Preconditioners for time-harmonic optimal control eddy-current problems*

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Abstract. Time-harmonic formulations enable solution of time-dependent PDEs without use of normally slow time-stepping methods. Two efficient preconditioners for the discretized parabolic and eddy current electromagnetic optimal control problems, one on block diagonal form and one utilizing the two by two block structure of the resulting matrix, are presented with simplified analysis and numerical illustrations. Both methods result in tight eigenvalue bounds for the preconditioned matrix and very few iterations that hold uniformly with respect to the mesh, problem and method parameters, with the exception of the dependence on reluctivity for the block diagonal preconditioner.

Keywords: preconditioning, Krylov subspace methods, optimal control, eddy currents, time-harmonic

1 Introduction

Time dependent partial differential equations are normally solved numerically with some time stepping method. Due to reasons of numerical stability, if an explicit method is used, one must choose very small time steps or use a stable implicit method which requires the solution of a large scale linear system at each step, both of which can be computationally costly.

However, for time-harmonic problems one can approximate the solution on the whole given global time interval by a truncated Fourier series of trigonometric functions and use a multiharmonic approach. For linear problems the arising problems for each frequency separate, which makes it possible to solve for all frequencies, $\omega = k\pi/T$, $k = 0, 1, 2, \cdots$, where T is the end time, in parallel. Furthermore, to achieve a sufficient numerical accuracy it suffices normally with the use of few terms in the Fourier series expansion.

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For nonlinear problems one can often use a two-grid method, which enables the solution of the nonlinear equation only on a coarse grid, see e.g. [4]. To illustrate the ideas, in this talk we mainly consider an optimal control problem for a linear parabolic heat equation and only shortly describe the application for an eddy current electromagnetic problem.

The main contribution of the paper is the presentation of two preconditioners for the arising two-by-two and four-by-four block matrix systems. The analysis and numerical tests show that they are very efficient, leading to few, mostly single digits iterations. One of the preconditioners has previously been presented by Kolmbauer and Langer [1] and the other is based on previously presented preconditioners for optimal control problems by Axelsson, Farouq, Neytcheva [2]. A simplified analysis of the methods will appear in [7].

We show first the derivation of the two-by-two block matrix system and show then that the two preconditioners lead to tight eigenvalue bounds which for the first type preconditioner hold uniformly under some restrictions and for the second one without any restrictions with respect to all parameters. To illustrate the methods, the paper ends with numerical tests. It is found that they need very few iterations, in particular for practical values of the problem and method parameters.

2 The optimal control problem

Following [1], consider the optimal control problem of finding the state $y(x,t)$ and control $u(x,t)$ that minimizes the functional,

$$J(y,u) = \frac{1}{2}\int_{\Omega\times(0,T)} |y(x,t) - y_d(x,t)|^2 dx\, dt + \frac{1}{2}\beta \int_{\Omega\times(0,T)} |u(x,t)|^2 dx\, dt,$$

subject to the time periodic parabolic, heat equation problem,

$$\begin{aligned} \frac{\partial y(x,t)}{\partial t} - \Delta y(x,t) &= u(x,t) \text{ in } \Omega\times(0,T), \\ y(x,t) = 0 \text{ in } \Gamma\times(0,T), \quad y(x,0) &= y(x,T) \text{ and } u(x,0) = u(x,T) \text{ in } \Omega. \end{aligned}$$

Here $\Gamma = \partial\Omega$, y_d is the desired state and $\beta > 0$ is the cost regularization parameter for the control function $u(x,t)$. The target function is assumed to be time-harmonic, $y_d(x,t) = y_d(x)e^{i\omega t}$ with frequency $\omega = 2\pi k/T$ for some non-negative integer k.

