A Boundary Element Method for Homogenization of Periodic Structures *

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Abstract

Homogenized coefficients of periodic structures are calculated via an auxiliary partial differential equation in the periodic cell. Typically a volume finite element discretization is employed for the numerical solution. In this paper we reformulate the problem as a boundary integral equation using Steklov–Poincaré operators. The resulting boundary element method only discretizes the boundary of the periodic cell and the interface between the materials within the cell. We prove that the homogenized coefficients converge super-linearly with the mesh size and we support the theory with examples in 2 and 3 dimensions.

Keywords: homogenization, boundary element method

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1 Introduction

Solving a boundary value problem which involves materials with composite microstructure is computationally demanding. Therefore, we look for homogeneous (constant) material coefficients imitating the original microstructure so that the solution to the original problem with a highly oscillating material function is in a sense close to the solution of a related problem with the constant material function. There are two well-known approaches to homogenization. In both approaches auxiliary boundary value problems in the so-called representative volume element (RVE) are solved. In the first approach the geometry of the microstructure does not need to be periodic and a large portion of the domain has to be covered in the auxiliary problems. In such cases energy methods [Tar09] are often employed. The methods provide effective coefficients that preserve the energy of the solution for a fixed type of boundary conditions. The second approach, which our paper actually deals with, assumes a periodic microstructure. As stated in [All92], energy methods do not take full advantage of the periodicity. The homogenized coefficients are instead determined by solutions to auxiliary problems with periodic boundary conditions. For the presentation of mathematical theory and methods for the periodic case we refer to [CD99].

As far as numerical methods for the solution to the auxiliary problems are concerned, volume discretization techniques such as finite element methods prevail in literature, cf. [RL10]. In this paper we consider boundary element methods (BEM) that rely on the fundamental solution for a given differential operator. Although BEM is advantageous due to the reduction in dimension, the resulting linear system matrices are densely populated due to the non-local nature of the fundamental solution and related integral operators. Fortunately, sparsification techniques such as the fast multipole methods (FMM) [Rok85] or adaptive cross approximation (ACA) [Beb00] reduce the computational complexity of a matrix action to almost linear. FMM relies on a hierarchical clustering of the geometry and a low-rank approximation of the fundamental solution for well-separated clusters. For periodic materials similar expansions have been constructed by an additional summation over periodic cells. As a result an FMM-BEM can be applied to the RVE including a large amount of periodic cells [ET93, GH98, YQ04]. A different BEM method is proposed in [Grz10], where the RVE is just one periodic cell and the homogeneous coefficients are computed by the energy method with prescribed constant elastic strains.

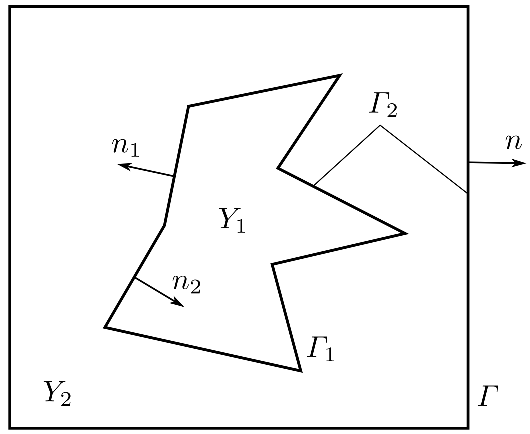


Figure 1: Periodic cell, notation.

In this paper we propose a BEM different from [Grz10] as we make use of the periodic structure in the spirit of [All92, BLP78, CD99]. Besides numerical experiments we provide an existence and convergence analysis of our boundary integral formulation and its boundary element counterparts, respectively. We consider the involved Dirichlet-to-Neumann maps as systems of boundary integral equations. Our analysis of the well-posedness of the auxiliary homogenization problems rely on inf-sup stability, as the bilinear forms are not elliptic. Here additional arguments are required as a multiply-connected domain is present. To our best knowledge this case has not been treated in literature yet. Note that the proposed homogenization problems are similar to those arising in boundary element tearing and interconnecting methods [LS03, LOS07].

We consider a scalar elliptic boundary value problem for the equation

$$-\mathrm{div}(a_\varepsilon(x)\,\nabla u_\varepsilon(x)) = f(x), \quad x \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a Lipschitz domain, f is a source term, $\varepsilon > 0$ is a geometrical period, and a_ε is a periodic material function. Note that (1) can be completed by usual types of linear boundary conditions, cf. [BLP78]. Let us adopt the notation of Fig. 1. We consider the periodic cell $Y := (0,1)^d$ to be decomposed into a simply connected Lipschitz inclusion Y_1 and the complement $Y_2 := Y \setminus \overline{Y_1}$. The respective boundaries shall be denoted by $\Gamma := \partial Y$, $\Gamma_1 := \partial Y_1$, and $\Gamma_2 := \partial Y_2 = \Gamma \cup \Gamma_1$. The material function is determined by $a_\varepsilon(x) := a(x/\varepsilon)$, where we define the Y-periodic function

$$a(y) := \begin{cases} a_1, \ y \in Y_1, \\ a_2, \ y \in Y_2, \end{cases}$$

with $a_1, a_2 > 0$. It is well-known, cf. [BLP78, CD99], that solutions u_ε of (1) converge weakly (as $\varepsilon \to 0_+$) in $H^1(\Omega)$ to the solution u_0 of the homogenized problem

$$-\mathrm{div}(\mathbf{A}_0 \, \nabla u_0(x)) = f(x), \quad x \in \Omega$$

completed with the same boundary conditions. The homogenized coefficients $\mathbf{A}_0 \in \mathbb{R}^{d\times d}$ are given by

$$(\mathbf{A}_0)_{ik} := \int_Y a(y) \left(\delta_{ik} - \frac{\partial \widetilde{\chi}^k}{\partial y_i}(y) \right) \mathrm{d}y \tag{2}$$

where δ_{ik} denotes the Kronecker delta and $\widetilde{\chi}^k$, $k = 1, \ldots, d$, are Y-periodic solutions to the auxiliary problems

$$-\mathrm{div}\left(a(y) \, \nabla \widetilde{\chi}^k(y)\right) = -\frac{\partial a}{\partial y_k}(y), \quad y \in Y, \, k = 1, \ldots, d$$

that is understood in the weak sense:

$$\begin{cases} \text{Find } \widetilde{\chi}^k \in H^1_{\mathrm{per}}(Y)\colon \\ \underbrace{\displaystyle\int_Y a(y) \, \nabla \widetilde{\chi}^k(y) \cdot \nabla \widetilde{v}(y) \, \mathrm{d}y}_{=:\widetilde{a}(\widetilde{\chi}^k, \widetilde{v})} = \displaystyle\int_Y a(y) \, \frac{\partial \widetilde{v}}{\partial y_k}(y) \, \mathrm{d}y \quad \forall \widetilde{v} \in H^1_{\mathrm{per}}(Y). \end{cases} \tag{3}$$

The space $\widetilde{\mathcal{V}} := H^1_{\mathrm{per}}(Y)$ comprises Y-periodic functions from $H^1(Y)$.

Finally, we recall the sense in which problem (3) is well-posed. Obviously, both the bilinear and linear forms are continuous on $H^1(Y)$. However, since $\widetilde{a}(1, 1) = 0$ the bilinear form $\widetilde{a}(\cdot, \cdot)$ is not elliptic on whole $\widetilde{\mathcal{V}}$. Owing to the Poincaré inequality, there exists some $\widetilde{c}_{\mathrm{P}} := \widetilde{c}_{\mathrm{P}}(Y_2) > 0$ such that

$$\int_Y |\nabla \widetilde{v}(y)|^2 \, \mathrm{d}y + \left(\int_{\Gamma_2} \widetilde{v}(y) \, \mathrm{d}s(y) \right)^2 \geq \widetilde{c}_{\mathrm{P}} \int_Y \widetilde{v}^2(y) \, \mathrm{d}y \quad \forall \, \widetilde{v} \in H^1(Y), \tag{4}$$

and $\widetilde{a}(\cdot, \cdot)$ is elliptic on the subspace $\widetilde{\mathcal{U}} := \left\{ \widetilde{v} \in \widetilde{\mathcal{V}} \colon \int_{\Gamma_2} \widetilde{v}(y) \, \mathrm{d}s(y) = 0 \right\}$, namely,

$$\widetilde{a}(\widetilde{v}, \widetilde{v}) \geq \underbrace{\min\{a_1, a_2\} \, \frac{\widetilde{c}_{\mathrm{P}}}{1 + \widetilde{c}_{\mathrm{P}}}}_{=:\widetilde{c}} \, \|\widetilde{v}\|^2_{H^1(Y)} \quad \forall \, \widetilde{v} \in \widetilde{\mathcal{U}}. \tag{5}$$

Therefore, by the Lax-Milgram theorem [Stei08, Th. 3.4], auxiliary problem (3) is well-posed on $\widetilde{\mathcal{U}}$.

The rest of the paper is organized as follows: In Section 2 we present a direct boundary integral formulation of the auxiliary problem and prove its well-posedness and equivalence to (3). In Section 3 we give a stable boundary element discretization. Super-linear convergence of the discretized homogenized coefficients is proven in Section 4 by means of the Aubin–Nitsche trick. In Section 5 we verify the theory on numerical examples in 2 and 3 dimensions. We conclude in Section 6.

