Preconditioning methods for eddy-current optimally controlled time-harmonic electromagnetic problems

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Abstract

Time-harmonic problems arise in many important applications, such as eddy current optimally controlled electromagnetic problems. Eddy current modelling can also be used in non-destructive testings of conducting materials. Using a truncated Fourier series to approximate the solution, for linear problems the equation for different frequencies separate, so it suffices to study solution methods for the problem for a single frequency.

The arising discretized system takes a two-by-two or four-by-four block matrix form. Since the problems are in general three-dimensional in space and hence of very large scale, one must use an iterative solution method. It is then crucial to construct efficient preconditioners.

It is shown that an earlier used preconditioner for optimal control problems is applicable here also and leads to very tight eigenvalue bounds and hence very fast convergence such as for a Krylov subspace iterative solution method. A comparison is done with an earlier used block diagonal preconditioner.

1 Introduction

Eddy current electromagnetic problems, i.e. modelling the interaction of magnetic and electric fields, arise as an application of Maxwell's equations. They can be used in many applications including non-destructive testings of conducting materials, see e.g. [1]. As has been shown e.g. in [2], [3] and [4], time-dependent eddy current electromagnetic and similar problems can be approximated efficiently by a truncated Fourier series expansion in time-harmonic terms. This enables replacing the, normally slow and expensive step by step time integration procedure with the solution of time-harmonic problems for a set of frequencies. For linear problems, they decouple and the different frequency problems can hence be solved independently in parallel. For nonlinear problems that arise because the coefficient in the differential equation may depend on the solution, see e.g. [5], and references therein, a possible solution method is to use a two level grid method where the nonlinear problem is solved on a coarse grid and the linearized equation is solved on the fine grid, see e.g. [6]. This will, however, not be considered in the present paper.

The sources are frequently harmonic alternating currents. For each frequency term one gets a coupled system involving state and costate regularized equations. Since the conductivity coefficient is zero in nonconducting regions one must in general regularize the problem to get a unique solution. This is not needed if the solution is divergence free and a classical $\inf - \sup$ condition holds.

Optimal control problems can be formulated via a Lagrange multiplier to impose the state equation which leads to a Lagrangian optimization functional. One of the necessary zero gradient conditions for this functional involves a simple relation between the control function and the Lagrangian multiplier, implying that one of the two can directly be eliminated. Using a real valued formulation involving $\cos(\omega t)$ and $\sin(\omega t)$, where ω is the frequency, as shown in [3, 4], after a finite element discretization the resulting reduced system can be written in a four-by-four block matrix operator form. For a general presentation of optimal control problems for partial differential equations, we refer to [7].

To solve the corresponding linear system, which is on a special saddle point form, direct methods are unfeasible since the problems are generally of large scale, so iterative solution methods must be used. Thereby the choice of preconditioner is crucial. It is shown that a previously used preconditioner for optimal control problems considered for a two-by-two block matrix problem with square blocks (see [8, 9, 10] and references therein) can be used here also, leading to two systems of two-by-two block form and for each two-by-two block, two simpler systems of elliptic form to be solved. The resulting eigenvalues are favourably bounded by $[1 - \alpha, 1]$, where α is less than $1/2$ and it is small for large frequencies.

The resulting block matrix to be solved involves still a matrix on saddle point form but the above preconditioning method can be applied here also, leading to an inner iteration method. Since the condition number of the preconditioned matrix is very small, few iterations are needed both for the outer and inner iterations.

In [3] another preconditioner, on block diagonal form, is used resulting in a spectral condition number bounded by 3, that holds for the square of the preconditioned matrix. Note that then the bound for the number of iterations becomes $2m$, where m corresponds to the iterations needed to reduce the residual to a desired relative small number for a matrix with condition number 3. Since our method leads to a condition number bound less than 2, and this holds for the matrix itself, i.e. not for the square of the matrix, it does not need this double amount of iterations and is therefore more efficient than the method in [3].

The remainder of the paper is composed as follows. For completeness of the presentation, in the next section we present a short introduction to Maxwell's equations for eddy current problems and show how the two-by-two block matrix structure arises. Then, in Section 3 we present a general outline of preconditioners for matrices on two-by-two block form with square matrices including an eigenvalue analysis. We show also that the degree of the minimal polynomial for a special form of block triangular normal matrix does not depend on the off-diagonal block, which has important consequences for the preconditioner presented in this paper. We present also a, compared to the presentation in [3], simplified analysis of preconditioners on block diagonal form.

In Section 4 we present an application of the preconditioner for optimal control problems of parabolic type with a time-harmonic target state. Following [3], in Section 5 we consider then an eddy current electromagnetic problem, which leads to a matrix on a double two-by-two block structure and a corresponding time-harmonic formulation. In Section 6 follows then an outline and analysis of the preconditioner for this problem, including the more difficult to handle case where the control and state functions are prescribed only on a subset of the whole domain of definition.

Section 7 contains extensive numerical tests that illustrate the efficiency of the methods. The paper ends with some concluding remarks. A shorter version of this paper has been published in the special LSSC'17 issue, see [11]. It does not give all details and proofs of the method and does not contain the important case where the control and state functions are prescribed only on a subset of the whole domain of definition.

2 Multigrid-FEM for the eddy current problem

We shall consider the linear eddy current case of Maxwell's equations in a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary Γ. The problem can be formulated as to find a time-dependent magnetic vector potential y such that

$$\begin{cases} \sigma \frac{\partial y}{\partial t} + \operatorname{curl}(\nu \operatorname{curl} y) = j & \text{in } \Omega \times (0,T), \\ y \times n = 0 & \text{on } \partial\Omega \times (0,T), \\ y = y_0 & \text{on } \partial\Omega \times \{0\}, \end{cases}$$

where σ, ν, and j denote the electrical conductivity, magnetic reluctivity, and the external current density, respectively, and where n is the unit outward normal vector to Ω.

Due to that σ is vanishing on a part of Ω to obtain uniqueness in the non-conducting regions one can prescribe the solution to be divergence-free so that a classical $\inf - \sup$ stability relation holds. Here we employ another regularization of the state equation, we add the term εy, $\varepsilon > 0$. The regularized problem takes then the form,

$$\begin{cases} \sigma \frac{\partial y}{\partial t} + \operatorname{curl}(\nu \operatorname{curl} y) + \varepsilon y = j & \text{in } \Omega \times (0,T), \\ y \times n = 0 & \text{on } \partial\Omega \times (0,T), \\ y = y_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{2.1}$$

As pointed out in [12, 13, 14], since j is by the charge conservation law divergence-free the control admits the form $j = \operatorname{curl} u$. At the same time pointwise state variable constraints are considered as well as nonlinear and nonsmooth constitutive laws. These topics fall outside the major concern of the present paper. We discuss them in Conclusion.

In the time-harmonic regime with angular frequency ω, $y(x,t) = Re\{\widehat{y}(x)e^{i\omega t}\}$, $j(x,t) = Re\{\widehat{j}(x)e^{i\omega t}\}$ it leads to find the complex-valued amplitude $\widehat{y}$ satisfying

$$\begin{cases} i\omega\sigma\widehat{y} + \operatorname{curl}(\nu \operatorname{curl} \widehat{y}) + \varepsilon\widehat{y} = \widehat{j} & \text{in } \Omega, \\ \widehat{y} \times n = 0 & \text{on } \partial\Omega. \end{cases}$$

The problem is formulated in sense of distributions to find $\widehat{y} \in H_0(\mathrm{curl};\Omega)$:

$$i\omega \int_\Omega \sigma(x)\,\widehat{y}(x)\cdot v(x)\,dx + \int_\Omega \left(\nu(x)\,\mathrm{curl}\,\widehat{y}(x)\cdot \mathrm{curl}\,v(x) + \varepsilon\,\widehat{y}(x)\cdot v(x)\right)\,dx$$
$$= \int_\Omega \widehat{j}(x)\cdot v(x)\,dx$$

for all complex-valued test functions $v \in H_0(\mathrm{curl};\Omega)$, where $H_0(\mathrm{curl};\Omega) := \{v \in L^2(\Omega)^3 :\ \mathrm{curl}\,v \in L^2(\Omega)^3,\ v\times n = 0 \text{ on } \Gamma\}$. Assuming $\widehat{j} \in L^2(\Omega)^3$ the linear form is bounded. Assuming further $\sigma,\nu \in L^\infty(\Omega)$, $\sigma(x) \geq 0$, and $\nu(x) \geq \nu_0 > 0$ the bilinear form is bounded and elliptic, therefore, the problem is uniquely solvable and the solution is continuously dependent on the data. Note that the problem can be equivalently viewed as a 2-by-2 real elliptic system solved for the cosine and sine parts, $\widehat{y}(x) = y^c(x) + iy^s(x)$.

Conforming finite element approximation of functions $v \in H_0(\mathrm{curl};\Omega)$ requires continuity of the traces $v\times n$. In [15] and [16] two classes of such elements were proposed. Here we use the lowest-order finite elements of the former class, which is referred to as Nédélec-I elements. On a tetrahedral mesh the finite element functions takes the local form $v(x) = a\times x + b$, $a,b \in R^3$. In order to preserve global continuity of the tangential components the degrees of freedom are tangential moments along edges. We arrive at the linear system of equations

$$(i\omega M + K)z = b, \tag{2.2}$$

where $M_{ij} := \int_\Omega \sigma\,\varphi_j\cdot\varphi_i$, $K_{ij} := \int_\Omega \nu\,\mathrm{curl}\,\varphi_j\cdot\mathrm{curl}\,\varphi_i + \varepsilon\,\varphi_j\cdot\varphi_i$, $b_i := \int_\Omega \widehat{j}\cdot\varphi_i$ with $\varphi_i(x)$, $i,j = 1,\ldots,n$, being the Nédélec-I basis functions. To avoid complex arithmetics we can rewrite (2.2) in real valued block matrix form

$$\begin{bmatrix} K & -\omega M \\ \omega M & K \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

where $z = x + iy$ and $b = \xi + i\eta$. As we shall see, such types of block matrices arise also in some optimal control problems for PDEs, including time-harmonic problems.