Remark 1. If the target function is not time-harmonic, one can approximate it by a truncated Fourier series of the form $y_d = \sum_{k=0}^{N}\left(y_{d,k}^c \cos(k\omega t) + y_{d,k}^s \sin(k\omega t)\right)$, where the Fourier coefficients are given by classical expressions. Since the equation is linear, the solution and the control are also time-harmonic, $y(x,t) = y(x)e^{i\omega t}$ and $u(x,t) = u(x)e^{i\omega t}$ and the problem separates for the different frequencies.

Therefore it suffices to consider the single frequency problem,

$$\text{minimize}_{y,u} \ \frac{1}{2}\int_{\Omega} |y(x) - y_d(x)|^2 dx + \frac{1}{2}\beta \int_{\Omega} |u(x)|^2 dx$$

subject to $i\omega y(x) - \Delta y(x) = u(x)$ in Ω. We assume that $y(x)$ and $y_d(x)$ are real valued but the control $u(x) = u_0(x) + iu_1(x)$ must be complex valued.

The state equation and hence also the minimization problem, has a unique solution. Using an appropriate finite element subspace V_h for both y and u and a complex-valued Lagrange multiplier vector $\underline{\zeta}$, the corresponding Lagrangian functional for the discretized constrained optimization problem becomes

$$L(y, \underline{u}, \underline{\zeta}) = \frac{1}{2}(y - y_d)^T M(y - y_d) + \frac{1}{2}\beta \underline{u}^* M \underline{u} + Re\{\underline{\zeta}^*(i\omega My + Ky - M\underline{u})\},$$

where M is the mass matrix and K is the negative discrete Laplacian. One of the first order necessary condition, $\nabla L(y, \underline{u}, \underline{\zeta}) = 0$, shows that $\beta M\underline{u} = -M\underline{\zeta}$ and lead to the reduced system, $\begin{bmatrix} M & \beta(K - i\omega M) \\ K + i\omega M & -M \end{bmatrix} \begin{bmatrix} y \\ \underline{u} \end{bmatrix} = \begin{bmatrix} My_d \\ 0 \end{bmatrix}$, using the relation $\underline{\zeta} = \beta\underline{u}$. Here we multiply the second equation with $\sqrt{\beta}$ and introduce $\tilde{u} = \sqrt{\beta}u$, which gives

$$\begin{bmatrix} M & \sqrt{\beta}(K - i\omega M) \\ \sqrt{\beta}(K + i\omega M) & -M \end{bmatrix} \begin{bmatrix} y \\ \underline{\tilde{u}} \end{bmatrix} = \begin{bmatrix} My_d \\ 0 \end{bmatrix}. \tag{1}$$

We present now two preconditioners for this block matrix.

3 A block diagonal preconditioner

We consider now a general form of two-by-two block matrices, for which the matrix in (1) is a special case. Let then $\mathcal{A} = \begin{bmatrix} A & E - iF \\ E + iF & -A \end{bmatrix}$, where A is spd and E and F are symmetric and positive semidefinite (spsd). Following [1], but with a simplified analysis, we consider block diagonal preconditioner. The following eigenvalue bounds hold for the preconditioned matrix.

Proposition 1. *Let* $\mathcal{D} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$, $D = A + E + F$ *and assume that* $ED^{-1}F = FD^{-1}E$. *This holds if* $F = \omega(A + \delta E)$, $\omega > 0$, $0 \le \delta \le 1$. *Then the matrix* $(\mathcal{D}^{-1}\mathcal{A})^2$ *is block diagonal, its eigenvalues are real and contained in the interval* $\frac{1}{4} \le \lambda\left((\mathcal{D}^{-1}\mathcal{A})^2\right) \le 1$. *If* $F = \omega A$, *then* $\frac{1}{3} \le \frac{1}{2(1+\omega/(1+\omega^2))} \le \lambda((\mathcal{D}^{-1}\mathcal{A})^2) \le 1$.

Proof. A proof is presented in [7]. Note that for ω small or large, the lower bound is close to its value $1/2$ taken for $\omega = 0$.