2 Boundary integral formulation

We shall arrive at a boundary integral formulation of (3). Referring to Fig. 1 we denote by n_1 and $n_2 = -n_1$ the exterior unit normal vectors of Y_1 and Y_2, respectively. We denote the Γ_i-trace and the Γ-trace of a function $\widetilde{v} \in H^1(Y)$ by v_i and v, respectively. Throughout the paper quantities overset with a tilde are related to the domains, while the others are related to the boundaries. By the Gauss theorem, we can evaluate the homogenized coefficients (2) from χ_1^k, the Γ_1-trace of $\widetilde{\chi}^k \in \widetilde{\mathcal{V}} = H^1_{\mathrm{per}}(Y)$,

$$(\mathbf{A}_0)_{ik} = \delta_{ik} \left\{ a_2 + (a_1 - a_2) \int_{\Gamma_1} y_i \; (n_1(y))_i \; \mathrm{d}s(y) \right\} \\ - \underbrace{(a_1 - a_2) \int_{\Gamma_1} \chi_1^k(y) \; (n_1(y))_i \; \mathrm{d}s(y)}_{=:b^i(\chi_1^k)} . \quad (6)$$

Similarly, the right-hand side of (3) can be reduced to Γ_1 using the Gauss theorem and the Γ-periodicity of $\widetilde{v}$

$$\int_Y a \frac{\partial \widetilde{v}}{\partial y_k} \mathrm{d}y = a_1 \int_{\Gamma_1} v_1 \, (n_1)_k \, \mathrm{d}s(y) + a_2 \int_{\Gamma_2} v_2 \, (n_2)_k \, \mathrm{d}s(y) \\ = (a_1 - a_2) \int_{\Gamma_1} v_1 \, (n_1)_k \, \mathrm{d}s(y). \quad (7)$$

In order to derive a boundary integral version of (3), we consider restrictions and traces of $\chi^k \in H^1_{\mathrm{per}}(Y)$

$$\widetilde{\chi}_1^k := \widetilde{\chi}^k_{|_{Y_1}}, \quad \widetilde{\chi}_2^k := \widetilde{\chi}^k_{|_{Y_2}}, \quad \chi_1^k := \widetilde{\chi}^k_{|_{\Gamma_1}} \in H^{1/2}(\Gamma_1), \quad \chi_2^k := \widetilde{\chi}^k_{|_{\Gamma_2}} \in H^{1/2}(\Gamma_2).$$

Note that $\chi^k_{2|\Gamma_1} = \chi^k_1$ and that $\widetilde{\chi}^k_i \in H^1(Y_i)$ is the harmonic extension of χ^k_i, i.e.,

$$\int_{Y_i} \nabla \widetilde{\chi}^k_i \cdot \nabla \widetilde{\varphi}_i \, \mathrm{d}y = 0 \quad \forall \widetilde{\varphi}_i \in H^1_0(Y_i). \tag{8}$$

Using a splitting $\widetilde{v} = \widetilde{\mathcal{E}} v_2 + \widetilde{\varphi}_1 + \widetilde{\varphi}_2$ for some continuous extension $\widetilde{\mathcal{E}} v_2$ of any $v_2 \in H^{1/2}(\Gamma_2)$ and the harmonic extensions (8), we can reduce the bilinear form in (3) to

$$\begin{aligned} \int_Y a \, \nabla \widetilde{\chi}^k \cdot \nabla \widetilde{v} \, \mathrm{d}y &= a_1 \int_{Y_1} \nabla \widetilde{\chi}^k_1 \cdot \nabla \widetilde{\mathcal{E}} v_2 \, \mathrm{d}y + a_2 \int_{Y_2} \nabla \widetilde{\chi}^k_2 \cdot \nabla \widetilde{\mathcal{E}} v_2 \, \mathrm{d}y \\ &= a_1 \, \langle S_1 \chi^k_1, v_{2|\Gamma_1} \rangle_{\Gamma_1} + a_2 \, \langle S_2 \chi^k_2, v_2 \rangle_{\Gamma_2} \qquad \forall v_2 \in H^{1/2}(\Gamma_2), \end{aligned} \tag{9}$$

where the Steklov–Poincaré operators $S_i \colon H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ are defined as Dirichlet to Neumann maps by Green's formulae in Y_1 and Y_2. By $\langle \cdot, \cdot \rangle_{\Gamma_i}$ we denote the duality pairing between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$. We set χ^k_2 as (χ^k_1, χ^k) and arrive at a boundary integral formulation of (3):

$$\begin{cases} \text{Find } \left(\chi^k_1, \chi^k\right) \in \mathcal{V} \colon \ \forall (v_1, v) \in \mathcal{V} \colon \\ \underbrace{a_1 \, \langle S_1 \chi^k_1, v_1 \rangle_{\Gamma_1} + a_2 \, \langle S_2 (\chi^k_1, \chi^k), (v_1, v) \rangle_{\Gamma_2}}_{=: a((\chi^k_1, \chi^k), (v_1, v))} = \underbrace{(a_1 - a_2) \int_{\Gamma_1} v_1 \, (n_1)_k \, \mathrm{d}s}_{=: b^k(v_1)}, \end{cases} \tag{10}$$

where $\mathcal{V} := H^{1/2}(\Gamma_1) \times H^{1/2}_{\mathrm{per}}(\Gamma)$. The space $H^{1/2}_{\mathrm{per}}(\Gamma)$ contains Γ-traces of functions from $H^1_{\mathrm{per}}(Y)$. The problem (10) is not uniquely solvable which we will take care of in formulation (13). We equip $\mathcal{V}$ with the norm

$$\|(v_1, v)\|^2_{\mathcal{V}} := \|v_1\|^2_{1/2, \Gamma_1} + \|v\|^2_{1/2, \Gamma},$$

where we consider the Sobolev-Slobodeckii inner products and norms, e.g.,

$$\|v_1\|^2_{1/2, \Gamma_1} := \underbrace{\int_{\Gamma_1} (v_1(y))^2 \mathrm{d}s(y)}_{=: \|v_1\|^2_{0, \Gamma_1}} + \underbrace{\int_{\Gamma_1} \int_{\Gamma_1} \frac{[v_1(x) - v_1(y)]^2}{|x - y|^d} \, \mathrm{d}s(y) \, \mathrm{d}s(x)}_{=: |v_1|^2_{1/2, \Gamma_1}}.$$

Let us first show that S_2 can be applied to a function from $\mathcal{V}$.

Lemma 2.1. *Assume that*

$$\mathrm{dist}(\Gamma_1, \Gamma) > 0, \tag{11}$$

then the space $H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma)$ is isomorphic to $H^{1/2}(\Gamma_2)$. In particular, for all $v_2 \equiv (v_1, v) \in H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma)$:

$$
\begin{aligned}
&\|v_1\|^2_{1/2,\Gamma_1} + \|v\|^2_{1/2,\Gamma} \leq \|v_2\|^2_{1/2,\Gamma_2} \leq \\
&\qquad\qquad \left(1 + \frac{4 \max\{|\Gamma_1|, |\Gamma|\}}{\operatorname{dist}^d(\Gamma_1, \Gamma)}\right) \left(\|v_1\|^2_{1/2,\Gamma_1} + \|v\|^2_{1/2,\Gamma}\right), \qquad (12)
\end{aligned}
$$

where $|\Gamma_1|$, $|\Gamma|$ are the measures of the respective manifolds.

Proof. The isomorphism is the identity, i.e., for $v_2 \in H^{1/2}(\Gamma_2)$: $v_1 := v_2|_{\Gamma_1}$, $v := v_2|_{\Gamma}$. The first inequality in (12) is straightforward. The other one follows from (11) and simple manipulations,

$$
\begin{aligned}
\|v_2\|^2_{1/2,\Gamma_2} &= \|v_1\|^2_{1/2,\Gamma_1} + \|v\|^2_{1/2,\Gamma} + 2 \int_{\Gamma_1} \int_{\Gamma} \frac{|v_1(x) - v(y)|^2}{\|x - y\|^d} \,\mathrm{d}s(y)\,\mathrm{d}s(x) \\
&\leq \|v_1\|^2_{1/2,\Gamma_1} + \|v\|^2_{1/2,\Gamma} + \frac{4}{\operatorname{dist}^d(\Gamma_1, \Gamma)} \left(|\Gamma|\,\|v_1\|^2_{0,\Gamma_1} + |\Gamma_1|\,\|v\|^2_{0,\Gamma}\right) \\
&\leq \left(1 + \frac{4 \max\{|\Gamma_1|, |\Gamma|\}}{\operatorname{dist}^d(\Gamma_1, \Gamma)}\right) \left(\|v_1\|^2_{1/2,\Gamma_1} + \|v\|^2_{1/2,\Gamma}\right).
\end{aligned}
$$

□

Theorem 2.2. *Let $\widetilde{\chi}^k \in \widetilde{\mathcal{V}}$ be a solution to (3), then the pair of traces $(\chi_1^k, \chi^k) \in \mathcal{V}$ solves (10). On the other hand, let $(\chi_1^k, \chi^k) \in \mathcal{V}$ solve (10), then the extension $\widetilde{\chi}^k \in \widetilde{\mathcal{V}}$ satisfying (8) is a solution to (3).*

Proof. The main steps of the proof of the first statement were given in the derivation of (10) above. Details are similar to [QV99, Lemma 1.2.1]. There exist solutions $\widetilde{\chi}^k$ to (3), which are unique up to an additive constant.