The resulting system can be solved efficiently by use of various iterative solution methods, based on block matrix preconditioning methods and algebraic or geometric multigrid methods. For the latter, as prolongation operator we choose the natural embedding operator. However, construction of a smoother is now specific due to the large kernel of the curl-operator, which includes gradient fields. Hiptmair [17] proposes a hybrid smoother which separately smooths out projections of the Nédélec functions onto a discrete gradient space. Here we employ another approach proposed by Arnold, Falk, and Winther [18]. The smoother is constructed as an overlapping Schwarz method, where a block represents a Dirichlet problem comprising the degrees of freedom adjacent to a given node. This can again treat the nodal-based discrete gradient fields. Note that we assemble the Schwarz blocks in the multiplicative, rather than additive, way.

3 A preconditioner for matrices in two-by-two block form of a special saddle-point form with square matrix blocks

We present now a preconditioner which is similar to the one used in earlier publications, such as [8, 9, 10], but give here a slightly different presentation and an improved eigenvalue estimate.

3.1 The two-by-two block matrix preconditioner and its eigenvalue analysis

Let

$$\mathcal{A} = \begin{bmatrix} A & B_2 \\ -B_1 & A \end{bmatrix} \tag{3.1}$$

be given where A, of order $n \times n$, is assumed to be symmetric and positive definite and $A + B_i$, $i = 1, 2$ are nonsingular. Let $\mathcal{C} = \begin{bmatrix} A + B_1 + B_2 & B_2 \\ -B_1 & A \end{bmatrix}$ be a preconditioner to $\mathcal{A}$, to be used in a Krylov subspace type of iteration method, such as GMRES [19] or MinRes [20]. Given a linear matrix preconditioning equation

$$\mathcal{C} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{3.2}$$

by adding the first to second equation, the system can be written in the equivalent form,

$$\begin{cases} (A + B_1 + B_2)x + B_2 y = f \\ (A + B_2)x + (A + B_2)y = f + g \end{cases}$$

i.e.,

$$\begin{bmatrix} A + B_1 & B_2 \\ 0 & A + B_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} f \\ f + g \end{bmatrix},$$

where $z = x + y$. Hence $z = (A + B_2)^{-1}(f + g)$ and $(A + B_1)x = f - B_2 z$. Therefore the algorithm to compute the solution of (3.2) can be written as

- Solve $(A + B_2)z = f + g$
- Compute $\widetilde{f} = f - B_2 z$
- Solve $(A + B_1)x = \widetilde{f}$
- Compute $y = z - x$

Hence, besides some vector additions, the algorithm involves a solution of a linear systems with $A+B_2$, a matrix vector multiplication with B_2 and a system with matrix $A+B_1$. In practice, the solution of the two linear systems contribute to the major cost of computing an action of $\mathcal{C}^{-1}$. As we shall see, in our applications they correspond to elliptic operators.

It is seen that the above procedure is equivalent to the use of the following form of the inverse of $\mathcal{C}$,

$$\mathcal{C}^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} (A + B_1)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (A + B_2)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}. \tag{3.3}$$

This form was already used in [10].

For the analysis of the rate of convergence of the preconditioned iteration method we need information about the eigenvalue distribution of the preconditioned matrix $\mathcal{C}^{-1}\mathcal{A}$. To derive this we make the following assumptions.

Assumption 3.1. *A and $B_1 + B_2$ are symmetric and positive semidefinite (spsd) and $A + B_1 + B_2$ is positive definite. Further $A + B_1$ and $A + B_2$ are nonsingular and B_1, B_2 commute, $B_1B_2 = B_2B_1$, in the sense that $x^T(B_1B_2 - B_2B_1)x = 0$ for all x.*

Clearly, the assumption is equivalent to that A is positive definite on the nullspace $\mathcal{N}(B_1 + B_2)$.

As we shall see, in our application $A = M$, where M is the mass matrix corresponding to the observation domain and $B_1 = \sqrt{\beta}(K - i\omega M)$, $B_2 = B_1^*$, where $\beta > 0$, K is a discretization of a selfadjoint second-order elliptic operator, and $\omega \geq 0$ is an angular frequency. Hence B_1 and B_2 commute. In the following we assume that $\beta = 1$, which is no restriction to the main result.

Proposition 3.1. *Assume that Assumption 3.1 holds. Then an eigenvalue λ of matrix $\mathcal{C}^{-1}\mathcal{A}$, where $\mathcal{A}$, $\mathcal{C}$ are defined in (3.1) respectively (3.2) satisfies*

$$\frac{1}{2} \leq 1 - \frac{4q}{(1+2q)^2 + 4\widetilde{\omega}^2(1+q)^2} = g(q) \leq \lambda \leq 1,$$

where $q = \sqrt{\frac{\widetilde{\omega}^2 + 1/4}{\widetilde{\omega}^2 + 1}}$ and where $\widetilde{\omega} = \omega\eta$ and η, $0 \leq \eta \leq 1$, is an eigenvalue of $D^{-1/2}AD^{-1/2}$, $D = A + B_1 + B_2$. A minimal value of λ is taken for $\omega = 0$. The dimension of the eigenvalue space for $\lambda = 1$ equals $n + n_0$, where $n \times n = \dim(A)$ and $n_0 = \dim \mathcal{N}(B_1 + B_2)$.

Proof. For the generalized eigenvalue problem,

$$\lambda \mathcal{C} \begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{A} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \|x\| + \|y\| \neq 0,$$

where $\mathcal{C} = \mathcal{A} + \begin{bmatrix} B_1 + B_2 & 0 \\ 0 & 0 \end{bmatrix}$, it holds

$$(1-\lambda) \begin{bmatrix} A + B_1 + B_2 & B_2 \\ -B_1 & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (B_1 + B_2)x \\ 0 \end{bmatrix}. \tag{3.4}$$

It follows that $\lambda = 1$ for eigenvectors (x, y) such that $\{x \in \mathcal{N}(B_1 + B_2), y \text{ arbitrary}\}$ so the dimension of the eigenvector space corresponding to $\lambda = 1$ is $n + n_0$.

Consider now eigenvalues $\lambda \neq 1$. An addition of the equations in (3.4) shows that

$$(1-\lambda)(A + B_2)(x + y) = (B_1 + B_2)x$$

and, hence, it follows from the first equation in (3.4) that

$$(1-\lambda)((A + B_1)x + B_2(x + y)) = (B_1 + B_2)x = (1-\lambda)(A + B_2)(x + y)$$

so eliminating $x + y$ yields

$$(1-\lambda)(A + B_1)x = A(A + B_2)^{-1}(B_1 + B_2)x. \tag{3.5}$$

Let $\widehat{x} = D^{1/2}x$. Then using a transformation from both sides with $D^{-1/2}$, (3.5) shows that

$$(1-\lambda)(I-\widehat{B}_2)\widehat{x} = (I-(\widehat{B}_1+\widehat{B}_2))(I-\widehat{B}_1)^{-1}(\widehat{B}_1+\widehat{B}_2)\widehat{x},$$

where $\widehat{B}_i = D^{-1/2}B_iD^{-1/2}$, $i=1,2$. By use of the commutativity assumption, it holds that

$$(1-\lambda)(I-\widehat{B}_1)(I-\widehat{B}_2)\widehat{x} = (I-(\widehat{B}_1+\widehat{B}_2))(\widehat{B}_1+\widehat{B}_2)\widehat{x}. \tag{3.6}$$

Here the left hand side and right hand side matrices are spd respectively spsd, so $\lambda \leq 1$.

Further, using the relation $\widehat{B}_1\widehat{B}_2 = \left(\frac{1}{2}(\widehat{B}_1+\widehat{B}_2)\right)^2 + \widetilde{\omega}^2\widehat{A} = D^{-1/2}AD^{-1/2}$, it follows from (3.6) that

$$(1-\lambda)\left(1-\xi+\frac{1}{4}\xi^2+\widetilde{\omega}^2\right) = (1-\xi)\xi,$$

where ξ is an eigenvalue of $\widehat{B}_1+\widehat{B}_2$, $(\widehat{B}_1+\widehat{B}_2)\widehat{x} = \xi\widehat{x}$, $0\leq\xi\leq 1$. Hence

$$1-\lambda = \frac{(1-\xi)\xi}{(1-\xi/2)^2+\widetilde{\omega}^2}, \tag{3.7}$$

which again shows $\lambda \leq 1$. Further

$$\lambda - \frac{1}{2} = \frac{1}{2} - \frac{(1-\xi)\xi}{(1-\xi/2)^2+\widetilde{\omega}^2}$$

so $\lambda \geq \frac{1}{2}$. A computation shows that the term $\frac{(1-\xi)\xi}{(1-\xi/2)^2+\widetilde{\omega}^2}$ is maximized, i.e. a minimal value of λ is taken for

$$\xi = \frac{1}{1+q}, \quad q = \sqrt{\frac{\widetilde{\omega}^2+1/4}{\widetilde{\omega}^2+1}}.$$

Further

$$\lambda_{\min} = g(q) = 1 - \frac{4q}{(1+2q)^2+4\widetilde{\omega}^2(1+q)^2}.$$

It follows from (3.7) that the minimal value of λ is taken for $\widetilde{\omega}=0$, for which the optimal value of ξ is $\xi = \frac{2}{3}$. □

The above shows that $\widetilde{\omega}^2$ determines the lower eigenvalue bound of $\mathcal{C}^{-1}\mathcal{A}$ and hence the rate of convergence of the preconditioned iterative solution method. For a large value of ω convergence of the iterative solution method will be exceptionally rapid.

It follows further from (3.5) that the eigenvalue problem has a full eigenvector solution space, i.e. the preconditioned matrix $\mathcal{C}^{-1}\mathcal{A}$ is a normal matrix.

