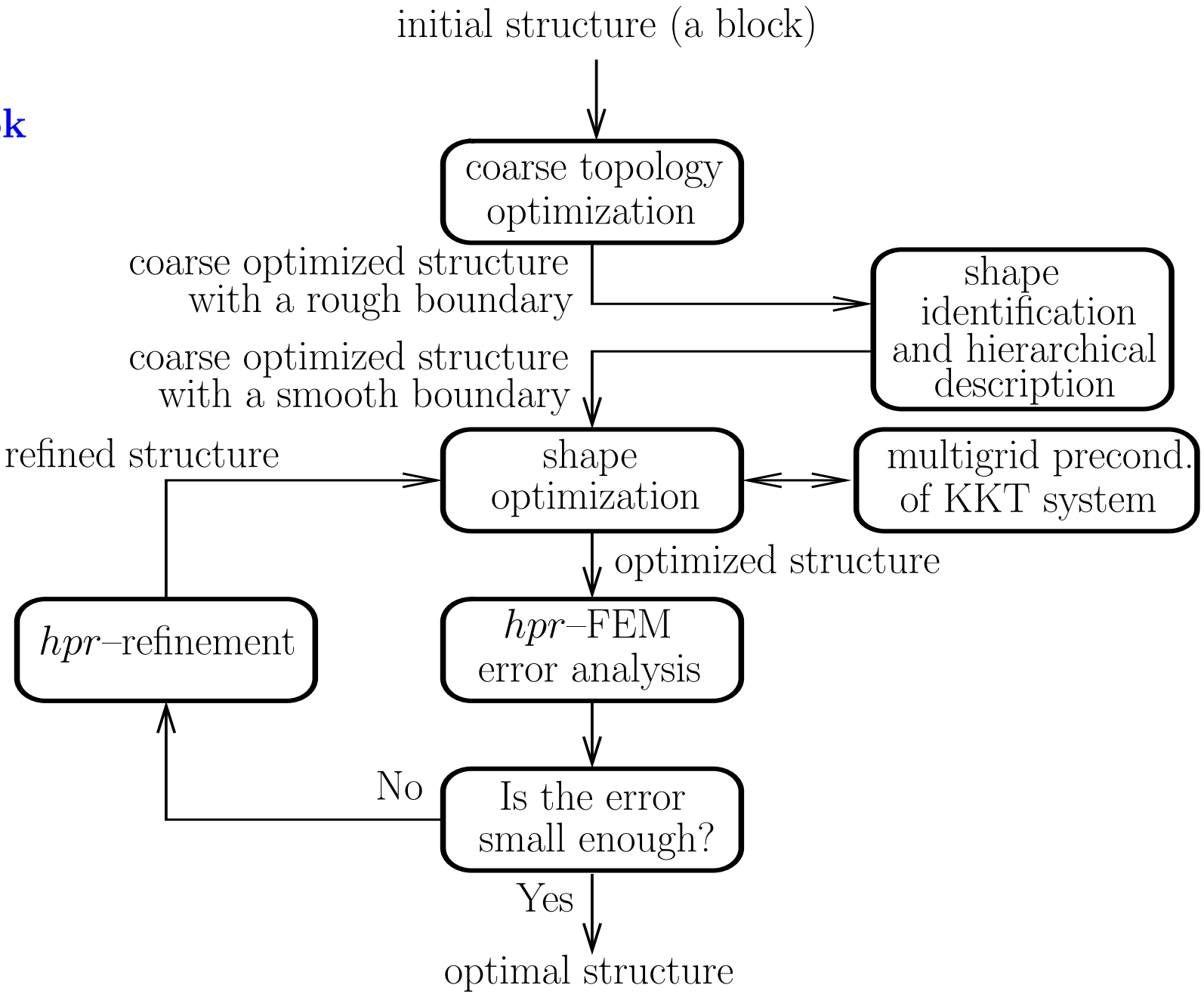
Education

- MATLAB tutorial code for 2d shape optimization (IMAMM '03)
- the thesis can serve as lecture notes

Outlook


initial structure (a block)
coarse topology optimization
coarse optimized structure with a rough boundary
shape identification and hierarchical description
coarse optimized structure with a smooth boundary
refined structure
shape optimization
multigrid precond. of KKT system
optimized structure
hpr–refinement
hpr–FEM error analysis
No
Is the error small enough?
Yes
optimal structure

Outlook

- Adaptive structural optimization
  - common aspects of topology and shape optimization

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  - smooth hierarchical parameterization of rough interfaces

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- Scientific software development
- Attracting some diploma students, industrial partners; cooperations abroad

Publications by Dalibor Lukáš

- L. - Shape optimization of homogeneous electromagnets. Lect. Notes Comp. Sci. Eng. 18, 2001.
- L., Kopřiva - Shape optimization of homogeneous electromagnets and their applications for measurements of magneto–optical effects. Rec. of COMPUMAG 2001.
- L., Mühlhuber, Kuhn - An object-oriented library for the shape optimization problems governed by systems of linear elliptic PDEs. Trans. of TU Ostrava 2001.
- L. - On the road – between Sobolev spaces and a manufacture of electromagnets. To appear in Trans. of TU Ostrava.
- L. - On solution to an optimal shape design problem in 3–dimensional linear magnetostatics. To appear in Appl. Math.
- L., Ciprian, Pištora, Postava, Foldyna - Multilevel solvers for 3–dimensional optimal shape design with an application to magneto–optics. To appear in SCIE.