On the other hand, $\widetilde{\chi}^k \in \widetilde{\mathcal{V}}$ are well-defined as harmonic extensions (8) of (χ_1^k, χ^k). To prove the second statement it is thus sufficient to show that a solution (χ_1^k, χ^k) to (10) is also unique up to an additive constant. Let $(\chi_1^k, \chi^k), (\xi_1^k, \xi^k) \in \mathcal{V}$ be two solutions of (10). Subtracting the two equations, choosing test functions $v_1 := \chi_1^k - \xi_1^k$, $v := \chi^k - \xi^k$, linearity and positive semidefiniteness of S_1, S_2 yield

$$\langle S_2(v_1, v), (v_1, v)\rangle_{\Gamma_2} = 0.$$

We complete the proof by $\operatorname{Ker} S_2 = \mathbb{R}$. □

Next we transfer (10) to a well-posed problem, see Theorem 2.5:

$$\begin{cases} \text{Find } (\chi_1^k, \chi^k) \in \mathcal{V}/\mathbb{R}: \\ a((\chi_1^k, \chi^k), (v_1, v)) = b^k(v_1) \quad \forall (v_1, v) \in \mathcal{V}/\mathbb{R}. \end{cases} \tag{13}$$

The quotient space is understood as the space of vectors $(v_1, v) \in \mathcal{V}$ that minimize the $\mathcal{V}$-norm in the equivalence class $\mathcal{C}_v := \{(v_1 + c, v + c) : \ c \in \mathbb{R}\}$. Here and in the following text we omit the indices for constant functions.

Lemma 2.3.

$$\mathcal{V}/\mathbb{R} = \left\{ (v_1, v) \in H^{1/2}(\mathit{\Gamma}_1) \times H^{1/2}_{\text{per}}(\mathit{\Gamma}) : \ \int_{\mathit{\Gamma}_1} v_1 \, \mathrm{d}s + \int_{\mathit{\Gamma}} v \, \mathrm{d}s = 0 \right\}.$$

Proof. For $(u_1, u) \in \mathcal{V}$ we shall find $v_1 := u_1 + c_u$ and $v := u + c_u$ such that the quadratic function $\varphi_{u_2}(c) := \|(u_1 + c, u + c)\|^2_{\mathcal{V}}$ is minimized over $c \in \mathbb{R}$. The minimum is uniquely attained at

$$c_{u_2} := -\frac{(u_1, 1)_{1/2,\mathit{\Gamma}_1} + (u, 1)_{1/2,\mathit{\Gamma}}}{\|1\|^2_{1/2,\mathit{\Gamma}_1} + \|1\|^2_{1/2,\mathit{\Gamma}}}.$$

Hence, (v_1, v) is the only element from $\mathcal{C}_u$ with vanishing minimizer c_v of φ_v,

$$0 = (v_1, 1)_{1/2,\mathit{\Gamma}_1} + (v, 1)_{1/2,\mathit{\Gamma}} = \int_{\mathit{\Gamma}_1} v_1 \, \mathrm{d}s + \int_{\mathit{\Gamma}} v \, \mathrm{d}s.$$

$\square$

Lemma 2.4. *Under assumption* (11) *the bilinear form* $a(\cdot, \cdot)$ *is bounded and elliptic on* $\mathcal{V}/\mathbb{R}$*, i.e., there exist* $C, c > 0$ *such that for all* $(u_1, u), (v_1, v) \in \mathcal{V}/\mathbb{R}$:

$$a((u_1, u), (v_1, v)) \le C \, \|(u_1, u)\|_{\mathcal{V}} \, \|(v_1, v)\|_{\mathcal{V}}$$

and

$$a((v_1, v), (v_1, v)) \ge c \, \|(v_1, v)\|^2_{\mathcal{V}}.$$

Proof. Both properties are inherited from the bilinear form $\widetilde{a}(\cdot, \cdot)$ of (3). First of all, the boundedness holds on the superspace $H^{1/2}(\mathit{\Gamma}_1) \times H^{1/2}(\mathit{\Gamma})$. Indeed, any couple of traces (v_1, v) can be extended by the harmonic extension (8) to $\widetilde{v} \in H^1(Y)$ such that

$$\|\widetilde{v}\|_{H^1(Y_1)} \le C_1^{\text{ext}} \, \|v_1\|_{1/2,\mathit{\Gamma}_1} \quad \text{and} \quad \|\widetilde{v}\|_{H^1(Y)} \le C^{\text{ext}} \, \|v\|_{1/2,\mathit{\Gamma}}$$

with positive constants C_1^{ext} and C^{ext} depending only on Y_1 and Y, respectively. Now the boundedness follows from (9)

$$\begin{aligned} a((u_1,u),(v_1,v)) = \widetilde{a}(\widetilde{u},\widetilde{v}) &\le \widetilde{C}\, \|\widetilde{u}\|_{H^1(Y)}\, \|\widetilde{v}\|_{H^1(Y)} \\ &\le \underbrace{\widetilde{C}\, \max\{C_1^{\text{ext}}, C^{\text{ext}}\}^2}_{=:C}\, \|(u_1,u)\|_{\mathcal{V}}\, \|(v_1,v)\|_{\mathcal{V}}, \end{aligned}$$

where $\widetilde{C}$ is the boundedness constant of the volume bilinear form $\widetilde{a}$ on $H^1(Y)$.

The ellipticity is shown along a similar line. From the trace theorem for each $(v_1,v) \in H^{1/2}(\mathit{\Gamma}_1) \times H^{1/2}(\mathit{\Gamma})$ and the corresponding harmonic extension $\widetilde{v} \in H^1(Y)$ we have

$$\|v_1\|_{1/2,\mathit{\Gamma}_1} \le c_1^{\text{tr}}\, \|\widetilde{v}\|_{H^1(Y_1)} \quad \text{and} \quad \|v\|_{1/2,\mathit{\Gamma}} \le c^{\text{tr}}\, \|\widetilde{v}\|_{H^1(Y)}. \tag{14}$$

with the positive constants c_1^{tr} and c^{tr} depending only on Y_1 and Y, respectively. Restricting to functions from $\mathcal{V}/\mathbb{R}$ and using (5) completes the proof

$$a((v_1,v),(v_1,v)) = \widetilde{a}(\widetilde{v},\widetilde{v}) \ge \widetilde{c}\, \|\widetilde{v}\|^2_{H^1(Y)} \ge \underbrace{\widetilde{c}\, \min\left\{\frac{1}{c_1^{\text{tr}}}, \frac{1}{c^{\text{tr}}}\right\}^2}_{=:c}\, \|(v_1,v)\|^2_{\mathcal{V}}.$$

□

Now the Lemma of Lax-Milgram yields

Theorem 2.5. *Problem* (13) *is well-posed, i.e., there exists a unique solution* $(\chi_1^k, \chi^k) \in \mathcal{V}/\mathbb{R}$ *to* (13) *such that*

$$\|(\chi_1^k,\chi^k)\|_{\mathcal{V}} \le \frac{1}{c} \sup_{\substack{(v_1,v)\in\mathcal{V}/\mathbb{R} \\ (v_1,v)\neq 0}} \frac{b^k(v_1)}{\|(v_1,v)\|_{\mathcal{V}}}.$$

2.1 Stabilized Formulation

We can further reformulate (13):

$$\begin{cases} \text{Find } (\chi_1^k,\chi^k) \in \mathcal{V}\colon\ \forall (v_1,v) \in \mathcal{V}\colon \\ \underbrace{a((\chi_1^k,\chi^k),(v_1,v)) + \underline{a}\left(\int_{\mathit{\Gamma}_1} \chi_1^k \,\mathrm{d}s + \int_{\mathit{\Gamma}} \chi^k \,\mathrm{d}s\right)\left(\int_{\mathit{\Gamma}_1} v_1 \,\mathrm{d}s + \int_{\mathit{\Gamma}} v \,\mathrm{d}s\right)}_{=:\widehat{a}(\chi_2^k, v_2)} = b^k(v_1), \end{cases} \tag{15}$$

where $\underline{a} := \min\{a_1, a_2\}$.

Lemma 2.6. *The bilinear form $\widehat{a}(\cdot,\cdot)$ is bounded and elliptic on $\mathcal{V}$.*

Proof. As the boundedness is straightforward using the trace theorem, we only show the ellipticity. For $v_2 := (v_1, v) \in \mathcal{V}$ we have

$$a(v_2, v_2) + \underline{a}\left(\int_{\Gamma_2} v_2 \,\mathrm{d}s\right)^2 = \widetilde{a}(\widetilde{v}, \widetilde{v}) + \underline{a}\left(\int_{\Gamma_2} \widetilde{v} \,\mathrm{d}s\right)^2$$
$$\geq \underline{a}\left\{\int_Y |\nabla \widetilde{v}|^2 + \left(\int_{\Gamma_2} \widetilde{v} \,\mathrm{d}s\right)^2\right\}.$$

Using (4) and (14), a straightforward calculation yields

$$a(v_2, v_2) + \underline{a}\left(\int_{\Gamma_2} v_2 \,\mathrm{d}s\right)^2 \geq \underline{a}\,\frac{\widetilde{c}_{\mathrm{P}}}{1+\widetilde{c}_{\mathrm{P}}}\,\min\left\{\frac{1}{c_1^{\mathrm{tr}}}, \frac{1}{c^{\mathrm{tr}}}\right\}^2 \|v_2\|_{\mathcal{V}}^2$$

□

Theorem 2.7. *Problems* (13) *and* (15) *are equivalent and well-posed.*

Proof. We first show that a solution to (15) also solves (13). Indeed, taking a test function $(v_1, v) \in \mathcal{V}/\mathbb{R}$, i.e., $\int_{\Gamma_1} v_1 + \int_{\Gamma} v = 0$ (Lemma 2.3), the variational identity (15) turns into (13). The transformation of the right-hand side is given in (7). Next, taking the test function $(v_1, v) := (1, 1)$, (15) becomes $\int_{\Gamma_1} \chi_1^k + \int_{\Gamma} \chi^k = 0$, i.e., $(\chi_1^k, \chi^k) \in \mathcal{V}/\mathbb{R}$.