We show now that the number of iterations for such a normal matrix is determined fully by the number of disjoint eigenvalues of the matrix.

3.2 The degree of the minimal polynomial

A computation using (3.3) shows that $\mathcal{C}^{-1}\mathcal{A}$ has a block triangular structure,

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} (A+B_1)^{-1} & B_2 \\ 0 & (A+B_2)^{-1} \end{bmatrix} \begin{bmatrix} B_1+B_2 & 0 \\ B_1+B_2 & 0 \end{bmatrix},$$

i.e. is of the form $H = \begin{bmatrix} D & 0 \\ E & I \end{bmatrix}$, where we assume that D has a full eigenvector space and that $D-I$ is nonsingular. If $D-I$ is singular, we can use a unitary transformation, involving corresponding singular eigenvectors of D, to get a new form of H for which it holds.

We note that

$$\begin{bmatrix} D & 0 \\ E & I \end{bmatrix} \begin{bmatrix} I & 0 \\ E(D-I)^{-1} & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ E(D-I)^{-1} & I \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}.$$

This shows that

$$\begin{bmatrix} I & 0 \\ E(D-I)^{-1} & I \end{bmatrix} \begin{bmatrix} v_i \\ e_i \end{bmatrix}, \quad i = 1, 2, \ldots, 2n,$$

where $Dv_i = \lambda_i v_i$, $i = 1, 2, \ldots, n$, $v_i \neq 0$, $i = n+1, \ldots, 2n$ and e_i is the $(n+i)'$th unit vector, give the eigenvectors of $\begin{bmatrix} D & 0 \\ E & I \end{bmatrix}$ and

$$\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v_i \\ e_i \end{bmatrix} = \begin{bmatrix} v_i \\ e_i \end{bmatrix} \lambda_i$$

give the corresponding eigenvalues, where $\lambda_i = 1$, $i = n+1, \ldots, 2n$. Clearly the eigenvectors form a complete subspace of $\mathcal{C}^{2n}$, so H is a normal matrix.

It is seen that the number of eigenvalues not equal to unity, equals the number of eigenvalues of D. Hence the degree m of the minimal polynomial $\mathcal{P}_m(H) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for H does not depend on matrix E and equals the number of distinct eigenvalues of D.

Corollary 3.1. *The degree of the minimal polynomial for the matrix* $\begin{bmatrix} D & 0 \\ E & I \end{bmatrix}$, *where D has a complete eigenvector space, equals m, where m is the number of distinct eigenvalues of D. In exact arithmetics, and if $E \neq 0$, the number of Krylov iterations equals $m+1$. If $D = I$ and $E \neq 0$, it suffices with two iterations.*

3.3 A block diagonal preconditioner

For a comparison of our preconditioner with the block diagonal preconditioner used in [3, 4], we present now a simplified analysis of the latter. Let $\mathcal{A} = \begin{bmatrix} A & E-iF \\ E+iF & -A \end{bmatrix}$, where A is spd and E, F are symmetric and positive semidefinite (spsd). As we shall see in a later section, matrices on this form arise in some time-harmonic optimal control problems. It is not applicable for the case where the control and state functions are prescribed only on a subset of the whole domain of definition.

Proposition 3.2. *Let* $\mathcal{A} = \begin{bmatrix} A & E - iF \\ E + iF & -A \end{bmatrix}$, *where* A *is spd and* E, F *are spsd, and let* $\mathcal{D} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$, $D = A + E + F$, *and assume that* $ED^{-1}F = FD^{-1}E$. *This holds if* $F = \omega(A + \delta E)$, $\omega > 0$, $\delta \le 1$. *Then matrix* $(\mathcal{D}^{-1}\mathcal{A})^2$ *is block diagonal and its eigenvalues are real and contained in the interval* $\frac{1}{4} \le \lambda\left((\mathcal{D}^{-1}\mathcal{A})^2\right) \le 1$. *This holds uniformly with respect to both of the discretization and problem parameters. If* $F = \omega A$, $\omega > 0$, *then* $\frac{1}{3} \le \frac{1}{2(1+\omega/(1+\omega^2))} \le \lambda((\mathcal{D}^{-1}\mathcal{A})^2) \le 1$.

Proof. Let $\mathcal{D} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$, where $D = A + E + F$. For the analysis of the spectrum of $\mathcal{D}^{-1}\mathcal{A}$, consider the spectrally equivalent eigensolution problem,

$$\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{bmatrix} \mathcal{A} \begin{bmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \|x\| + \|y\| \neq 0$$

that is,

$$\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{B} \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$\begin{aligned} \mathcal{B} &= \begin{bmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{bmatrix} \begin{bmatrix} D - E - F & E - iF \\ E + iF & -(D - E - F) \end{bmatrix} \begin{bmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{bmatrix} = \\ &= \begin{bmatrix} I - \widehat{E} - \widehat{F} & \widehat{E} - i\widehat{F} \\ \widehat{E} + i\widehat{F} & -I + \widehat{E} + \widehat{F} \end{bmatrix}, \end{aligned}$$

and $\widehat{E} = D^{-1/2}ED^{-1/2}$, $\widehat{F} = D^{-1/2}FD^{-1/2}$. A computation shows that

$$\mathcal{B} = \begin{bmatrix} (I - \widehat{E} - \widehat{F})^2 + \widehat{E}^2 + \widehat{F}^2 + iG & (1+i)G \\ (-1+i)G & (I - \widehat{E} - \widehat{F})^2 + \widehat{E}^2 + \widehat{F}^2 - iG \end{bmatrix}$$

where $G = \widehat{E}\widehat{F} - \widehat{F}\widehat{E}$. We assume now that $\widehat{E}$ and $\widehat{F}$ commute, i.e. $G = 0$.

This holds if $F = \omega(A + \delta E)$, $\omega > 0$, because then $D = (1+\omega)A + (1+\omega\delta)E$ and $\widehat{F} = \omega(\widehat{A} + \delta\widehat{E})$, where $\widehat{A} = D^{-1/2}AD^{-1/2}$. Then $\widehat{A} = \frac{1}{1+\omega}\left(I - (1+\omega\delta)\widehat{E}\right)$ and

$$\widehat{F} = \frac{\omega}{1+\omega}(I - (1+\omega\delta)\widehat{E} + (1+\omega)\delta\widehat{E}) = \frac{\omega}{1+\omega}(I - (1-\delta)\widehat{E}).$$

Hence

$$\widehat{E}\widehat{F} = \frac{\omega}{1+\omega}(I - (1-\delta)\widehat{E})\widehat{E} = \widehat{F}\widehat{E}.$$

Therefore $G = 0$ and $\mathcal{B}$ is block diagonal, with identical blocks,

$$\mathcal{B} = \begin{bmatrix} (I - \widehat{E} - \widehat{F})^2 + \widehat{E}^2 + \widehat{F}^2 & 0 \\ 0 & (I - \widehat{E} - \widehat{F})^2 + \widehat{E}^2 + \widehat{F}^2 \end{bmatrix}.$$

Here, since $0 \le \widehat{E} < I$, $0 \le \widehat{F} < I$, and $0 \le \widehat{E} + \widehat{F} < I$,

$$(I - \widehat{E} - \widehat{F})^2 + \widehat{E}^2 + \widehat{F}^2 = I - (\widehat{E} + \widehat{F})(I - (\widehat{E} + \widehat{F})) - \widehat{E}(I - \widehat{E}) - \widehat{F}(I - \widehat{F}) \le I.$$

Further

$$x^T(\widehat{E}+\widehat{F})x \le \frac{1}{4}x^Tx + x^T(\widehat{E}+\widehat{F})^2x \tag{3.8}$$

so

$$x^T(\widehat{E}+\widehat{F})(I-(\widehat{E}+\widehat{F}))x \le \frac{1}{4}x^Tx,$$

and

$$x^T\widehat{E}x \le \frac{1}{4}x^Tx + x^T\widehat{E}^2x, \quad x^T\widehat{F}x \le \frac{1}{4}x^Tx + x^T\widehat{F}^2x. \tag{3.9}$$

Hence

$$(I-\widehat{E}-\widehat{F})^2 + \widehat{E}^2 + \widehat{F}^2 \ge \left(1-\frac{3}{4}\right)I = \frac{1}{4}I$$

and the spectral condition number of B^2 equals 4. If $F = \omega A$, then $\widehat{F} = \frac{\omega}{1+\omega}(I-\widehat{E})$ and

$$\begin{aligned}(I-\widehat{E}-\widehat{F})^2 + \widehat{E}^2 + \widehat{F}^2 &= \left(\frac{1}{1+\omega}\right)^2(I-\widehat{E})^2 + \widehat{E}^2 + \left(\frac{\omega}{1+\omega}\right)^2(I-\widehat{E})^2 = \\ &= \frac{1+\omega^2}{(1+\omega)^2}(I-\widehat{E})^2 + \widehat{E}^2 = \frac{1}{1+\xi}(I-\widehat{E})^2 + \widehat{E}^2,\end{aligned}$$

where $\xi = \frac{2\omega}{1+\omega^2}$, i.e. $0 \le \xi \le 1$. Since $0 \le \widehat{E} < I$, a computation shows that this expression is bounded below by

$$\frac{1}{1+\xi}\left(1-\frac{1}{2+\xi}\right)^2 + \left(\frac{1}{2+\xi}\right)^2 = \frac{1}{2+\xi} = \frac{1}{2(1+\omega/(1+\omega^2))} \ge \frac{1}{3},$$

which completes the proof. □

This proposition shows that the eigenvalues of the diagonally block preconditioned matrix in Section 3, i.e. $\left\{\begin{bmatrix} D^{-1} & 0 \\ 0 & D^{-1}\end{bmatrix}\begin{bmatrix} M & \sqrt{\beta}(K-i\omega M) \\ \sqrt{\beta}(K+i\omega M) & -M\end{bmatrix}\right\}^2$ are bounded below by $\frac{1}{2(1+\omega/(1+\omega^2))}$. Hence, for small or large values of ω the lower bound is close to $1/2$, the value taken for $\omega = 0$.