To solve a system with $\mathcal{A}$ using the block diagonal preconditioner in a Krylov subspace iteration method, as well known requires then $2m$ iterations so that the residual r^{2m} satisfies $\|r^{2m}\|/\|r^0\| \le \frac{2q^m}{1+q^{2m}}$, where $q = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{1}{2+\sqrt{3}}$. This is guaranteed for $2m$ proportional to the logarithm of the relative precision.

4 A preconditioner for a two-by-two block matrix of special form with square matrix blocks

Consider now a matrix $\mathcal{A} = \begin{bmatrix} A & B_2 \\ B_1 & -A \end{bmatrix}$ and preconditioner $\mathcal{C} = \begin{bmatrix} A + B_1 + B_2 & B_2 \\ B_1 & -A \end{bmatrix}$, where A, of order $n \times n$, is spd and $A + B_i$, $i = 1, 2$ are nonsingular. The preconditioner and the given matrix will be used in a Krylov subspace type of iteration method. We show first that linear systems $\mathcal{C} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$ can be readily solved. For this reason, change the sign of the second equation and add the first, which results in $\begin{bmatrix} A + B_1 & B_2 \\ 0 & A + B_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} f \\ f - g \end{bmatrix}$, where $z = x + y$. Hence the algorithm to compute the solution (x, y) can be written: 1. Solve $(A + B_2)z = f - g$, compute $\tilde{f} = f - B_2 z$. 2. Solve $(A + B_1)x = \tilde{f}$, compute $y = z - x$. Therefore, besides a matrix vector multiplication with matrix B_2 and some vector additions, the algorithm involves a solution with matrix $A + B_2$ and with $A + B_1$. In our problems, they will be discretized elliptic type of matrices.

It is seen that the above algorithm is equivalent to computing the action of the following form of the inverse of $\mathcal{C}$,

$$\mathcal{C}^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} (A + B_1)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -(A + B_2)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}.$$

This form was already given in [2].

For the computation of the rate of convergence of the iteration method we need information about the eigenvalue distribution of $\mathcal{C}^{-1}\mathcal{A}$.

Proposition 2. *Let* $\mathcal{A} = \begin{bmatrix} A & B_2 \\ -B_1 & A \end{bmatrix}$, $\mathcal{C} = \mathcal{A} + \begin{bmatrix} B_1 + B_2 & 0 \\ 0 & 0 \end{bmatrix}$, *where* A, *of order* $n \times n$ *is spd and* $B_2 = B_1^*$, $B_1 = B$, $B + B^*$ *is positive semidefinite and* $A + B$ *is nonsingular. Then the eigenvalues* λ *of* $\mathcal{C}^{-1}\mathcal{A}$ *are real and satisfy* $\frac{1}{2} \le \frac{1}{1+\alpha} \le \lambda \le 1$, *where* $\alpha = \max_{\mu} \{Re(\mu)/|\mu|\}$, *and* μ *are eigenvalues of* $Bz = \mu A z$, $z \neq 0$. *The eigenvector space is complete, so* $\mathcal{C}^{-1}\mathcal{A}$ *is a normal matrix. Here* $\lambda = 1$ *is an eigenvalue of dimension* $n + n_0$, *where* $n_0 = dim\{\mathcal{N}(B + B^*)\}$.

Proof. For a proof, see [7].

This proposition shows that the relative size, $Re(\mu)/|\mu|$ of the real part of the eigenvalues of $\mu A z = Bz$, $\|z\| \neq 0$, determines the lower bound of $\mathcal{C}^{-1}\mathcal{A}$.