Now it is enough to notice that (15) is well-posed by means of the Lax-Milgram theorem. □

2.2 Mixed formulation

We recall that the Steklov–Poincaré operators S_i in (10) admit the form [Stei08]

$$S_i = D_i + B_i^T\,(V_i)^{-1}\,B_i, \quad B_i := \frac{1}{2}\,I_i + K_i,\; i = 1, 2 \tag{16}$$

where we denote by $V_i \colon H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$, $K_i \colon H^{1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$, and $D_i \colon H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ the single-layer, double-layer, and hypersingular boundary integral operators on Γ_i, respectively. By $(\cdot)^T$ we denote the adjoint (transpose) operator and I_i is the identity on $H^{1/2}(\Gamma_i)$. Note that in two dimensions $\mathrm{diam}(Y_i) < 1$ ensures the invertibility of V_i. This is not a restriction, since one can easily shrink Y and, correspondingly, rescale b^k and the solution.

The analysis of S_1 is standard, see e.g. [Stei08], while the one of S_2 is more involved. This is due to $\ker D_2 = \operatorname{span}\{(1,0),(0,1)\} = \operatorname{span}\{(1,0),1\}$, see [McL00, Th. 8.20], while $\ker S_2 = \ker B_2 = \operatorname{span}\{1\}$. We will take care of this technical difficulity in Lemmata 2.8, 2.10, and 2.11.

Next we will replace the operators S_i by the related systems of integral equations and apply an Aubin–Nitsche trick on the systems for our main result in Theorem 4.2. An alternative would be to replace S_i by a discrete approximation, as we are not able to realize V_i^{-1} on the discrete level. In this case the Aubin–Nitsche trick would be much more involved.

We exploit (16) via the substitution $t_i^k := (V_i)^{-1} B_i(\chi_i^k)$, $i = 1, 2$. Thus problem (15) is equivalent to the following:

$$\begin{cases} \text{Find } (t_1^k, t^k, \chi_1^k, \chi^k) \in \mathcal{W}: \\ A((\underbrace{t_1^k, t^k}_{=:t_2^k}, \underbrace{\chi_1^k, \chi^k}_{=:\chi_2^k}), (\underbrace{\tau_1, \tau}_{=:\tau_2}, \underbrace{v_1, v}_{=:v_2})) = b^k(v_1) \quad \forall (\tau_1, \tau, v_1, v) \in \mathcal{W}, \end{cases} \tag{17}$$

where $\mathcal{W} := H^{-1/2}(\Gamma_1) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma_1) \times H^{1/2}_{\mathrm{per}}(\Gamma)$ and

$$\begin{aligned} A((t_2^k, \chi_2^k), (\tau_2, v_2)) := \sum_{i=1}^{2} a_i \Big\{ \langle \tau_i, V_i\, t_i^k \rangle_{\Gamma_i} - \big\langle \tau_i, B_i\, \chi_i^k \big\rangle_{\Gamma_i} + \big\langle B_i^{\,T}\, t_i^k, v_i \big\rangle_{\Gamma_i} \\ + \langle D_i\, \chi_i^k, v_i \rangle_{\Gamma_i} \Big\} + \underline{a} \int_{\Gamma_2} \chi_2^k \, \mathrm{d}s \int_{\Gamma_2} v_2 \, \mathrm{d}s. \end{aligned} \tag{18}$$

We equip $\mathcal{W}$ with the norm

$$\|(t_1, t, u_1, u)\|_{\mathcal{W}}^2 := \|t_1\|_{-1/2,\Gamma_1}^2 + \|t\|_{-1/2,\Gamma}^2 + \|u_1\|_{1/2,\Gamma_1}^2 + \|u\|_{1/2,\Gamma}^2.$$

For ease of notation we define

$$\begin{aligned} \langle \tau_2, \widetilde{V}_2\, t_2 \rangle_{\Gamma_2} &:= \sum_{i=1}^{2} a_i \, \langle \tau_i, V_i\, t_i \rangle_{\Gamma_i}, \quad \langle \tau_2, \widetilde{B}_2\, u_2 \rangle_{\Gamma_2} := \sum_{i=1}^{2} a_i \, \langle \tau_i, B_i\, u_i \rangle_{\Gamma_i}, \\ \langle \widetilde{D}_2\, u_2, v_2 \rangle_{\Gamma_2} &:= \sum_{i=1}^{2} a_i \, \langle D_i\, u_i, v_i \rangle_{\Gamma_i} + \underline{a} \int_{\Gamma_2} u_2 \, \mathrm{d}s \int_{\Gamma_2} v_2 \, \mathrm{d}s. \end{aligned}$$

We will prove stability and later on convergence of approximations of the solution to (17). Obviously, the bilinear form $A(\cdot, \cdot)$ is bounded on $\mathcal{W}$ as all boundary integral operators are bounded. However, we cannot employ the usual Lax-Milgram theorem since $A(\cdot,\cdot)$ is no longer $\mathcal{W}$-elliptic. Namely, for $v_2 := (\alpha,\beta)$ with $0 \neq \alpha,\beta \in \mathbb{R}$ such that $\alpha\,|\Gamma| + \beta\,|\Gamma_1| = 0$ we have $A((0,v_2),(0,v_2)) = 0$. Instead we will use the characterizaton of bijective operators by inf-sup stability conditions [Nec62, BaB72].

We start with a specific $H^{1/2}(\Gamma_2)$-ellipticity estimate which we will use to prove stability of formulation (18). To our best knowledge the following result is novel for Γ_2 being the boundary of a doubly-connected domain.

Lemma 2.8. *There exists* $\widehat{c} > 0$ *such that*

$$\langle \widetilde{D}_2\, v_2, v_2\rangle_{\Gamma_2} + 2\alpha_{v_2}^2 \langle 1, \widetilde{V}_2\, 1\rangle_{\Gamma_2} \geq \widehat{c}\, \|v_2\|^2_{1/2,\Gamma_2} \qquad \forall v_2 \in H^{1/2}(\Gamma_2)$$

where

$$\alpha_{v_2} = \frac{\langle 1, \widetilde{B}_2\, v_2\rangle_{\Gamma_2}}{\langle 1, \widetilde{V}_2\, 1\rangle_{\Gamma_2}}. \tag{19}$$

Proof. We factorize $H^{1/2}(\Gamma_2)$ by $X^0 := \operatorname{Ker} D_2 = \operatorname{span}\{(1,0),(0,1)\}$ with respect to the inner product $(.,.)_{1/2,\Gamma_2}$,

$$H^{1/2}(\Gamma_2) = X^* \oplus X^0.$$

Correspondingly we denote function components, $H^{1/2}(\Gamma_2) \ni v_2 = v_2^* + v_2^0$. This factorization and the non-negativity of D_1 yield

$$\begin{aligned}
&\langle \widetilde{D}_2\, v_2, v_2\rangle_{\Gamma_2} + 2\alpha_{v_2}^2 \langle 1, \widetilde{V}_2 1\rangle_{\Gamma_2} \\
&\qquad\qquad \geq a_2 \langle D_2\, v_2^*, v_2^*\rangle_{\Gamma_2} + \underbrace{2\frac{\langle 1, \widetilde{B}_2\, v_2\rangle^2_{\Gamma_2}}{\langle 1, \widetilde{V}_2\, 1\rangle_{\Gamma_2}} + \underline{a}\left(\int_{\Gamma_2} v_2 \,\mathrm{d}s\right)^2}_{=:f(v_2)^2} \\
&\qquad\qquad \geq a_2\, c_{D_2}\, \|v_2^*\|^2_{1/2,\Gamma_2} + f(v_2)^2 =: \|v_2\|^2_{1/2,\Gamma_2;f}.
\end{aligned}$$

In the last estimate we have used the semi-ellipticity of D_2 for our setting of two components Γ_1 and Γ of Γ_2, see [McL00, Theorem 8.21]. Indeed, the latter is an equivalent norm on $H^{1/2}(\Gamma_2)$. To show it we slightly modify the proof of the norm equivalence theorem of Sobolev, see, e.g., [Stei08, Th. 2.6].