4 An application for an optimal control parabolic problem with a time-harmonic target state and distributed control

Following [3] and [10], consider the optimal control problem of finding the state $y(x,t)$ and the control $u(x,t)$ that minimizes the functional

$$J(y,u) = \frac{1}{2}\int_0^T\int_\Omega |y(x,t)-y_d(x,t)|^2 dx\,dt + \frac{1}{2}\beta\int_0^T\int_\Omega |u(x,t)|^2 dx\,dt,$$

subject to the time-dependent parabolic heat equation

$$\begin{aligned}\frac{\partial y(x,t)}{\partial t} - \Delta y(x,t) &= u(x,t) && \text{in } \Omega\times(0,T) \\ y(x,t) &= 0, && x\in\Gamma\times(0,T) \\ y(x,0) &= y(x,T) && \text{in } \Omega \\ u(x,0) &= u(x,T) && \text{in } \Omega.\end{aligned}$$

Here $\Gamma = \partial\Omega$, y_d is the desired state and $\beta > 0$ is a regularization parameter for the cost of the control function. The target function is assumed to be time-harmonic, $y_d(x,t) = y_d(x)e^{i\omega t}$ with period $\omega = 2\pi k/T$ for some non-negative integer k. Even if the target function is not time-harmonic with a single frequency, one can approximate it by a truncated Fourier series expansion of the form

$$y_d(\cdot,t) = \sum_{k=0}^{N} \left(y_{d,k}^c \cos(k\omega t) + y_{d,k}^s \sin(k\omega t)\right),$$

where the Fourier coefficients are given by the classical expressions, $y_{d,k}^c = \frac{2}{T}\int_0^T y_d \cos(k\omega t)dt$ and $y_{d,k}^s = \frac{2}{T}\int_0^T y_d \sin(k\omega t)dt$. Then, since the equations are linear, the solution and the control are also time-harmonic, $y(x,t) = y(x)e^{i\omega t}$ and $u(x,t) = u(x)e^{i\omega t}$. Furthermore, the equations for the different frequencies separate and can hence be solved independently in parallel. For the further analysis, it suffices therefore to consider a single frequency problem. Hence $y(x)$, $u(x)$ are time-independent solutions of the optimal control problem, given $y_d(\cdot)$,

$$\text{minimize } \frac{1}{2}\int_\Omega |y(x) - y_d(x)|^2 dx + \frac{1}{2}\beta \int_\Omega |u(x)|^2 dx$$

subject to

$$\begin{cases} i\omega y(x) - \Delta y(x) = u(x) & \text{in } \Omega \\ y(x) = 0, & x \in \Gamma. \end{cases}$$

We assume that $y(x)$ and $y_d(x)$ are real valued but the control $u(x)$ must be complex valued,

$$u(x,t) = u_0(x) + iu_1(x).$$

The state equation and hence also the minimization problem, has a unique solution. Using an appropriate finite element subspace V_h for both y_d and u and a complex-valued Lagrange multiplier vector ζ, the corresponding Lagrangian functional for the discretized constrained optimization problem becomes

$$L(y,u,\zeta) = \frac{1}{2}(y - y_d)^T M(y - y_d) + \frac{1}{2}\beta u^* M u + \zeta^*(i\omega M y + Ky - Mu),$$

where M is the mass matrix, corresponding to the L_2-inner product in V_h and K is the negative discrete Laplacian.

The first order necessary conditions, $\nabla_{(y,u,\zeta)} L(y,u,\zeta) = 0$, which are also sufficient for the existence of a solution, give the algebraic system,

$$\begin{bmatrix} M & 0 & K - i\omega M \\ 0 & \beta M & -M \\ K + i\omega M & -M & 0 \end{bmatrix} \begin{bmatrix} y \\ u \\ \zeta \end{bmatrix} = \begin{bmatrix} My_d \\ 0 \\ 0 \end{bmatrix}.$$

Using the relation $\zeta = \beta u$, leads to the reduced system

$$\begin{bmatrix} M & \beta(K - i\omega M) \\ K + i\omega M & -M \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} My_d \\ 0 \end{bmatrix}.$$

We recall that y is a real valued vector but u is complex valued. Here we multiply the second equation with $\sqrt{\beta}$ and introduce $\widetilde{u} = \sqrt{\beta}u$, which gives

$$\begin{bmatrix} M & \sqrt{\beta}(K - i\omega M) \\ \sqrt{\beta}(K + i\omega M) & -M \end{bmatrix} \begin{bmatrix} y \\ \widetilde{u} \end{bmatrix} = \begin{bmatrix} My_d \\ 0 \end{bmatrix},$$

i.e. a system on the form (3.1).

By applying the preconditioner

$$\mathcal{C} = \begin{bmatrix} M + 2\sqrt{\beta}K & \sqrt{\beta}(K - i\omega M) \\ \sqrt{\beta}(K + i\omega M) & -M \end{bmatrix},$$

it follows from Proposition 3.1 that we get the eigenvalue bounds $1 \geq \lambda \geq g(q) \geq \frac{1}{2}$ of $\mathcal{C}^{-1}\mathcal{A}$. The preconditioned Krylov subspace iteration method converges particularly fast for large values of ω.

Remark 4.1. *Following an approach used in earlier papers [21, 22], a preconditioner of the same type as used for optimal control problems, can also be used to avoid complex arithmetics and to rewrite the problem in real valued form, leading to a system on 4×4 block matrix form of a similar type as considered in this paper. This will however not be done here.*

5 Preconditioning of a double two-by-two block matrix arising in eddy current electromagnetic problems

5.1 Optimal control problem

Following [3] one can consider a multiharmonic method to numerically solve eddy current problems. As pointed out in [2, 3], an efficient method to solve time dependent eddy current problems that avoids costly standard time-stepping methods, is namely to use a truncated Fourier series expansion method in time, leading to a multiharmonic problem. Normally it suffices then to use few harmonic terms to get a sufficiently accurate time integration on a given time interval $[0, T]$. As shown in [4], see also [23], one can use this technique also for distributed optimal control problems where the state equation is a parabolic equation as in the previous section. Due to the classical orthogonality relation between the sine and cosine functions and since the given problem is linear, the multiharmonic problem decouples into several separate harmonic problems, which can be straightforwardly solved independently in parallel. For the analysis of the efficiency of the preconditioned method to be presented, it suffices then to consider a single harmonic problem as is done in this paper.

For eddy current electromagnetic problems, as follows from Section 2, the vector Laplace operator used in the previous section is replaced by a curl curl operator. We consider therefore first the problem, find the state y and control u that minimizes the cost functional,

$$J(y, u) = \frac{1}{2}\int_{\Omega\times[0,T]} |y - y_d|^2 dx\, dt + \frac{\beta}{2}\int_{\Omega\times(0,T)} |u|^2 dx\, dt$$

subject to the regularized state equation (2.1).

Here y_d is the desired state and $\beta > 0$ is a cost regularization parameter. It is assumed that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. Further the reluctivity $\nu \in L^\infty(\Omega)$ is uniformly positive and independent of $|\text{curl } y|$, i.e.we assume that the eddy current problem is linear. The conductivity $\sigma \in L^\infty(\Omega)$ is piecewise constant, positive in conducting and zero in nonconducting subdomains.

Applying a Lagrange multiplier w to impose the state equation, the Lagrangian functional becomes

$$\mathcal{L}(y,u,w) = J(y,u) + \int_{\Omega\times(0,T)} \left(\sigma \frac{\partial y}{\partial t} + \mathrm{curl}(\nu\,\mathrm{curl}\, y) + \varepsilon y - u\right) w \, dx\, dt.$$

The first order necessary conditions $\nabla_{(y,u,w)}\mathcal{L}(y,u,w) = 0$ are then applied. Here $\nabla_u \mathcal{L}(y,u,w) = 0$ gives the relation $\beta u - w = 0$ in $\Omega \times (0,T)$ which enables elimination of the control variable, resulting in the reduced optimality system,

$$\begin{cases} \sigma \frac{\partial y}{\partial t} + \mathrm{curl}(\nu\,\mathrm{curl}\, y) + \varepsilon y - \beta^{-1} w = 0 & \text{in } \Omega\times(0,T) \\ -\sigma \frac{\partial w}{\partial t} + \mathrm{curl}(\nu\,\mathrm{curl}\, w) + \varepsilon w + y = y_d & \text{in } \Omega\times(0,T) \\ y \times n = 0 & \text{on } \partial\Omega\times(0,T) \\ w \times n = 0 & \text{on } \partial\Omega\times(0,T) \\ y = y_0 & \text{on } \partial\Omega\times\{0\} \\ w = 0 & \text{on } \partial\Omega\times\{T\} \end{cases}. \tag{5.1}$$

5.2 Time harmonic equation

For the time-harmonic problem the aim is to compute a periodic steady state solution (y,w) that satisfies (5.1) but not necessarily the initial conditions $y = y_0$ and $w = 0$. Including instead the periodicity condition, $y(0) = y(T)$, the state equation takes the form

$$\begin{cases} \sigma \frac{\partial y}{\partial t} + \mathrm{curl}(\nu\,\mathrm{curl}\, y) + \varepsilon y = u & \text{in } \Omega\times(0,T) \\ y \times n = 0 & \text{on } \partial\Omega\times(0,T) \\ y(0) = y(T) & \text{in } \Omega \end{cases}.$$

