Multigrid Method for Coupled Optimal Topology and Shape Design in Nonlinear Magnetostatics

D. Lukáš

with P. Chalmovianský, U. Langer, R. Stainko, and J. Pištora

Dep. of Applied Mathematics, Institute of Physics,
VŠB–Technical University Ostrava, CZ
SFB F013 "Numerical and Symbolic Scientific Computing",
University Linz, AT

email: dalibor.lukas@vsb.cz

Outline

- Benchmark problem
  - 2D/3D linear/nonlinear magnetostatics
- Topology optimization for nonlinear magnetostatics
  - Nonlinear state sensitivity analysis
- Shape optimization for nonlinear magnetostatics
  - Multilevel solver
- Sequential 2D topology–shape optimization
- Outlook

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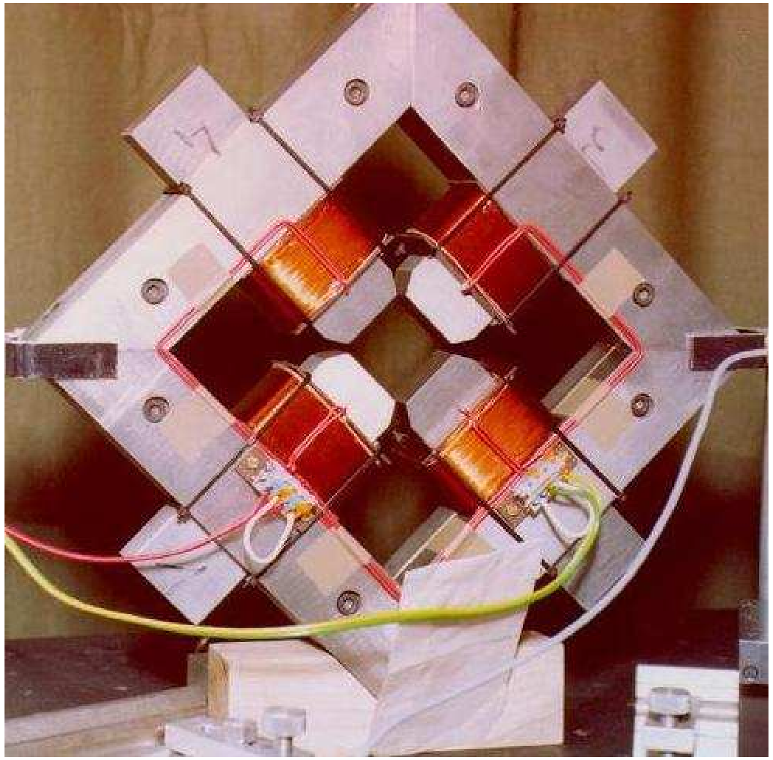
Magnetostatic benchmark problem

Maltese cross electromagnet



- is used for measurements of magnetooptic effects,
- produces magnetic field constant in the middle,
- is capable to rotate the magnetic field,
- is produced at Institute of Physics, VŠB–TU Ostrava,
- is also used at INSA Toulouse, University Paris VI, Simon Fraser University Vancouver, Charles University Prague

Magnetostatic benchmark problem

Optimization problem

Find optimal geometry α of the electromagnet in order to minimize inhomogeneities of the magnetic field in the middle area Ω_{m} among the pole heads.