The upper bound for $\|v_2\|_{1/2,\Gamma_2;f}$ is straightforward since f is bounded as the operator $\widetilde{B}_2$ is. Next we prove the lower bound by contradiction. Assume that for each positive integer n there exists $v_{2,n} \in H^{1/2}(\Gamma_2)$ such that

$$\|v_{2,n}\|_{1/2,\Gamma_2;f} \leq \frac{1}{n}\, \|v_{2,n}\|_{1/2,\Gamma_2}.$$

Set $\overline{v}_{2,n} := v_{2,n}/\|v_{2,n}\|_{1/2,\Gamma_2}$, then

$$\|\overline{v}_{2,n}\|_{1/2,\Gamma_2} = 1 \tag{20}$$

and $\|\overline{v}_{2,n}\|_{1/2,\Gamma_2;f} \leq 1/n$. Thus

$$\overline{v}^*_{2,n} \to 0 \text{ in } H^{1/2}(\Gamma_2), \quad \langle 1, \widetilde{B}_2\,\overline{v}_{2,n}\rangle_{\Gamma_2} \to 0, \quad \text{and} \quad \int_{\Gamma_2} \overline{v}_{2,n}\,\mathrm{d}s \to 0. \tag{21}$$

Since $H^{1/2}(\Gamma_2)$ is compactly embedded in $L^2(\Gamma_2)$, (20) yields that there exists a subsequence, still denoted by $(\overline{v}_{2,n})$, which converges in $L^2(\Gamma_2)$. Let us denote the limit by $\overline{v}_2$. By the first term in (21) and the fact that X^0 is finite dimensional, thus, closed subspace of $H^{1/2}(\Gamma_2)$, we have

$$\overline{v}^0_{2,n} = \overline{v}_{2,n} - \overline{v}^*_{2,n} \to \overline{v}_2 \text{ in } H^{1/2}(\Gamma_2) \quad \text{and} \quad \overline{v}_2 = (\overline{v}_1, \overline{v}) \in \mathbb{R}^2 \equiv X^0.$$

From the kernel properties of $\widetilde{B}_2$,

$$\widetilde{B}_2(1,0) = (1,0) \quad \text{and} \quad \widetilde{B}_2(0,1) = (-1,0)$$

and from the second term in (21) we deduce that $\overline{v}_2 = \alpha(1,0) + \beta(0,1)$ is constant. Finally, the only constant for which the third term of (21) holds is zero, i.e. $\overline{v}_2 = 0$, which is a contradiction to (20). $\square$

Below we will make use of a simple but useful observation of minimization problems, see, e.g. [Pe13, Lemma 1.6].

Lemma 2.9. *Let X be a Hilbert space. Let $U \subset W \subset X$. For a symmetric, bounded, and X-elliptic bilinear form $d(\cdot,\cdot)\colon X \times X \to \mathbb{R}$ there holds*

$$d(w,w) \geq d(u,u)$$

for the solutions of the corresponding variational problems

$$\begin{aligned} w \in W\colon \quad d(w,p) &= f(p) \quad \forall p \in W, \\ u \in U\colon \quad d(u,p) &= f(p) \quad \forall p \in U. \end{aligned}$$

As the bilinear form $A(\cdot,\cdot)$ in (18) is not elliptic, we use the inf-sup conditions in the following lemmata to characterize bijective operators, see e.g. [Nec62, BaB72] and [OD10, Th.6.6.1].

Lemma 2.10. *There exists a $c_A > 0$ such that for all $(t_2, u_2) \in \mathcal{W}$ it holds that*

$$\sup_{0 \neq (\tau_2, v_2) \in \mathcal{W}} \frac{A((t_2, u_2), (\tau_2, v_2))}{\|(\tau_2, v_2)\|_{\mathcal{W}}} \geq c_A \, \|(t_2, u_2)\|_{\mathcal{W}}.$$

Proof. The bilinear form (18) can be seen as follows:

$$A((t_2, u_2), (\tau_2, v_2)) = \langle \tau_2, \widetilde{V}_2 \, t_2 \rangle_{\Gamma_2} - \langle \tau_2, \widetilde{B}_2 \, u_2 \rangle_{\Gamma_2} + \langle t_2, \widetilde{B}_2 \, v_2 \rangle_{\Gamma_2} + \langle \widetilde{D}_2 \, u_2, v_2 \rangle_{\Gamma_2},$$

Obviously, all new operators are linear and bounded. The selfadjointness of V_i and D_i implies the selfadjointness of $\widetilde{V}_2$ and $\widetilde{D}_2$. In addition, $a_1, a_2 > 0$ and the ellipticity of V_1 and V_2 yield the ellipticity of $\widetilde{V}_2$.

We prove the estimate by a special choice

$$\tau_2 = \frac{t_2}{2} - q_{u_2}, \quad v_2 = u_2$$

where $q_{u_2} \in H^{-1/2}(\Gamma_2)$ is the unique solution (Lemma of Lax-Milgram) of

$$\langle p_2, \widetilde{V}_2 \, q_{u_2} \rangle_{\Gamma_2} = \frac{1}{2} \langle p_2, \widetilde{B}_2 \, u_2 \rangle_{\Gamma_2} \quad \forall p_2 \in H^{-1/2}(\Gamma_2) \tag{22}$$

and $\|q_{u_2}\|_{H^{-1/2}(\Gamma_2)} \leq c \|u_2\|_{H^{1/2}(\Gamma_2)}$. Thus we have

$$\|(\tau_2, v_2)\|_{\mathcal{W}} \leq c \, \|(t_2, u_2)\|_{\mathcal{W}}. \tag{23}$$

For our specific choice $v_2 = u_2$ and $\tau_2 = \frac{t_2}{2} - q_{u_2}$ we observe by using the symmetry of $\widetilde{V}_2$ and (22)

$$\begin{aligned} A((t_2, u_2), (\tau_2, v_2)) &= \frac{1}{2} \langle t_2, \widetilde{V}_2 \, t_2 \rangle_{\Gamma_2} + \langle q_{u_2}, \widetilde{B}_2 \, u_2 \rangle_{\Gamma_2} + \langle \widetilde{D}_2 \, u_2, u_2 \rangle_{\Gamma_2} \\ &\qquad - \langle q_{u_2}, \widetilde{V}_2 \, t_2 \rangle_{\Gamma_2} - \frac{1}{2} \langle t_2, \widetilde{B}_2 \, u_2 \rangle_{\Gamma_2} + \langle t_2, \widetilde{B}_2 \, u_2 \rangle_{\Gamma_2} \\ &= \frac{1}{2} \langle t_2, \widetilde{V}_2 \, t_2 \rangle_{\Gamma_2} + \langle \widetilde{D}_2 \, u_2, u_2 \rangle_{\Gamma_2} + 2 \langle q_{u_2}, \widetilde{V}_2 \, q_{u_2} \rangle_{\Gamma_2}. \end{aligned}$$

Next we choose $U = \operatorname{span}\{1\} \subset W = H^{-1/2}(\Gamma_2) = X$ and $\alpha_{u_2} \in U$ as defined in (19) to apply Lemma 2.9 for $d(\cdot, \cdot) = \langle \cdot, \widetilde{V}_2 \cdot \rangle_{\Gamma_2}$:

$$\begin{aligned} A((t_2, u_2), (\tau_2, v_2)) &\geq \frac{1}{2} \langle t_2, \widetilde{V}_2 \, t_2 \rangle_{\Gamma_2} + \langle \widetilde{D}_2 \, u_2, u_2 \rangle_{\Gamma_2} + 2 \alpha_{u_2}^2 \langle 1_2, \widetilde{V}_2 \, 1_2 \rangle_{\Gamma_2} \\ &\geq \frac{c_1^V}{2} \|t_2\|_{H^{-1/2}(\Gamma_2)} + \widehat{c} \, \|u_2\|_{1/2, \Gamma_2}^2 \\ &\geq \underbrace{\min\left\{ \frac{c_1^V}{2}, \widehat{c} \right\}}_{=: c_A} \|(t_2, u_2)\|_{\mathcal{W}}^2, \end{aligned}$$

where we used the ellipticity of $\widetilde{V}_2$ and Lemma 2.8. Now the assertion follows by (23) and by the injectivity of the mapping $(u_2, t_2) \mapsto (v_2, \tau_2)$. $\square$

Next we check the adjoint operator.

Lemma 2.11. *There exists a $c_A > 0$ such that for all $(\tau_2, v_2) \in \mathcal{W}$ it holds that*

$$\sup_{0 \neq (t_2, u_2) \in \mathcal{W}} \frac{A((t_2, u_2), (\tau_2, v_2))}{\|(t_2, u_2)\|_{\mathcal{W}}} \geq C_A \|(\tau_2, v_2)\|_{\mathcal{W}}.$$

Proof. For a given $(\tau_2, v_2) \in \mathcal{W}$ we take

$$t_2 := \frac{\tau_2}{2} + q_{v_2}, \quad u_2 := v_2,$$

where

$$q_{v_2} \in H^{-1/2}(\Gamma_2): \quad \langle p_2, \widetilde{V}_2\, q_{v_2} \rangle_{\Gamma_2} = \frac{1}{2} \langle p_2, \widetilde{B}_2\, v_2 \rangle_{\Gamma_2} \quad \forall p_2 \in H^{-1/2}(\Gamma_2).$$

The rest follows the lines of the proof of Lemma 2.10 as

$$A((t_2, u_2), (\tau_2, v_2)) \geq \frac{1}{2} \langle \tau_2, \widetilde{V}_2\, \tau_2 \rangle_{\Gamma_2} + \langle \widetilde{D}_2\, v_2, v_2 \rangle_{\Gamma_2} + 2\alpha_{v_2}^2 \langle 1, \widetilde{V}_2\, 1 \rangle_{\Gamma_2}.$$

$\square$

Finally, a general theory on bijective operators [OD10, Th. 6.6.1] implies

Theorem 2.12. *There exists a unique solution $(t_1^k, t^k, \chi_1^k, \chi^k) \in \mathcal{W}$ to (17) and*

$$\|(t_1^k, t^k, \chi_1^k, \chi^k)\|_{\mathcal{W}} \leq \frac{1}{c_A} \sup_{0 \neq (\tau_1, \tau, v_1, v) \in \mathcal{W}} \frac{b^k(v_1)}{\|(\tau_1, \tau, v_1, v)\|_{\mathcal{W}}}.$$