Similarly, the condition $w(T) = 0$ is replaced by the periodicity condition, $w(0) = w(T)$. We consider then a time-harmonic desired state,

$$y_d(x,t) = y_d^c(x)\cos(wt) + y_d^s(x)\sin(\omega t).$$

Due to the linearity of the problem, the state y, the Lagrange multiplier, i.e. costate w and the control u are time-harmonic as well with the same frequency ω,

$$\begin{aligned} y(x,t) &= y^c(x)\cos(\omega t) + y^s(x)\sin(\omega t) \\ u(x,t) &= u^c(x)\cos(\omega t) + u^s(x)\sin(\omega t) \\ w(x,t) &= w^c(x)\cos(\omega t) + w^s(x)\sin(\omega t). \end{aligned}$$

The Fourier coefficients $u^c(x)$ and $u^s(x)$ are then related to the corresponding coefficients $w^c(x)$, $w^s(x)$ as before. Using the above time-harmonic representation of the solution and the replacement of the initial and end conditions with the periodicity conditions, as is shown in [3], the optimality system (5.1) can be written

$$
\begin{aligned}
\omega\sigma y^s + \operatorname{curl}(\nu \operatorname{curl} y^c) + \varepsilon y^c - \beta^{-1} w^c &= 0 \quad \text{in } \Omega \\
-\omega\sigma y^c + \operatorname{curl}(\nu \operatorname{curl} y^s) + \varepsilon y^s - \beta^{-1} w^s &= 0 \quad \text{in } \Omega \\
-\omega\sigma w^s + \operatorname{curl}(\nu \operatorname{curl} w^c) + \varepsilon w^c + y^c &= y_d^c \quad \text{in } \Omega \\
\omega\sigma w^c + \operatorname{curl}(\nu \operatorname{curl} w^s) + \varepsilon w^s + y^s &= y_d^s \quad \text{in } \Omega \\
y^c \times n &= 0 \quad \text{on } \partial\Omega \\
y^s \times n &= 0 \quad \text{on } \partial\Omega \\
w^c \times n &= 0 \quad \text{on } \partial\Omega \\
w^s \times n &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\tag{5.2}
$$

6 The finite element matrix and its preconditioners

6.1 Distributed control function

Consider first the problem where the control and observation functions are defined on the whole space.

For the finite element discretization of the variational formulation of the equations in (5.2), we use the edge elements, as described in Section 2. After a reordering of the equations, this yields the following system of linear equations,

$$
\begin{bmatrix} M & 0 & K & -M_\omega \\ 0 & M & M_\omega & K \\ K & M_\omega & -\beta^{-1}M & 0 \\ -M_\omega & K & 0 & -\beta^{-1}M \end{bmatrix}
\begin{bmatrix} y^c \\ y^s \\ w^c \\ w^s \end{bmatrix} =
\begin{bmatrix} y_d^c \\ y_d^s \\ 0 \\ 0 \end{bmatrix},
$$

where $M_\omega := \omega\, M$ and

$$
(y_d^c)_i = \int_\Omega y_d^c v_i dx, \quad (y_d^s)_i = \int_\Omega y_d^s v_i dx.
$$

In a similar way as was done before, we modify the system by multiplying the last two equations with $\sqrt{\beta}$ and scale the multiplier variable to $\begin{bmatrix} \widetilde{w}^c \\ \widetilde{w}^s \end{bmatrix} = \frac{1}{\sqrt{\beta}} \begin{bmatrix} w^c \\ w^s \end{bmatrix}$, resulting in the system

$$
\begin{bmatrix} M & 0 & \widetilde{K} & -\widetilde{M_\omega} \\ 0 & M & \widetilde{M_\omega} & \widetilde{K} \\ \widetilde{K} & \widetilde{M_\omega} & -M & 0 \\ -\widetilde{M_\omega} & \widetilde{K} & 0 & -M \end{bmatrix}
\begin{bmatrix} y^c \\ y^s \\ \widetilde{w}^c \\ \widetilde{w}^s \end{bmatrix} =
\begin{bmatrix} y_d^c \\ y_d^s \\ 0 \\ 0 \end{bmatrix},
$$

where $\widetilde{K} = \sqrt{\beta} K$ and $\widetilde{M_\omega} = \sqrt{\beta} M_\omega$.

Using the same type of preconditioning as in Section 4, obtained by adding the off-diagonal blocks to the primary diagonal block, we get

$$
\mathcal{C} = \begin{bmatrix} M + 2\widetilde{K} & 0 & \widetilde{K} & -\widetilde{M_\omega} \\ 0 & M + 2\widetilde{K} & \widetilde{M_\omega} & \widetilde{K} \\ \widetilde{K} & \widetilde{M} & -M & 0 \\ -\widetilde{M} & \widetilde{K} & 0 & -M \end{bmatrix}.
$$

To get the same form of the matrix as in Section 3, let

$$A = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \qquad B = \begin{bmatrix} \widetilde{K} & \widetilde{M}_\omega \\ -\widetilde{M}_\omega & \widetilde{K} \end{bmatrix}.$$

Then

$$\mathcal{A} = \begin{bmatrix} A & B^* \\ B & -A \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} A + 2\widetilde{K}' & B^* \\ B & -A \end{bmatrix},$$

where $\widetilde{K}' = \begin{bmatrix} \widetilde{K} & 0 \\ 0 & \widetilde{K} \end{bmatrix}$. Then $B + B^* = 2\widetilde{K}'$ which is spsd and it follows from Proposition 3.1 that the eigenvalues λ of $\mathcal{C}^{-1}\mathcal{A}$ satisfy $\frac{1}{2} \leq \lambda \leq 1$.

The arising inner systems with the block matrix $\begin{bmatrix} M + \widetilde{K} & \widetilde{M}_\omega \\ -\widetilde{M}_\omega & M + \widetilde{K} \end{bmatrix}$ can also be solved by iteration using the same type of preconditioner as for the outer system, that is with $\begin{bmatrix} M + \widetilde{K} + 2\widetilde{M}_\omega & \widetilde{M}_\omega \\ -\widetilde{M}_\omega & M + \widetilde{K} \end{bmatrix}$.

The eigenvalue bounds for the corresponding preconditioned matrix follows from Proposition 3.1. In this case they follow also more directly, since the corresponding eigenvalues $\widetilde{\lambda}$ satisfy

$$(\widetilde{\lambda} - 1) \begin{bmatrix} M + \widetilde{K} + 2\widetilde{M}_\omega & \widetilde{M}_\omega \\ -\widetilde{M}_\omega & M + \widetilde{K} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} 2\widetilde{M}_\omega & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

from which it follows that $\widetilde{\lambda} \leq 1$ and

$$(\widetilde{\lambda} - 1)x^T(M + \widetilde{K} + 2\widetilde{M}_\omega + \widetilde{M}_\omega(M + \widetilde{K})^{-1}\widetilde{M}_\omega)x = -2x^T\widetilde{M}_\omega x,$$

i.e.

$$(\widetilde{\lambda} - 1)\widehat{x}^T(I + 2\widehat{\widetilde{M}}_\omega + \widehat{\widetilde{M}}_\omega^2)\widehat{x} = -2\widehat{x}^T\widehat{M}_\omega\widehat{x},$$

where $\widehat{\widetilde{M}}_\omega = (M + \widetilde{K})^{-1/2}\widetilde{M}_\omega(M + \widetilde{K})^{-1/2}$ and $\widehat{x} = (M + \widetilde{K})^{1/2}x$. It follows that $\widetilde{\lambda} - 1 \geq -\frac{1}{2}$, i.e. $\widetilde{\lambda} \geq \frac{1}{2}$, so $\frac{1}{2} \leq \widetilde{\lambda} \leq 1$.

Remark 6.1. *In Section 2.2 in [4] it is stated that the spectral condition number of two block diagonal preconditioned matrices $\mathcal{D}^{-1}\mathcal{A}$ are bounded by $\sqrt{3}$. As $\mathcal{A}$ is a saddle point type of matrix, this is a misleading statement. As has been shown in Section 3.3 of our paper a computation shows that under a certain conditions, the square matrix, $(\mathcal{D}^{-1}\mathcal{A})^2$ is block diagonal and has a condition number bound $\kappa((\mathcal{D}^{-1}\mathcal{A})^2) \leq 3$. As is correctly stated in Theorem 4.2 in [3], there are then actually $2m$ iterations needed to reduce the relative residual $\|r^{2m}\|/\|r^0\| \leq \frac{2q^m}{1+q^{2m}}$, where $q \leq \frac{\sqrt{3}-1}{\sqrt{3}+1}$ for the above preconditioner. Hence the number of iterations is doubled compared to the solution of a symmetric and positive definite problem with condition number 3. As iterative acceleration method the MinRes method, see [20], see also [4], was used. Note also that the derivation of the above bounds hold under certain restrictions of the type of problem considered. They are not applicable for the problems consider in the next section.*

In our method the condition number for the outer iterations is bounded by $1/(1 - \alpha)$, where $\alpha \leq 1/2$. For the inner iterations a similar bound holds.

The relative stopping criteria for the inner iterations can be given by a fixed, not very small amount ε. Hence, the total number of inner iterations for all outer iterations will be proportional to the number of outer iterations. The total cost will in general be proportional to the cost of solving the inner-most elliptic systems. For this one can use an AMG type of solver, see e.g. [24], [25] and [26] for which an optimal order of computational complexity holds. Alternatively, one can use a geometric multigrid method such as [18], see also references therein. It follows that the total computational cost will be proportional to the number of outer iterations, which is determined by a condition number $1/(1-\alpha)$, where $\alpha \le 1/2$. This holds uniformly with respect to the discretization parameter h, the problem parameters ν, σ, ω and the method, cost parameters β and ε.