For the matrix in (1) it follows that

$$\mathcal{C} = \begin{bmatrix} M + 2\sqrt{\beta}K & \sqrt{\beta}(K - i\omega M) \\ \sqrt{\beta}(K + i\omega M) & -M \end{bmatrix},$$

and $\sqrt{\beta}(K - i\omega M)z = \mu M z$, $\|z\| \neq 0$, so $\alpha = \frac{Re(\mu)}{\sqrt{Re(\mu)^2 + \omega^2}} \le 1$. The correspondingly preconditioned Krylov subspace iteration method converges therefore fast with a rate determined by the narrow eigenvalue bounds $\frac{1}{2} \le \frac{1}{1+\alpha} \le \lambda \le 1$, and will be particularly fast for large values of ω where α gets small.

5 A double two-by-two block matrix arising in eddy current electromagnetic problems

Following [1], consider now the multiharmonic method to numerically solve an eddy current problem. Here the vector Laplacian operator in Section 2 is replaced by a curl curl operator. It is assumed that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$. The reluctivity $\nu \in L^\infty(\Omega)$ is uniformly positive and we assume that it does not depend on the solution y, so the problem is linear. The conductivity $\sigma \in L^\infty(\Omega)$ is piecewise constant, positive in conducting and zero in nonconducting subdomains. Due to the discontinuity of σ and to obtain uniqueness in the nonconducting domains, the state equation must be regularized. This is done here by adding a positive term εy, $\varepsilon > 0$ to the state equation. However, for the case of divergence free vector solutions, this is not needed. The regularized optimal control problem takes then the form,

$$\text{minimize}_{(y,u)} \frac{1}{2} \int_{\Omega\times(0,T)} |y - y_d|^2 dx\, dt + \frac{\beta}{2} \int_{\Omega\times(0,T)} |u|^2 dx\, dt,$$

subject to the state equation,

$$\begin{cases} \sigma \frac{\partial y}{\partial t} + \operatorname{curl}(\nu \operatorname{curl} y) + \varepsilon y = u \;\text{ in } \Omega \times (0,T) \\ y \times \underline{n} = 0 \;\text{ on } \Gamma \times (0,T), \qquad y = y_0 \;\text{ on } \Gamma \times \{0\}. \end{cases}$$

For a time-harmonic problem, the initial condition is replaced by the periodicity equation, $y(0) = y(T)$, in Ω. Applying a Lagrange multiplier w to impose the state equation, the Lagrangian functional becomes

$$\mathcal{L}(y,u,w) = J(y,u) + \int_{\Omega\times(0,T)} \left(\sigma \frac{\partial y}{\partial t} + \operatorname{curl}(\nu \operatorname{curl} y) + \varepsilon y - u \right) w\, dx\, dt.$$

The first order necessary condition $\nabla_w \mathcal{L}(y,u,w) = 0$ gives the relation $\beta u = w$ in $\Omega \times (0,T)$, which enables elimination of the control variable. As before we use a truncated Fourier series expansion for y and u, which decouple the equations so that it suffices to consider only one frequency.

For the finite element discretization we use the lowest order tetrahedral edge elements, originally introduced in Nèdèlec [3]. After a reordering of the equations, this yields the following system of linear equations,

$$\begin{bmatrix} M & 0 & K & -M_\omega \\ 0 & M & M_\omega & K \\ K & M_\omega & -\beta^{-1} M & 0 \\ -M_\omega & K & 0 & -\beta^{-1} M \end{bmatrix} \begin{bmatrix} y^c \\ y^s \\ w^c \\ w^s \end{bmatrix} = \begin{bmatrix} y_d^c \\ y_d^s \\ 0 \\ 0 \end{bmatrix},$$

where $M = [M_{ij}]$, $M_{ij} = \int_\Omega u_j v_i dx$, $(M_\omega)_{ij} = \int_\Omega \sigma\omega u_j v_i dx$, $i,j = 1,2,\cdots,n$, $K = [K_{ij}]$, $K_{ij} = \int_\Omega \nu \operatorname{curl} u_j \operatorname{curl} v_i + \varepsilon \int_\Omega u_j v_i dx$. Further u_i, v_i are taken from the set of finite element basis functions in $H_0(\text{curl}) = \{v \in L^2(\Omega) :\ \text{curl } v \in$