3 Boundary element method

We consider conforming (finite element) discretizations of Γ_1 and Γ into m_1 and m elements (line segments in 2d, triangles in 3d), respectively. The discretization parameter h is maximal element diameter. We introduce the discontinuous element-wise constant basis functions $(\psi_{1,j})_{j=1}^{m_1}$ and $(\psi_j)_{j=1}^{m}$ and the globally continuous, element-wise linear basis functions $(\varphi_{1,j})_{j=1}^{n_1}$ and $(\varphi_j)_{j=1}^{n}$, where n_1 and n denote the respective numbers of nodes. To simplify the notation we further denote the number of elements and nodes along $\mathit{\Gamma}_2$ by $m_2 := m_1 + m$ and $n_2 := n_1 + n$, respectively, and we denote by $(\psi_{2,j})_{j=1}^{m_2}$ and $(\varphi_{2,j})_{j=1}^{n_2}$ the basis functions associated to $\mathit{\Gamma}_2$ such that

$$\psi_{2,j} := \begin{cases} \psi_{1,j}, & j \leq m_1, \\ \psi_{j-m_1}, & j > m_1, \end{cases} \quad \text{and} \quad \varphi_{2,j} := \begin{cases} \varphi_{1,j}, & j \leq n_1, \\ \varphi_{j-n_1}, & j > n_1. \end{cases} \tag{24}$$

The considered conforming discrete trial spaces are denoted by

$$\begin{aligned} Q_i^h &:= \operatorname{span}(\psi_{i,j})_{j=1}^{m_i} \subset H^{-1/2}(\mathit{\Gamma}_i), & Q^h &:= \operatorname{span}(\psi_j)_{j=1}^{m} \subset H^{-1/2}(\mathit{\Gamma}), \\ V_i^h &:= \operatorname{span}(\varphi_{i,j})_{j=1}^{n_i} \subset H^{1/2}(\mathit{\Gamma}_i), & V^h &:= \operatorname{span}(\varphi_j)_{j=1}^{n} \subset H^{1/2}(\mathit{\Gamma}) \end{aligned}$$

and $V_{\mathrm{per}}^h \subset V^h$, $\mathcal{V}^h := V_1^h \times V_{\mathrm{per}}^h$, $\mathcal{W}^h := Q_1^h \times Q^h \times V_1^h \times V_{\mathrm{per}}^h$.

The Galerkin approximation of (25) reads

$$\begin{cases} \text{Find } (t_1^{k,h}, t^{k,h}, \chi_1^{k,h}, \chi^{k,h}) \in \mathcal{W}^h: \\ A((t_1^{k,h}, t^{k,h}, \chi_1^{k,h}, \chi^{k,h}), (\tau_1^h, \tau^h, v_1^h, v^h)) = b^k(v_1^h) \quad \forall (\tau_1^h, \tau^h, v_1^h, v_1) \in \mathcal{W}^h. \end{cases} \tag{25}$$

The well-posedness of (25) relies on discrete inf-sup conditions.

Lemma 3.1. *There exists some $\widehat{c}_A > 0$ independent of h such that for all $(t_2^h, u_2^h) \in \mathcal{W}^h$ it holds that*

$$\sup_{0 \neq (\tau_2^h, v_2^h) \in \mathcal{W}^h} \frac{A((t_2^h, u_2^h), (\tau_2^h, v_2^h))}{\|(\tau_2^h, v_2^h)\|_{\mathcal{W}}} \geq \widehat{c}_A \, \|(t_2^h, u_2^h)\|_{\mathcal{W}}.$$

Further, for all $0 \neq (\tau_2^h, v_2^h) \in \mathcal{W}^h$ there holds

$$\sup_{(t_2^h, u_2^h) \in \mathcal{W}^h} A((t_2^h, u_2^h), (\tau_2^h, v_2^h)) > 0.$$

Proof. The proof is the same as the one of Lemma 2.10 but we choose discrete functions $(\tau_2^h, v_2^h) \in \mathcal{W}^h$ as

$$\tau_2^h := \frac{t_2^h}{2} - q_{u_2^h}^h, \quad v_2^h := u_2^h, \tag{26}$$

with the discrete counterpart of (22):

$$q_{u_2^h}^h \in Q_2^h\colon \quad \langle p_2^h, \widetilde{V}_2 \, q_{u_2^h}^h \rangle_{\mathit{\Gamma}_2} = \frac{1}{2} \langle p_2^h, \widetilde{B}_2 \, u_2^h \rangle_{\mathit{\Gamma}_2} \quad \forall p_2^h \in Q_2^h \tag{27}$$

and $\|q^h_{u^h_2}\|_{H^{-1/2}(\Gamma_2)} \leq c\|u^h_2\|_{H^{1/2}(\Gamma_2)}$. We end up with

$$A((t^h_2, u^h_2), (\tau^h_2, v^h_2)) \geq \frac{1}{2}\langle t^h_2, \widetilde{V}_2\, t^h_2\rangle_{\Gamma_2} + \langle \widetilde{D}_2\, u^h_2, u^h_2\rangle_{\Gamma_2} + 2\alpha^2_{u^h_2}\langle 1, \widetilde{V}_2\, 1\rangle_{\Gamma_2}$$

as $U = \operatorname{span}\{1\} \subset W = Q^h_2$ in Lemma 2.9. By the ellipticity of $\widetilde{V}_2$ and Lemma 2.8 the proof of the first statement is complete.

Concerning the second statement we choose similarly to the proof of Lemma 2.11 for a given $(\tau^h_2, v^h_2) \in \mathcal{W}^h$

$$t^h_2 := \frac{\tau^h_2}{2} + q^h_{v^h_2} \in Q^h_2, \quad u_2 := v_2 \in \mathcal{V}^h$$

where

$$q^h_{v^h_2} \in Q^h_2 \colon \quad \langle p^h_2, \widetilde{V}_2\, q^h_{v^h_2}\rangle_{\Gamma_2} = \frac{1}{2}\langle p^h_2 \widetilde{B}_2\, v^h_2, \rangle_{\Gamma_2} \quad \forall p^h_2 \in Q^h_2.$$

The rest follows the lines of the proof of Lemma 2.10. □

With the discrete inf-sup conditions we can again state:

Theorem 3.2. *There exists a unique solution* $(t^{k,h}_1, t^{k,h}, \chi^{k,h}_1, \chi^{k,h}) \in \mathcal{W}^h$ *to* (25) *and*

$$\|(t^{k,h}_1, t^{k,h}, \chi^{k,h}_1, \chi^{k,h})\|_{\mathcal{W}} \leq \frac{1}{\widehat{c}_A} \sup_{0\neq(\tau_1,\tau,v_1,v)\in\mathcal{W}} \frac{b^k(v_1)}{\|(\tau_1,\tau,v_1,v)\|_{\mathcal{W}}}.$$

The Galerkin discretizations of the operators V_i, K_i, D_i, and I_i in (17) lead to matrices $\mathbf{V}_i \in \mathbb{R}^{m_i\times m_i}$, $\mathbf{K}_i \in \mathbb{R}^{m_i\times n_i}$, $\mathbf{D}_i \in \mathbb{R}^{n_i\times n_i}$, and $\mathbf{M}_i \in \mathbb{R}^{m_i\times n_i}$,

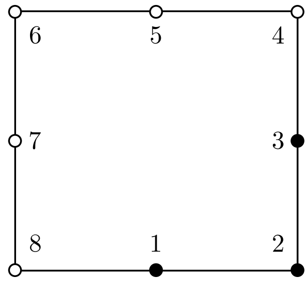
$$\begin{aligned} (\mathbf{V}_i)_{jk} &:= \langle \psi_{i,j}, V_i\psi_{i,k}\rangle_{\Gamma_i}, & (\mathbf{K}_i)_{jk} &:= \langle \psi_{i,j}, K_i\varphi_{i,k}\rangle_{\Gamma_i},\\ (\mathbf{D}_i)_{jk} &:= \langle D_i\varphi_{i,k}, \varphi_{i,j}\rangle_{\Gamma_i}, & (\mathbf{M}_i)_{jk} &:= \langle \psi_{i,j}, \varphi_{i,k}\rangle_{\Gamma_i}. \end{aligned}$$

After eliminating $t^{k,h}_1$ and $t^{k,h}$ from (25) the Stekov–Poincaré operators S_i are approximated by

$$\mathbf{S}_i = \mathbf{D}_i + \left(\frac{1}{2}\mathbf{M}_i + \mathbf{K}_i\right)^T (\mathbf{V}_i)^{-1} \left(\frac{1}{2}\mathbf{M}_i + \mathbf{K}_i\right).$$

The discretization of b^k in (6) leads to the vector $\mathbf{b}^k_1 \in \mathbb{R}^{n_1}$,

$$(\mathbf{b}^k_1)_j := (a_1 - a_2) \int_{\operatorname{supp}\varphi_{1,j}} \varphi_{1,j}(y)\,(n_1(y))_k \,\mathrm{d}s(y).$$



$$\mathbf{P} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Figure 2: A 2d periodic cell ($n := 8$, $\widehat{n} := 3$) and the related periodicity matrix.