6.2 Control and state functions prescribed on a subset

Consider now the case where the control u is prescribed only on a subset, such as an electric coil, i.e. not distributed on the whole domain Ω. It vanishes then outside this subregion. Although this is frequently not satisfied in practice, for simplicity we assume that the observation region is also restricted to this subdomain, Ω_d, which is defined by a characteristic function,

$$\tau(x) = \begin{cases} 1, & x \in \Omega_d \\ 0, & x \in \Omega \setminus \Omega_d. \end{cases}$$

The corresponding cost functional is then

$$J(y,u) = \frac{1}{2}\int_{\Omega\times(0,T)} \tau(x)|y-y_d|^2 dx\,dt + \frac{1}{2}\beta\int_{\Omega\times(0,T)} \tau(x)|u|^2 dx\,dt$$

and the optimization problem is subject to the state equation,

$$\begin{cases} \sigma\frac{\partial y}{\partial t} + \operatorname{curl}(\nu \operatorname{curl} y) + \varepsilon\, y &= \tau(x)u & \text{in } \Omega\times(0,T) \\ y\times n &= 0 & \text{on } \partial\Omega\times(0,T) \\ y(0) &= y(T) & \text{in } \Omega. \end{cases}$$

After a similar transformation with $\sqrt{\beta}$ as has been done previously, the corresponding finite element matrix takes now the form,

$$\mathcal{A} = \begin{bmatrix} M_0 & 0 & K & -M_\omega \\ 0 & M_0 & M_\omega & K \\ K & M_\omega & -M_0 & 0 \\ -M_\omega & K & 0 & -M_0 \end{bmatrix}$$

where M_0 is the mass matrix corresponding to the subdomain Ω_d and has zero entries at nodepoints in $\Omega\setminus\Omega_d$. For ease of presentation from now on we denote $\widehat{K}$ and $\widehat{M_\omega}$ by K and M_ω, respectively.

Following the previously presented general approach, the first step in constructing a preconditioner is to add

$$\begin{bmatrix} K & -M_\omega \\ M_\omega & K \end{bmatrix} + \begin{bmatrix} K & M_\omega \\ -M_\omega & K \end{bmatrix} = 2\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$$

to the primal two-by-two block diagonal matrix, which gives

$$\mathcal{C}_0 = \begin{bmatrix} M_0+2K & 0 & K & -M_\omega \\ 0 & M_0+2K & M_\omega & K \\ K & M_\omega & -M_0 & 0 \\ -M_\omega & K & 0 & -M_0 \end{bmatrix}. \tag{6.1}$$

In a similar way as has been done for two-by-two block matrices with square blocks (cf (3.3)), we transform $\mathcal{C}_0$ to $\widetilde{\mathcal{C}}_0 = L\mathcal{C}_0 L$ where

$$L = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & I & 0 \\ 0 & -I & 0 & I \end{bmatrix},$$

which gives

$$\begin{aligned} \widetilde{\mathcal{C}}_0 &= L \begin{bmatrix} M_0+2K & 0 & K & -M_\omega \\ 0 & M_0+2K & M_\omega & K \\ K+M_0 & M_\omega & -M_0 & 0 \\ -M_\omega & K+M_0 & 0 & -M_0 \end{bmatrix} L = \\ &= \begin{bmatrix} M_0+K & M_\omega & K & -M_\omega \\ -M_\omega & M_0+K & M_\omega & K \\ 0 & 0 & -(M_0+K) & M_\omega \\ 0 & 0 & -M_\omega & -(M_0+K) \end{bmatrix}, \end{aligned} \tag{6.2}$$

i.e. a block triangular form.

Here we can replace the two block diagonal matrices

$$\begin{bmatrix} M_0+K & M_\omega \\ -M_\omega & M_0+K \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} M_0+K+2M_\omega & M_\omega \\ -M_\omega & M_0+K \end{bmatrix}$$

and

$$\begin{bmatrix} -(M_0+K) & M_\omega \\ -M_\omega & -(M_0+K) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} -(M_0+K) & M_\omega \\ -M_\omega & -(M_0+K+2M_\omega) \end{bmatrix},$$

respectively. The final preconditioner on the transformed level would then become

$$\widetilde{\mathcal{C}} = \begin{bmatrix} M_0+K+2M_\omega & M_\omega & K & -M_\omega \\ -M_\omega & M_0+K & M_\omega & K \\ 0 & 0 & -(M_0+K) & M_\omega \\ 0 & 0 & -M_\omega & -(M_0+K+2M_\omega) \end{bmatrix}. \tag{6.3}$$

Let $\widetilde{\mathcal{C}}$ act on a vector $\begin{bmatrix} x \\ y \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. We can reduce $\widetilde{\mathcal{C}}$ to a fully block triangular form by adding row 1 to row 2, row 4 to row 3 and introducing the vectors $z_1 = x_1 + x_2$ and $z_2 = y_1 + y_2$. An elementary computation shows that this results in that the preconditioner equation,

$$\widetilde{\mathcal{C}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

can be written in the form

$$\begin{bmatrix} D_0 & M_\omega & K & -(M_\omega + K) \\ 0 & D_0 & M_\omega & K - M_\omega \\ 0 & 0 & -D_0 & 0 \\ 0 & 0 & -M_\omega & -D_0 \end{bmatrix} \begin{bmatrix} x_1 \\ z_1 \\ z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_1 + f_2 \\ g_1 + g_2 \\ g_2 \end{bmatrix}$$

where $D_0 = M_0 + K + M_\omega$. Here we can permute the last two rows and columns, and the vectors $z_2 \leftrightarrow y_2$ to get a fully block triangular form. Hence, besides some matrix vector products, a computation of the action of the preconditioner, involves four consecutive solutions of elliptic problems with matrix D_0.

However, there is an undesirable problem with this preconditioner. In general it introduces complex eigenvalues of the preconditioned matrix. This can be seen by transforming $\widetilde{\mathcal{C}}$ back to the original, untransformed form. We get then

$$\mathcal{C} = L^{-1} \widetilde{\mathcal{C}} L^{-1} = \mathcal{C}_0 + 2 \begin{bmatrix} M_\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_\omega & 0 & 0 & 0 \\ 0 & -M_\omega & 0 & -M_\omega \end{bmatrix}$$

where $L^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ I & 0 & I & 0 \\ 0 & I & 0 & I \end{bmatrix}$. The generalized eigenvalue problem for this matrix $\mathcal{C}$ is,

$$\lambda \mathcal{C} \begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{A} \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$\mathcal{C} = \mathcal{A} + 2 \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} M_\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_\omega & 0 & 0 & 0 \\ 0 & -M_\omega & 0 & -M_\omega \end{bmatrix},$$

and gives now in general complex valued eigensolutions, which may slow done the rate of convergence.

To avoid this we prefer instead to solve systems with $\widetilde{\mathcal{C}}_0$ in (6.2) using a coupled inner-outer iteration method. Then the arising systems with the block diagonal matrix $\begin{bmatrix} M_0 + K & M_\omega \\ -M_\omega & M_0 + K \end{bmatrix}$ are solved with the preconditioner

$$\mathcal{A}_0 := \begin{bmatrix} M_0 + K + 2M_\omega & M_\omega \\ -M_\omega & M_0 + K \end{bmatrix} = \begin{bmatrix} D_0 + M_\omega & M_\omega \\ -M_\omega & M_0 + K \end{bmatrix}.$$

Following the previously used approach, to solve an equation $\mathcal{A}_0 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$ we rewrite the equations as

$$\begin{bmatrix} D_0 & M_\omega \\ 0 & D_0 \end{bmatrix} \begin{bmatrix} x \\ x + y \end{bmatrix} = \begin{bmatrix} f \\ f + g \end{bmatrix},$$

which involves two solutions with the elliptic matrix $D_0 = M_0 + K + M_\omega$. The arising systems with the other block diagonal matrix in $\widetilde{\mathcal{C}}_0$ i.e. with

$\begin{bmatrix} -(M_0+K) & M_\omega \\ -M_\omega & -(M_0+K) \end{bmatrix}$, are solved in a similar way. Even though one now gets coupled inner-outer iterations, which multiply up, this is a viable approach since the arising condition numbers for both the outer and inner iterations are bounded by a not large number $1/(1-\alpha)$, for different positive values of α less than or equal to $1/2$. Hence there will be few iterations. Furthermore, in practice it suffices to solve the inner systems to a fairly rough relative accuracy, say 10^{-2} to get the smallest or nearly the smallest number of outer iterations, so that there will be very few iterations. This is clearly demonstrated in the next section.

7 Numerical illustrations

To illustrate the performance of the preconditioning methods an extensive set of test problems have been run. All tests are done for two different values of h. This suffices since it turns out that the number of iterations does not vary, or varies only little with h. We list the number of iterations and for coupled outer-inner iterations, the number of outer and, in brackets, the total number of inner iterations.

For illustration of how the number of iterations depend on the parameters involved, we consider $\log_{10}\beta^{-1} = 10, 8, 6, 4, 2, 0$ combined with $\log_{10}\omega = -8, -4, 0, 4, 8$ in Tables 1, 2, 5, the same numbers of $\log_{10}\nu$ in Table 3 and same numbers of $\log_{10}\sigma_2$ in Table 4.

First we consider the 2-by-2 complex-valued system arising from optimal control parabolic problem described in Section 4. The computational domain $\Omega = (0,1)^3$ is discretized using linear nodal finite elements with steps $h = 1/16, 1/32$, which leads to 28819 and 243431 non-Dirichlet nodal complex-valued DOFs (nodes), respectively. For the solution we employ flexible GMRES (FGMRES), see [19]. In Table 1 we document robustness of the number of iterates (equal to number of preconditioner actions) with respect to β, ω, and h.

Table 1: Heat equation optimal control, 2-by-2 problem: Robustness of FGMRES iterations with respect to β, ω, and h for the relative precision 10^{-8}.