$L^2(\Omega), v \times \underline{n} = 0$ on $\Gamma\}$ on edges i, j. The values on the right hand side are given by $(y^c_d)_i = \int_\Omega y^c_d v_i dx, \quad (y^s_d)_i = \int_\Omega y^s_d v_i dx.$

In a similar way as was done before, we modify the system by multiplying the last two equations with $\sqrt{\beta}$ and scale the multiplier variable to $\begin{bmatrix} \tilde{w}^c \\ \tilde{w}^s \end{bmatrix} = \frac{1}{\sqrt{\beta}} \begin{bmatrix} w^c \\ w^s \end{bmatrix}$. Using the same type of preconditioning as in Section 4, obtained by adding the off-diagonal blocks to the primary diagonal block, we get

$$\mathcal{C} = \begin{bmatrix} M + 2\tilde{K} & 0 & \tilde{K} & -\tilde{M}_\omega \\ 0 & M + 2\tilde{K} & \tilde{M}_\omega & \tilde{K} \\ \tilde{K} & \tilde{M} & -M & 0 \\ -\tilde{M} & \tilde{K} & 0 & -M \end{bmatrix},$$

where $\tilde{K} = \sqrt{\beta}K$ and $\tilde{M}_\omega = \sqrt{\beta}M_\omega$. To get the same form of the matrix as in Section 4, let $A = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$, $B = \begin{bmatrix} \tilde{K} & \tilde{M}_\omega \\ -\tilde{M}_\omega & \tilde{K} \end{bmatrix}$. Then $\mathcal{A} = \begin{bmatrix} A & B^* \\ B & -A \end{bmatrix}$ and $C = \begin{bmatrix} A + 2\tilde{K}' & B^* \\ B & -A \end{bmatrix}$, where $\tilde{K}' = \begin{bmatrix} \tilde{K} & 0 \\ 0 & \tilde{K} \end{bmatrix}$. Then $B + B^* = 2\tilde{K}'$, which is spd and it follows from Proposition 2 that the eigenvalues λ of $\mathcal{C}^{-1}\mathcal{A}$ satisfy $\frac{1}{1+\alpha} \leq \lambda \leq 1$, where α is the ratio, $\alpha = Re(\mu)/|\mu|$ and μ is eigenvalue of the generalized eigenvalue problem,

$$\begin{bmatrix} \tilde{K} & \tilde{M}_\omega \\ -\tilde{M}_\omega & \tilde{K} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mu \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } \begin{bmatrix} \hat{K} & \hat{M}_\omega \\ -\hat{M}_\omega & \hat{K} \end{bmatrix} \begin{bmatrix} M^{1/2}x \\ M^{1/2}y \end{bmatrix} = \mu \begin{bmatrix} M^{1/2}x \\ M^{1/2}y \end{bmatrix},$$

where $\hat{K} = M^{-\frac{1}{2}}\tilde{K}M^{-\frac{1}{2}}$, $\hat{M}_\omega = M^{-\frac{1}{2}}\tilde{M}_\omega M^{-\frac{1}{2}}$. Hence $\alpha = \frac{\|\hat{K}^{-1/2}\hat{M}_\omega\hat{K}^{-1/2}\|}{1+\|\hat{K}^{-1/2}\hat{M}_0\hat{K}^{-1/2}\|}$.

The arising inner systems with the block matrix $\begin{bmatrix} M + \tilde{K} & \tilde{M}_\omega \\ -\tilde{M}_\omega & M + \tilde{K} \end{bmatrix}$ can also be solved by iteration using the same type of preconditioner as for the outer system, i.e. with $\begin{bmatrix} M + \tilde{K} + 2\tilde{M}_\omega & \tilde{M}_\omega \\ -\tilde{M}_\omega & M + \tilde{K} \end{bmatrix}$.