Following the decomposition (24) of $\varphi_{2,j}$ with respect to degrees of freedom related to Γ_1 and Γ we introduce a two-by-two block structure of $\mathbf{S}_2$, e.g., $\mathbf{S}_2^{\Gamma_1,\Gamma} \in \mathbb{R}^{n_1 \times n}$. After eliminating $t_1^{k,h}$ and $t^{k,h}$ our discrete problem (25) is equivalent to the linear system

$$\begin{pmatrix} a_1\,\mathbf{S}_1 + a_2\left(\mathbf{S}_2^{\Gamma_1,\Gamma_1} + \mathbf{e}_1 \otimes \mathbf{e}_1\right) & a_2\left(\mathbf{S}_2^{\Gamma_1,\Gamma} + \mathbf{e}_1 \otimes \mathbf{e}\right)\mathbf{P} \\ a_2\,\mathbf{P}^T\left(\mathbf{S}_2^{\Gamma,\Gamma_1} + \mathbf{e} \otimes \mathbf{e}_1\right) & a_2\,\mathbf{P}^T\left(\mathbf{S}_2^{\Gamma,\Gamma} + \mathbf{e} \otimes \mathbf{e}\right)\mathbf{P} \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi}_1^k \\ \widehat{\boldsymbol{\chi}}^k \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^k \\ \mathbf{0} \end{pmatrix}, \tag{28}$$

with $(\mathbf{e}_1)_i := |\operatorname{supp}\varphi_{1,i}|/d$, and $(\mathbf{e})_i := |\operatorname{supp}\varphi_i|/d$, and the representations $\chi_1^{k,h}(y) = \sum_{j=1}^{n_1}(\boldsymbol{\chi}_1^k)_j\,\varphi_{1,j}(y)$, $\chi^{k,h}(y) = \sum_{j=1}^{n}(\mathbf{P}\,\widehat{\boldsymbol{\chi}}^k)_j\,\varphi_j(y)$. By $\widehat{\boldsymbol{\chi}}^k$ we denote the vector of coefficients of a discrete periodic function in V_{per}^h and $\mathbf{P} \in \mathbb{R}^{n \times \widehat{n}}$, $\widehat{n} \approx n/2$, implements the Γ-periodicity, see Fig. 2.

According to (6), the approximated homogenized coefficients are computed as

$$(\mathbf{A}_0^h)_{ik} := \delta_{ik}\left\{a_2 + (a_1 - a_2)\int_{\Gamma_1} y_i\,(n_1)_i\right\} - b^i(\chi_1^{k,h}). \tag{29}$$

4 Convergence analysis

Lemma 4.1. *Let the solution to* (17) *satisfy* $t_1^k \in H^{\beta-1}(\Gamma_1)$, $t^k \in H^{\beta-1}(\Gamma)$, $\chi_1^k \in H^\beta(\Gamma_1)$, *and* $\chi^k \in H^\beta(\Gamma)$ *for each* $k \in \{1,\ldots,d\}$ *and some* $\beta \in (\frac{d-1}{2}, 2]$, *then there holds for any conforming, shape-regular, and quasi-uniform family of boundary meshes*

$$\left\|\left(t_1^k - t_1^{k,h}, t^k - t^{k,h}, \chi_1^k - \chi_1^{k,h}, \chi^k - \chi^{k,h}\right)\right\|_{\mathcal{W}} \le \left(1 + \frac{C_A}{c_A}\right) C_\beta^k\, h^{\beta-\frac{1}{2}},$$

where C^k_β is the following Euclidean norm

$$C^k_\beta := \left\| \begin{pmatrix} C_{-1/2,\Gamma_1,\alpha}\, |t^k_1|_{\alpha,\Gamma_1} \\ C_{-1/2,\Gamma,\alpha}\, |t^k|_{\alpha,\Gamma} \\ C_{1/2,\Gamma_1,\beta}\, |\chi^k_1|_{\beta,\Gamma_1} \\ C_{1/2,\Gamma,\beta}\, |\chi^k|_{\beta,\Gamma} \end{pmatrix} \right\|. \tag{30}$$

Proof. For solutions $(t^k_1, t^k, \chi^k_1, \chi^k)$ and $(t^{k,h}_1, t^{k,h}, \chi^{k,h}_1, \chi^{k,h})$ to (17) and (25), respectively, there holds by Céa's-type lemma, cf. [Stei08, Th. 8.4], that

$$\begin{aligned} &\big\|\big(t^k_1 - t^{k,h}_1, t^k - t^{k,h}, \chi^k_1 - \chi^{k,h}_1, \chi^k - \chi^{k,h}\big)\big\|_{\mathcal{W}} \\ &\quad \le \left(1 + \frac{C_A}{c_A}\right) \inf_{(\tau^h_1,\tau^h,v^h_1,v^h)\in\mathcal{W}^h} \big\|\big(t^k_1 - \tau^h_1, t^k - \tau^h, \chi^k_1 - v^h_1, \chi^k - v^h\big)\big\|_{\mathcal{W}}, \end{aligned}$$

where $C_A, c_A > 0$ are the $\mathcal{W}$-boundedness and $\mathcal{W}$-ellipticity constants of $A(\cdot,\cdot)$, respectively. Now the assertion follows from the approximation property ([Stei08, Th. 10.4], [Sau11, Th. 4.3.20]) of piecewise constant functions

$$\inf_{\psi^h\in\Psi^h} \|t - \psi^h\|_{-1/2,\gamma} \le C_{-1/2,\gamma,\beta-1}\, h^{\beta-\frac{1}{2}}\, |t|_{\beta-1,\gamma}$$

for $\gamma \in \{\Gamma_1, \Gamma_2, \Gamma\}$ and the error estimate of the interpolation $I_h u$ by piecewise linear and globally continuous functions ([Stei08, Th. 10.9], [Sau11, Th. 4.3.22])

$$\|u - I_h u\|_{+1/2,\gamma} \le C_{1/2,\gamma,\beta}\, h^{\beta-\frac{1}{2}}\, |u|_{\beta,\gamma}$$

Here we use interpolation to get a discrete periodic function in V^h_{per}. Therefore we require $u \in H^\beta(\gamma)$ for $\beta \in (\frac{d-1}{2}, 2]$. □

We will employ the Aubin-Nitsche trick to prove a higher rate of convergence for the homogenized coefficients. To this end we consider the following adjoint problem: Find $(s^k_1, s^k, \xi^k_1, \xi^k) \in \mathcal{W}$ such that

$$A((\tau_1,\tau,v_1,v), (\underbrace{s^k_1, s^k}_{=:s^k_2}, \underbrace{\xi^k_1, \xi^k}_{=:\xi^k_2})) = b^k(v_1) \quad \forall(\tau_1,\tau,v_1,v) \in \mathcal{W} \tag{31}$$

and its Galerkin approximation to find $(s^{k,h}_1, s^{k,h}, \xi^{k,h}_1, \xi^{k,h}) \in \mathcal{W}^h$ such that

$$A((\tau^h_1,\tau^h,v^h_1,v^h), (\underbrace{s^{k,h}_1, s^{k,h}}_{=:s^{k,h}_2}, \underbrace{\xi^{k,h}_1, \xi^{k,h}}_{=:\xi^{k,h}_2})) = b^k(v^h_1) \quad \forall(\tau^h_1,\tau^h,v^h_1,v^h) \in \mathcal{W}^h. \tag{32}$$

As the bilinear form $A(\cdot,\cdot)$ satisfies both inf-sup conditions (Lemmata 2.10 and 2.11), these adjoint problems are well-posed. After elimination of (s_1^k, s^k) and $(s_1^{k,h}, s^{k,h})$ from (31) and (32), respectively, we arrive at the problem (15), or equivalently (13), and at the discrete problem (28). Thus, we conclude

$$\xi_1^k = \chi_1^k,\ \xi^k = \chi^k,\ \xi_1^{k,h} = \chi_1^{k,h},\ \text{and } \xi^{k,h} = \chi^{k,h} \tag{33}$$

and, consequently,

$$s_1^k = -t_1^k,\ s^k = -t^k,\ s_1^{k,h} = -t_1^{k,h}, \text{and } s^{k,h} = -t^{k,h}. \tag{34}$$

Theorem 4.2. *We consider some conforming, shape-regular, and quasi-uniform family of boundary meshes. Let the solution to* (17) *satisfy* $t_1^k \in H^{\beta-1}(\Gamma_1)$*,* $t^k \in H^{\beta-1}(\Gamma)$*,* $\chi_1^k \in H^\beta(\Gamma_1)$*, and* $\chi^k \in H^\beta(\Gamma)$ *for each* $k \in \{1,\dots,d\}$ *and some* $\beta \in (\frac{d-1}{2}, 2]$*, then there exits a* $\widetilde{C} > 0$ *independent of* h *such that*

$$\max_{i,k} \left|\left(\mathbf{A}_0 - \mathbf{A}_0^h\right)_{ki}\right| \le \widetilde{C}\, h^{2\beta-1}.$$

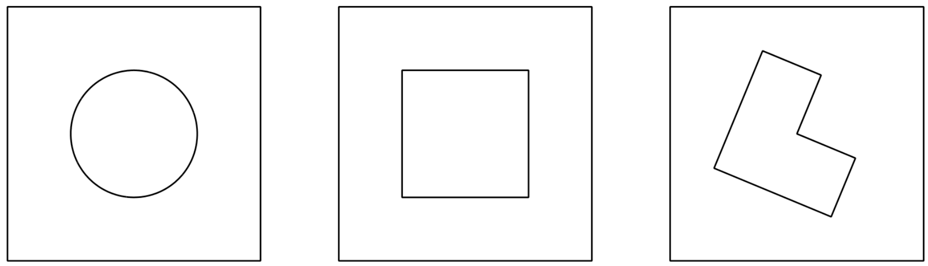
Proof. We rewrite the approximation error of the homogenized coefficients by (6), (29), the adjoint problem (31), and the Galerkin orthogonality of (17) and (25)

$$\begin{aligned}\left|\left(\mathbf{A}_0 - \mathbf{A}_0^h\right)_{ki}\right| &= \left|b^k(\chi_1^i - \chi_1^{i,h})\right| = \left|A((t_2^i - t_2^{i,h}, \chi_2^i - \chi_2^{i,h}), (s_2^k, \xi_2^k))\right| \\ &= \left|A((t_2^i - t_2^{i,h}, \chi_2^i - \chi_2^{i,h}), (s_2^k - s_2^{k,h}, \xi_2^k - \xi_2^{k,h}))\right|.\end{aligned}$$

Using relations (33) and (34), the boundeness of A, and Lemma 4.1, the error reads

$$\begin{aligned}\left|\left(\mathbf{A}_0 - \mathbf{A}_0^h\right)_{ki}\right| &= \left|A((t_2^i - t_2^{i,h}, \chi_2^i - \chi_2^{i,h}), (t_2^{k,h} - t_2^k, \chi_2^k - \chi_2^{k,h}))\right| \\ &\le \underbrace{C_A \left(1 + \frac{C_A}{c_A}\right)^2 C_\beta^i\, C_\beta^k}_{=:\widetilde{C}_{ik}}\, h^{2\beta-1},\end{aligned}$$

where the constants C_β^i and C_β^k are defined by (30). The proof is completed with $\widetilde{C} := \max_{i,k} \widetilde{C}_{ik}$. □



(a) Circle. (b) Square. (c) L-shape.