β	$h=1/32$					$h=1/64$				
	ω					ω				
	10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}	10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
10^{-10}	9	9	9	9	2	10	10	10	10	2
10^{-8}	10	10	10	9	1	11	11	11	9	1
10^{-6}	10	10	10	5	1	10	10	10	5	1
10^{-4}	10	10	10	3	1	10	10	10	3	1
10^{-2}	7	7	7	3	1	7	7	7	3	1
10^{0}	4	4	4	2	1	4	4	4	2	1

Further, we consider the 4-by-4 real-valued system arising from the eddy current optimal control problem described in Section 6.1. For simplicity reasons we set the regularization parameter $\varepsilon = 0$ so that K is spsd, which still satisfies Assumption 3.1. Later on in this section we shall demonstrate robustness with respect to ε. The computational domain $\Omega = (0,1)^3$ is now decomposed into subdomains $\Omega_2 = (1/4, 3/4)^3$ and $\Omega_1 = \Omega \backslash \Omega_2$, where we prescribe jumping conductivity σ_2 and σ_1, respectively. For the finite element discretization with steps $h = 1/16, 1/32$ we choose lowest-order Nédélec-I tetrahedral elements leading to 25602 and 214612 real-valued non-Dirichlet DOFs (edges), respectively. The tables 2-4 correspond to the tables 7.1-7.3, 7.4 and 7.5, respectively, in [27], the dissertation of M. Kolmbauer, where diagonal preconditioners are used. The outer and inner iterations were solved with the FGMRES method. The elliptic systems on the inner-most levels were solved with a direct solution method. It can be seen that our method never gives more iterations than those reported there but mostly fewer, and for some combinations of parameter values, significantly smaller number of iterations.

For the coupled outer-inner iterations we list also number of iterations for two values, 10^{-2} and 10^{-6} of the relative stopping criteria for the inner iterations. It is seen that, although there can be somewhat fewer number of outer iterations for the stricter stopping tolerance for almost all combinations of parameters, the total number of inner iterations are fewer for the more rough stopping tolerance.

Table 2: Eddy current optimal control, 4-by-4 system: Robustness of outer and total inner (in brackets) FGMRES iterations with respect to β, ω, and h, while fixing $\nu = \sigma_2 = 1$ and outer rel. prec. 10^{-8}.

h	β	inner rel. prec. 10^{-2}					inner rel. prec. 10^{-6}				
		ω					ω				
		10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}	10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
1/16	10^{-10}	10(20)	10(20)	10(20)	10(40)	3(11)	10(20)	10(20)	10(29)	10(87)	2(12)
	10^{-8}	11(22)	11(22)	11(22)	10(57)	3(11)	11(22)	11(22)	11(33)	9(125)	2(12)
	10^{-6}	11(22)	11(22)	11(22)	6(48)	3(11)	11(22)	11(22)	11(36)	5(80)	2(12)
	10^{-4}	9(18)	9(18)	9(18)	6(48)	3(11)	9(18)	9(18)	9(45)	4(72)	2(12)
	10^{-2}	5(10)	5(10)	6(23)	7(56)	3(11)	5(10)	5(15)	5(30)	3(55)	2(12)
	10^{0}	4(8)	4(8)	5(19)	7(56)	3(11)	4(8)	4(12)	4(24)	3(56)	2(12)
1/32	10^{-10}	10(20)	10(20)	10(20)	11(42)	4(15)	10(20)	10(20)	10(29)	10(89)	2(14)
	10^{-8}	11(22)	11(22)	11(22)	10(58)	4(15)	11(22)	11(22)	11(33)	10(144)	2(13)
	10^{-6}	11(22)	11(22)	11(22)	6(48)	4(15)	11(22)	11(22)	11(36)	5(80)	2(14)
	10^{-4}	9(18)	9(18)	9(18)	7(56)	4(15)	9(18)	9(18)	9(45)	4(72)	2(14)
	10^{-2}	5(10)	5(10)	6(23)	7(56)	4(15)	5(10)	5(14)	5(30)	3(55)	2(14)
	10^{0}	4(8)	4(8)	5(19)	7(56)	4(15)	4(8)	4(12)	4(24)	3(55)	2(14)

In the final tests, for the 4-by-4 eddy current problem with a restricted control and observation domain, we further consider the control-observation domain $\Omega_d = (1/4, 3/4)^3$. This now does not align with jumps in conductivity, for which we prescribe $\Omega_2 := (0,1)^2 \times (0, 1/2)$. The lowest-order Nédélec-I discretization into tetrahedra for $h = 1/16, 1/32, 1/64$ now leads to 24498, 204516, and 1670856 real-valued non-Dirichlet DOFs, respectively. Now we solve the inner-most systems with the matrix $D_0 = M_0 + K + M_\omega$ inexactly using conjugate gradients iterations preconditioned with geometric multigrid. We employ direct solver on the coarsest level, $h = 1/16$, and we employ the natural interpolation operator and three Arnold-Falk-Winther [18] presmoothing and postsmoothing steps on

Table 3: Eddy current optimal control, 4-by-4 system: Robustness of outer and total inner (in brackets) FGMRES iterations with respect to β, ν, and h, while fixing $\omega = \sigma_2 = 1$ and outer rel. prec. 10^{-8}.

h	β	inner rel. prec. 10^{-2}					inner rel. prec. 10^{-6}				
		ν					ν				
		10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}	10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
1/16	10^{-10}	1(2)	2(4)	10(20)	5(10)	2(4)	1(2)	2(4)	10(29)	5(15)	2(4)
	10^{-8}	2(4)	3(6)	11(22)	4(8)	3(6)	2(6)	3(6)	11(33)	4(12)	3(9)
	10^{-6}	2(4)	4(8)	11(22)	3(6)	3(6)	2(8)	4(16)	11(36)	3(11)	3(11)
	10^{-4}	3(6)	6(12)	9(18)	4(8)	4(8)	2(8)	6(30)	9(45)	2(8)	3(12)
	10^{-2}	2(8)	10(40)	6(23)	4(13)	5(17)	2(8)	10(87)	5(30)	2(8)	3(18)
	10^{0}	2(8)	10(57)	5(19)	3(12)	5(20)	2(8)	9(125)	4(24)	3(12)	5(40)
1/32	10^{-10}	1(2)	2(4)	10(20)	5(10)	3(6)	1(2)	2(4)	10(29)	5(14)	3(6)
	10^{-8}	2(4)	3(6)	11(22)	4(8)	3(6)	2(6)	3(6)	11(33)	4(12)	3(9)
	10^{-6}	2(4)	5(10)	11(22)	3(6)	3(6)	2(8)	5(20)	11(36)	3(11)	3(11)
	10^{-4}	3(6)	9(18)	9(18)	4(8)	4(8)	2(8)	9(45)	9(45)	2(8)	3(12)
	10^{-2}	2(8)	11(42)	6(23)	4(13)	5(16)	2(9)	10(89)	5(30)	3(12)	5(30)
	10^{0}	3(12)	10(58)	5(19)	3(12)	5(20)	3(13)	10(144)	4(24)	3(12)	5(40)

Table 4: Eddy current optimal control, 4-by-4 system: Robustness of outer and total inner (in brackets) FGMRES iterations with respect to β, σ_2, and h, while fixing $\omega = \nu = 1$ and outer rel. prec. 10^{-8}.

h	β	inner rel. prec. 10^{-2}					inner rel. prec. 10^{-6}				
		σ_2					σ_2				
		10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}	10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
1/16	10^{-10}	10(20)	10(20)	10(20)	10(38)	6(24)	10(30)	10(30)	10(29)	10(90)	6(47)
	10^{-8}	11(22)	11(22)	11(22)	11(65)	6(23)	11(33)	11(33)	11(33)	11(175)	6(42)
	10^{-6}	11(22)	11(22)	11(22)	12(75)	6(23)	11(43)	11(43)	11(36)	10(161)	5(34)
	10^{-4}	10(21)	10(21)	9(18)	10(80)	5(19)	9(45)	9(45)	9(45)	9(160)	4(28)
	10^{-2}	7(24)	7(24)	6(23)	7(56)	3(12)	6(54)	6(54)	5(30)	6(112)	3(21)
	10^{0}	7(32)	7(32)	5(19)	6(46)	2(8)	4(51)	4(51)	4(24)	4(69)	2(13)
1/32	10^{-10}	10(20)	10(20)	10(20)	10(38)	7(30)	10(30)	10(30)	10(29)	10(90)	6(48)
	10^{-8}	11(22)	11(22)	11(22)	12(71)	6(24)	11(33)	11(33)	11(33)	11(175)	6(45)
	10^{-6}	11(22)	11(22)	11(22)	12(76)	6(25)	11(43)	11(43)	11(36)	10(164)	5(35)
	10^{-4}	10(21)	10(21)	9(18)	10(80)	5(22)	9(45)	9(45)	9(45)	8(144)	4(28)
	10^{-2}	8(25)	8(25)	6(23)	7(56)	4(18)	6(54)	6(54)	5(30)	6(108)	3(21)
	10^{0}	8(43)	8(43)	5(19)	6(46)	2(9)	4(51)	4(51)	4(24)	4(69)	2(14)

the two finer levels. The relative precision of the inner-most multigrid-PCG iterations as well as the one of the FGMRES inner (2-by-2 system) iterations is 10^{-2}, while the relative precision of the outer-most (4-by-4 system) FGMRES iterations is 10^{-8}. Robustness of our approach is numerically documented in

Tables 5-7, being the counterparts to Tables 2-4, respectively. Recall that two systems with D_0 are solved within an inner iteration, thus, the total number of the inner-most iterations (multigrid actions) is twice the total number of inner iterations whenever one multigrid-PCG iteration suffices to achieve the tolerance.