The corresponding eigenvalues $\tilde{\lambda}$ satisfy $(\tilde{\lambda}-1) \begin{bmatrix} M + \tilde{K} + 2\tilde{M}_\omega & \tilde{M}_\omega \\ -\tilde{M}_\omega & M + \tilde{K} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 2\tilde{M}_\omega & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, from which it follows that $\tilde{\lambda} \leq 1$ and $(\tilde{\lambda}-1)x^T(M + \tilde{K} + 2\tilde{M}_\omega + \tilde{M}_\omega(M + \tilde{K})^{-1}\tilde{M}_\omega)x = -2x^T\tilde{M}_\omega x$, i.e. $(\tilde{\lambda}-1)\hat{x}^T(I + 2\hat{\hat{M}}_\omega + \hat{\hat{M}}_\omega^2)\hat{x} = -2\hat{x}^T\hat{M}_\omega\hat{x}$, where $\hat{\hat{M}}_\omega = (M + \tilde{K})^{-1/2}\tilde{M}_\omega(M + \tilde{K})^{-1/2}$ and $\hat{x} = (M + \tilde{K})^{1/2}x$. It follows that $\tilde{\lambda} - 1 \geq -\frac{1}{2}$, i.e. $\tilde{\lambda} \geq \frac{1}{2}$, so $\frac{1}{2} \leq \tilde{\lambda} \leq 1$.

In practice mostly the control and observation are restricted to subdomains of Ω. Due to limitation of space we do not consider this here, but it can be shown that our preconditioner performs as well for these problems also. However, as reported in [5], the performance of the block diagonal preconditioner deteriorates for small values of ν and β.

6 Numerical illustrations

In order to demonstrate the efficiency of our method, which holds uniformly with respect to all model and method parameters involved, some numerical tests were done. To enable a comparison with the block diagonal preconditioner used in the thesis by M. Kolmbauer [6], we test our method on the same problems as done there. Due to limitations we thereby choose only a subset of the problems.

We demonstrate this only on the eddy current electromagnetic problem with constant and with jump of the conductivity coefficient. In Table 1 we show how the uniformly bounded and low number of flexible GMRES (FGMRES) iterations varies for different values of the frequency ω and of the control cost parameter β. This is done for two values of the mesh size parameter h. In Table 2

Table 1. Robustness of outer and total inner (in brackets) FGMRES iterations with respect to β, ω, and h, while fixing $\nu = \sigma_2 = 1$ and outer rel. prec. 10^{-8}.

h	β	inner rel. prec. 10^{-2}					inner rel. prec. 10^{-6}				
		ω					ω				
		10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}	10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
1/16	10^{-10}	10(20)	10(20)	10(20)	10(40)	3(11)	10(20)	10(20)	10(29)	10(87)	2(12)
	10^{-8}	11(22)	11(22)	11(22)	10(57)	3(11)	11(22)	11(22)	11(33)	9(125)	2(12)
	10^{-6}	11(22)	11(22)	11(22)	6(48)	3(11)	11(22)	11(22)	11(36)	5(80)	2(12)
	10^{-4}	9(18)	9(18)	9(18)	6(48)	3(11)	9(18)	9(18)	9(45)	4(72)	2(12)
	10^{-2}	5(10)	5(10)	6(23)	7(56)	3(11)	5(10)	5(15)	5(30)	3(55)	2(12)
	10^{0}	4(8)	4(8)	5(19)	7(56)	3(11)	4(8)	4(12)	4(24)	3(56)	2(12)
1/32	10^{-10}	10(20)	10(20)	10(20)	11(42)	4(15)	10(20)	10(20)	10(29)	10(89)	2(14)
	10^{-8}	11(22)	11(22)	11(22)	10(58)	4(15)	11(22)	11(22)	11(33)	10(144)	2(13)
	10^{-6}	11(22)	11(22)	11(22)	6(48)	4(15)	11(22)	11(22)	11(36)	5(80)	2(14)
	10^{-4}	9(18)	9(18)	9(18)	7(56)	4(15)	9(18)	9(18)	9(45)	4(72)	2(14)
	10^{-2}	5(10)	5(10)	6(23)	7(56)	4(15)	5(10)	5(14)	5(30)	3(55)	2(14)
	10^{0}	4(8)	4(8)	5(19)	7(56)	4(15)	4(8)	4(12)	4(24)	3(55)	2(14)