Figure 3: 2d inclusions.

Table 1: Convergence table for 2d experiments.

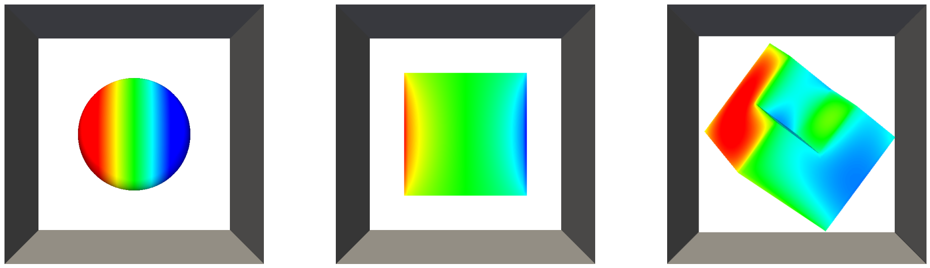
m	e^h_{circle}	eoc	e^h_{square}	eoc	e^h_{lshape}	eoc
64	$1.62 \cdot 10^{-3}$	—	$7.47 \cdot 10^{-3}$	—	$1.14 \cdot 10^{-2}$	—
128	$3.89 \cdot 10^{-4}$	2.05	$2.70 \cdot 10^{-3}$	1.47	$3.77 \cdot 10^{-3}$	1.60
256	$9.70 \cdot 10^{-5}$	2.00	$9.67 \cdot 10^{-4}$	1.48	$1.31 \cdot 10^{-3}$	1.52
512	$2.40 \cdot 10^{-5}$	2.02	$3.40 \cdot 10^{-4}$	1.51	$4.56 \cdot 10^{-4}$	1.53
1024	$5.71 \cdot 10^{-6}$	2.07	$1.12 \cdot 10^{-4}$	1.60	$1.50 \cdot 10^{-4}$	1.60
2048	$1.14 \cdot 10^{-6}$	2.32	$2.99 \cdot 10^{-5}$	1.91	$3.99 \cdot 10^{-5}$	1.91

5 Numerical results

To verify the above described boundary element approach to homogenization we present numerical experiments performed both in 2 and 3 dimensions.

In 2d we consider three examples with the inclusion Y_1 represented by a circle, a square, and an L-shaped domain rotated by $\pi/8$ clockwise. The inclusions are placed in the center of the reference cell Y, see Fig. 3. As already mentioned in Sect. 2.2, the reference domain is scaled to $Y := (0, 1/4)^2$ to ensure invertibility of the single-layer operators V_i. The material parameters are set to $a_1 := 1$, $a_2 := 10$. Since analytic solutions are not known, we use matrices computed on the finest discretization level instead of $\mathbf{A}_0$ for the convergence study. The finest level effective coefficients for the respective inclusions read $\mathbf{A}_{0,\text{circle}} = \operatorname{diag}(7.2310,\ 7.2310)$, $\mathbf{A}_{0,\text{square}} = \operatorname{diag}(6.4758,\ 6.4758)$, and

$$\mathbf{A}_{0,\text{lshape}} = \begin{pmatrix} 6.8083 & -0.1882 \\ -0.1882 & 7.2661 \end{pmatrix}.$$



(a) Sphere. (b) Cube. (c) Fichera corner.

Figure 4: 3d inclusions with χ_1^2.

The results are presented in Tab. 1 with m denoting the number of elements on Γ (the discretization parameter h for the inclusions is chosen accordingly), $e_\bullet^h$ denoting the error with respect to the finest approximation, and $\mathrm{eoc} := \log_2 e_\bullet^h / e_\bullet^{h/2}$ standing for the estimated order of convergence. Regularity results for transmision problems with polygonal interfaces provide $\beta \approx 1.23169$ for the square and the L-shaped inclusion, see [CS85, Section 6]. In these cases, Theorem 4.2 suggests an order of convergence of about 1.46 as essentially observed in Tab. 1. A higher convergence rate is attained for the smooth inclusion represented by a circle. For the corresponding convergence plots see Fig. 5a.

For the 3d experiments we consider higher dimensional equivalents of the 2d inclusions, namely, a ball, a cube, and a Fichera corner rotated by $\pi/3$ with respect to the axis $(1, 1/2, 1/4)$. The respective configurations are displayed in Fig. 4 together with the auxiliary functions χ_1^2 computed by the boundary element method. The material parameters are again set to $a_1 := 1$, $a_2 := 10$. The effective coefficients $\mathbf{A}_{0,\mathrm{sphere}} = \mathrm{diag}(9.1819,\ 9.1819,\ 9.1819)$, $\mathbf{A}_{0,\mathrm{cube}} = \mathrm{diag}(8.3998,\ 8.3998,\ 8.3998)$, and

$$\mathbf{A}_{0,\mathrm{fichera}} = \begin{pmatrix} 8.4830 & -0.0182 & -0.0791 \\ -0.0182 & 8.5758 & -0.0170 \\ -0.0791 & -0.0170 & 8.5381 \end{pmatrix}$$

computed on the finest level serve as the reference matrices $\mathbf{A}_0$ for the evaluation of the errors $e_\bullet^h$.

Tab. 2 summarizes the convergence results with m denoting the number of surface triangles on the boundary of Γ, $e_\bullet^h$ denoting the error and eoc standing for the estimated order of convergence. Similarly as in the 2d case we observe convergence rates according to Theorem 4.2 with higher order

Table 2: Convergence table for 3d experiments.

m	e^h_{sphere}	eoc	e^h_{cube}	eoc	e^h_{fichera}	eoc
192	$1.01 \cdot 10^{-1}$	—	$5.15 \cdot 10^{-2}$	—	$1.02 \cdot 10^{-1}$	—
768	$2.66 \cdot 10^{-2}$	1.92	$1.91 \cdot 10^{-2}$	1.43	$2.77 \cdot 10^{-2}$	1.89
3072	$6.44 \cdot 10^{-3}$	2.04	$6.37 \cdot 10^{-3}$	1.59	$8.48 \cdot 10^{-3}$	1.71
12288	$1.30 \cdot 10^{-3}$	2.31	$1.69 \cdot 10^{-3}$	1.91	$2.16 \cdot 10^{-3}$	1.97

acquired for the spherical inclusion. See Fig. 5 (right) for the graphs of the convergence results.

6 Conclusion

We considered a homogenization problem for a scalar elliptic boundary value problem with a periodic material. We derived an equivalent direct boundary integral formulation using Steklov–Poincaré operators and proved its well-posedness. A boundary element discretization was proposed and analyzed. We proved that the discretized homogenized coefficients converge super-linearly to the true ones, which was confirmed by numerical experiments in 2d and 3d.

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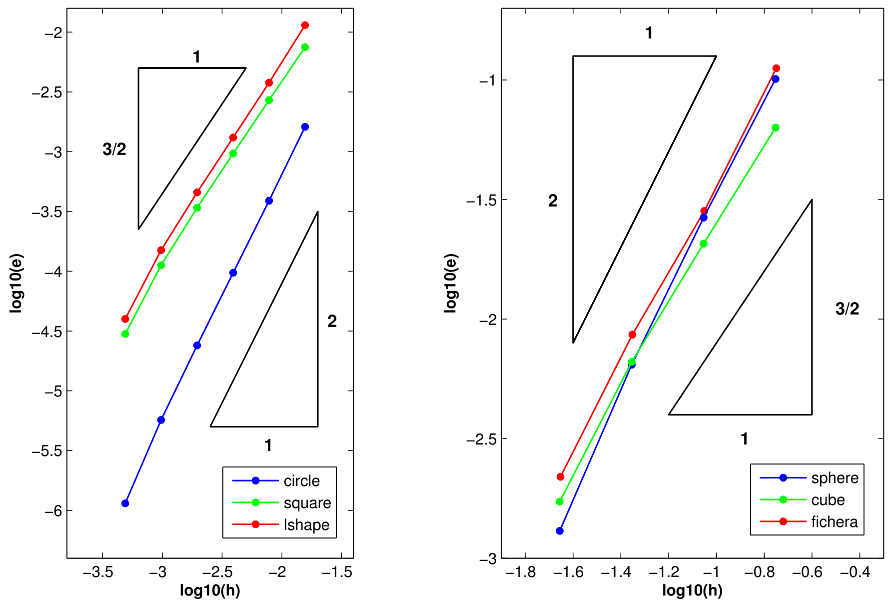

(a) 2d.

(b) 3d.

Figure 5: Convergence graphs for 2d and 3d experiments.

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