Table 5: Eddy current optimal control on a subset, 4-by-4 system: Robustness of outer FGMRES iterations, total inner (first number in brackets) FGMRES iterations, and total inner-most (second number in brackets) multigrid-preconditioned PCG iterations with respect to β, ω, and h, while fixing $\nu = \sigma_2 = 1$, outer rel. prec. 10^{-8}, and inner rel. prec. as well as inner-most rel. prec. 10^{-2}.

h	β	ω				
		10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
1/16	10^{-10}	1(2,4)	2(4,8)	7(25,50)	9(59,118)	3(8,16)
	10^{-8}	1(2,4)	3(6,12)	9(26,52)	10(68,136)	3(8,16)
	10^{-6}	1(2,4)	5(10,20)	11(35,70)	6(43,86)	2(5,10)
	10^{-4}	1(2,4)	5(10,20)	11(41,82)	6(42,84)	2(5,10)
	10^{-2}	1(2,4)	4(8,16)	9(51,102)	6(42,84)	2(5,10)
	10^{0}	2(4,8)	4(8,16)	8(45,90)	6(42,84)	2(4,8)
1/32	10^{-10}	7(14,28)	7(16,46)	8(23,87)	11(65,130)	3(8,16)
	10^{-8}	8(18,44)	8(20,55)	10(29,109)	10(64,128)	3(8,16)
	10^{-6}	10(21,74)	10(21,76)	11(37,141)	7(50,100)	3(8,16)
	10^{-4}	8(16,64)	8(16,64)	11(41,163)	6(42,84)	3(8,16)
	10^{-2}	6(13,52)	6(13,52)	9(49,196)	7(47,94)	3(7,14)
	10^{0}	6(15,60)	6(16,64)	7(42,168)	7(47,94)	3(7,14)
1/64	10^{-10}	7(16,47)	7(16,51)	9(27,104)	12(71,142)	4(12,24)
	10^{-8}	9(18,64)	9(18,68)	10(27,108)	11(67,134)	4(12,24)
	10^{-6}	8(17,68)	8(17,68)	11(35,140)	7(51,102)	4(12,24)
	10^{-4}	9(19,94)	9(19,94)	11(42,172)	6(44,88)	4(12,24)
	10^{-2}	6(12,68)	6(12,68)	9(49,197)	6(47,94)	4(11,22)
	10^{0}	6(12,68)	6(12,68)	8(52,230)	6(47,94)	4(11,22)

At the very end we note on robustness of our preconditioner with respect to the Maxwell regularization parameter $\varepsilon > 0$. We keep the setup of Table 6 now with the fixed $\beta := 10^{-6}$. We hardly observe any change in number of iterations, which is due to the robustness of the multigrid preconditioner for D_0. The results are shown in Table 8.

8 Concluding remarks

The most interesting application in this paper involves the optimal control problem where the control and observation functions are defined on a subset of the given domain. To avoid complex eigenvalues of the preconditioned four-by-four matrix arising when solving an eddy current electromagnetic problem and hence a slower rate of convergence we have used a coupled outer-inner iteration method, i.e. the arising two-by-two block matrices with square blocks are solved

Table 6: Eddy current optimal control on a subset, 4-by-4 system: Robustness of outer FGMRES iterations, total inner (first number in brackets) FGMRES iterations, and total inner-most (second number in brackets) multigrid-preconditioned PCG iterations with respect to β, ν, and h, while fixing $\omega = \sigma_2 = 1$, outer rel. prec. 10^{-8}, and inner rel. prec. as well as inner-most rel. prec. 10^{-2}.

h	β	ν				
		10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
1/16	10^{-10}	2(4,8)	4(17,34)	7(25,50)	4(8,16)	3(6,12)
	10^{-8}	2(4,8)	4(17,34)	9(26,52)	4(8,16)	4(8,16)
	10^{-6}	3(6,12)	5(24,48)	11(35,70)	5(10,20)	5(10,20)
	10^{-4}	3(10,20)	6(32,64)	11(41,82)	7(25,50)	7(25,50)
	10^{-2}	4(23,46)	9(59,118)	9(51,102)	7(38,76)	7(38,76)
	10^{0}	4(24,48)	10(68,136)	8(45,90)	6(31,62)	6(31,62)
1/32	10^{-10}	2(4,8)	4(17,34)	8(23,87)	6(13,52)	6(16,64)
	10^{-8}	2(4,8)	5(23,46)	10(29,109)	6(16,64)	6(15,60)
	10^{-6}	3(6,12)	6(30,60)	11(37,141)	6(12,48)	6(12,48)
	10^{-4}	3(10,20)	8(42,84)	11(41,163)	7(26,104)	7(26,104)
	10^{-2}	4(23,46)	11(65,130)	9(49,196)	8(43,172)	8(43,172)
	10^{0}	4(25,50)	10(64,128)	7(42,168)	7(38,152)	7(39,156)
1/64	10^{-10}	2(4,8)	4(17,34)	9(27,104)	6(12,68)	6(12,68)
	10^{-8}	2(4,8)	6(29,58)	10(27,108)	6(12,68)	6(12,68)
	10^{-6}	3(6,12)	6(28,56)	11(35,140)	4(13,64)	4(12,58)
	10^{-4}	3(10,20)	10(48,96)	11(42,172)	8(29,125)	8(28,125)
	10^{-2}	4(23,46)	12(71,142)	9(49,197)	8(43,188)	8(44,199)
	10^{0}	4(24,48)	11(67,134)	8(52,230)	8(44,225)	8(44,228)

by inner iterations. Even though the number of iterations multiplies up, since the condition numbers of the preconditioned matrices are small, bounded by $1/(1-\alpha)$, where $0 < \alpha \leq 1/2$ and α is particularly small for large frequencies, the total number of iterations will still be modest. Furthermore, one can use a not very small relative stopping criteria for the inner systems, in this way decreasing the number of inner iterations without increasing the number of outer iterations or with only a few additional iterations. The condition number bound holds uniformly with respect to all parameters involved, namely the discretization parameter h, the frequency ω, the conductivity σ, the reluctivity ν and the control cost parameter β.

For a comparison, in [4] it is found that the block diagonal preconditioner used leads to a rate of convergence factor which deteriorates to unity when $\beta\nu_{\min}^2$ takes small values. Since the cost parameter β in practical applications is small and the reluctivity ν can be small in certain parts of the domain, this method is less useful in many problems. Our method converges faster for most problems and is applicable uniformly for all parameter values.

Finally, we note that our method can be extended to treat the divergence-free condition, with which no additional regularization is required [12]. Together with the physical divergence-free control variable the Lagrange multipliers van-

Table 7: Eddy current optimal control on a subset, 4-by-4 system: Robustness of outer FGMRES iterations, total inner (first number in brackets) FGMRES iterations, and total inner-most (second number in brackets) multigrid-preconditioned PCG iterations with respect to β, σ_2, and h, while fixing $\omega = \nu = 1$, outer rel. prec. 10^{-8}, and inner rel. prec. as well as inner-most rel. prec. 10^{-2}.

h	β	σ_2: 10^{-8}	10^{-4}	10^{0}	10^{4}	10^{8}
1/16	10^{-10}	6(17,34)	6(17,34)	7(25,50)	9(57,114)	9(18,36)
	10^{-8}	9(24,48)	9(24,48)	9(26,52)	11(73,146)	10(20,40)
	10^{-6}	10(29,58)	10(29,58)	11(35,70)	12(84,168)	10(21,42)
	10^{-4}	11(37,74)	11(37,74)	11(41,82)	11(86,172)	10(20,40)
	10^{-2}	9(49,98)	9(49,98)	9(51,102)	7(54,108)	7(19,38)
	10^{0}	8(45,90)	8(45,90)	8(45,90)	6(42,84)	7(15,30)
1/32	10^{-10}	7(19,68)	7(19,68)	8(23,87)	11(65,161)	10(20,41)
	10^{-8}	10(30,110)	10(30,110)	10(29,109)	12(74,195)	10(20,40)
	10^{-6}	11(36,140)	11(36,140)	11(37,141)	13(91,229)	11(41,94)
	10^{-4}	11(37,147)	11(37,147)	11(41,163)	11(86,240)	10(36,131)
	10^{-2}	9(48,191)	9(48,191)	9(49,196)	8(63,199)	7(32,110)
	10^{0}	8(49,195)	8(49,195)	7(42,168)	7(54,172)	6(30,111)
1/64	10^{-10}	8(22,88)	8(22,88)	9(27,104)	11(65,199)	10(25,68)
	10^{-8}	11(30,118)	11(30,118)	10(27,108)	12(74,229)	11(38,101)
	10^{-6}	11(33,132)	11(33,132)	11(35,140)	13(92,330)	11(42,162)
	10^{-4}	11(39,161)	11(39,161)	11(42,172)	12(93,349)	10(38,154)
	10^{-2}	9(46,186)	9(46,186)	9(49,197)	8(63,253)	7(27,114)
	10^{0}	8(52,232)	8(52,232)	8(52,230)	7(54,219)	6(30,148)

ish. After adding a consistent term to the semi-elliptic operator the structure is no longer saddle-point, but elliptic. We can again eliminate the control variable from the KKT system and arrive at a 2x2 system. However, now the block (2,2) is only equivalent to the mass matrix in the block (1,1). Correspondingly the block (2,2) of our preconditioner will slightly change. It will be a part of a forthcoming paper.

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Table 8: Eddy current optimal control on a subset, 4-by-4 system: Robustness of outer FGMRES iterations, total inner (first number in brackets) FGMRES iterations, and total inner-most (second number in brackets) multigrid-preconditioned PCG iterations with respect to ε, ν, and h, while fixing $\beta := 10^{-6}$, $\omega = \sigma_2 = 1$, outer rel. prec. 10^{-8}, and inner rel. prec. as well as inner-most rel. prec. 10^{-2}.

h	ε	ν		
		10^{-8}	10^{-4}	10^{0}
1/16	10^{-8}	3(6,12)	5(24,48)	11(35,70)
	10^{-4}	3(6,12)	5(24,48)	11(35,70)
	10^{0}	4(9,18)	5(22,44)	11(33,66)
1/32	10^{-8}	3(6,12)	6(30,60)	11(37,141)
	10^{-4}	3(6,12)	6(30,60)	11(37,141)
	10^{0}	4(9,18)	6(29,58)	12(47,183)
1/64	10^{-8}	3(6,12)	6(28,56)	11(35,140)
	10^{-4}	3(6,12)	6(28,56)	11(35,140)
	10^{0}	4(10,20)	6(29,58)	11(43,174)

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