(left) it is shown how the number of iterations vary with respect to the values of reluctivity and ω, and in Table 2 (right) how they vary with respect to conductivity. Thereby the conductivity is fixed in the domain $\Omega = [0,1]^3$, respectively takes a constant positive value σ_2 in the subcube $\Omega_2 = [1/4, 3/4]^3$ and $\sigma_1 = 1$ in the rest of the domain $\Omega_1 = \Omega \setminus \Omega_2$. The listed number of degrees of freedom (DOFs) for the lowest order Nédélec elements on tetrahedron is equal to one per edge.

All tests demonstrate a remarkable, uniformly low number of iterations. For the problem with no jump in the conductivity coefficient the number of iteration decreases for large values of the frequency, which is in accordance with the theoretical bound of the condition number. It can be seen that the number of iterations demonstrate a more favourable performance of our method as compared to the block diagonal preconditioner. Furthermore, the choice of elliptic operator problem to be solved on the innermost level is straightforward in our method while it is somewhat more involved for the block diagonal preconditioner used in [5].

Table 2. Robustness of outer and total inner (in brackets) FGMRES iterations with respect to: a) β, ν, and h, while fixing $\omega = \sigma_2 = 1$ (left); b) β, σ_2, and h, while fixing $\omega = \nu = 1$ (right). In both cases we fix outer rel. prec. 10^{-8} and inner rel. prec. 10^{-2}.

h	β	ν					σ_2				
		10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}	10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
	10^{-10}	1(2)	2(4)	10(20)	5(10)	2(4)	10(20)	10(20)	10(20)	10(38)	6(24)
	10^{-8}	2(4)	3(6)	11(22)	4(8)	3(6)	11(22)	11(22)	11(22)	11(65)	6(23)
1/16	10^{-6}	2(4)	4(8)	11(22)	3(6)	3(6)	11(22)	11(22)	11(22)	12(75)	6(23)
	10^{-4}	3(6)	6(12)	9(18)	4(8)	4(8)	10(21)	10(21)	9(18)	10(80)	5(19)
	10^{-2}	2(8)	10(40)	6(23)	4(13)	5(17)	7(24)	7(24)	6(23)	7(56)	3(12)
	10^{0}	2(8)	10(57)	5(19)	3(12)	5(20)	7(32)	7(32)	5(19)	6(46)	2(8)
	10^{-10}	1(2)	2(4)	10(20)	5(10)	3(6)	10(20)	10(20)	10(20)	10(38)	7(30)
	10^{-8}	2(4)	3(6)	11(22)	4(8)	3(6)	11(22)	11(22)	11(22)	12(71)	6(24)
1/32	10^{-6}	2(4)	5(10)	11(22)	3(6)	3(6)	11(22)	11(22)	11(22)	12(76)	6(25)
	10^{-4}	3(6)	9(18)	9(18)	4(8)	4(8)	10(21)	10(21)	9(18)	10(80)	5(22)
	10^{-2}	2(8)	11(42)	6(23)	4(13)	5(16)	8(25)	8(25)	6(23)	7(56)	4(18)
	10^{0}	3(12)	10(58)	5(19)	3(12)	5(20)	8(43)	8(43)	5(19)	6(46)	2(9)

Tables 1 and 2 correspond to tables 7.1 - 7.3, 7.4 and 7.5 in [6] with the same stopping tolerance. It is seen that our method never gives more iterations than those reported there but mostly fewer and in some cases significantly smaller.

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