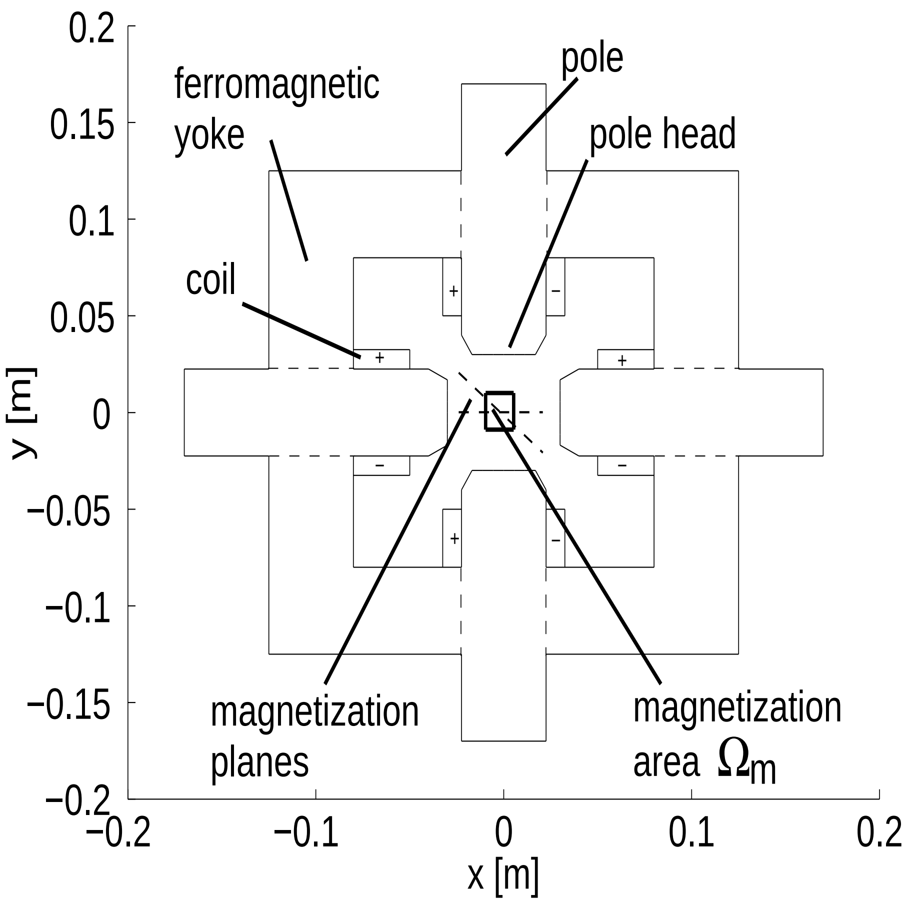
$$\min_{\alpha} \int_{\Omega_{\mathrm{m}}} \left|\mathbf{B}_\alpha(\mathbf{x}) - \mathbf{B}_\alpha^{\mathrm{avg}}\right|^2 \, d\mathbf{x}$$

$$\text{s.t. } \mathbf{B}_\alpha^{\mathrm{avg}} \geq \mathbf{B}^{\mathrm{min}},$$

where

$\mathbf{B}_\alpha(\mathbf{x})$ ...the magnetic flux density,

$\mathbf{B}_\alpha^{\mathrm{avg}}$ ...the average mag. flux density over Ω_{m}



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3D nonlinear magnetostatic state problem

Maxwell equations for magnetostatics

$$\begin{cases} \mathbf{curl}\,(\mathbf{H}(\mathbf{x})) = \mathbf{J}(\mathbf{x}) & \text{in } \mathbf{R}^3 \\ \mathbf{H}(\mathbf{x}) = \nu\,(\|\mathbf{B}(\mathbf{x})\|, \mathbf{x})\,\mathbf{B}(\mathbf{x}) & \text{in } \mathbf{R}^3 \\ \mathbf{B}(\mathbf{x}) \to \mathbf{0} & \|\mathbf{x}\| \to \infty \end{cases}$$

3D nonlinear magnetostatic state problem

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Magnetic vector potential

$$\mathbf{B}(\mathbf{x}) = \mathbf{curl}(\mathbf{u}(\mathbf{x}))$$

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Boundary value problem

$$\begin{cases} \mathbf{curl}\,(\nu(\|\mathbf{curl}(\mathbf{u}(\mathbf{x}))\|, \mathbf{x})\mathbf{curl}(\mathbf{u}(\mathbf{x}))) = \mathbf{J}(\mathbf{x}) & \text{in } \Omega \\ \mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}) = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

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Regularized weak formulation ($\varepsilon > 0$ small)

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{H_0}(\mathbf{curl};\Omega) : \\ \displaystyle\int_\Omega \nu\,(\|\mathbf{curl}(\mathbf{u})\|)\,\mathbf{curl}(\mathbf{u}) \cdot \mathbf{curl}(\mathbf{v}) + \varepsilon \mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{J} \cdot \mathbf{v}\, d\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H_0}(\mathbf{curl};\Omega), \end{cases}$$

where $0 < \nu_0 \leq \nu(\mathbf{x}) \leq \nu_1$, Lipschitz continuous a.e. in Ω and $\mathbf{J} \in \mathbf{Ker}_0(\mathrm{div};\Omega)$

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Discretization by FEM

using the lowest order Nédélec edge elements on tetrahedra

2D nonlinear magnetostatic state problem

Reduced 2D state problem

$$\begin{cases} -\mathrm{div}\left(\nu(\|\mathbf{grad}(u)\|, \mathbf{x})\mathbf{grad}(u(\mathbf{x}))\right) = J(\mathbf{x}) \text{ in } \Omega \\ \hfill u(\mathbf{x}) = 0 \qquad \text{on } \partial\Omega \end{cases}$$

and $\mathbf{B}(\mathbf{x}) := \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}, 0\right)$, $\mathbf{J}(\mathbf{x}) = (0, 0, J(\mathbf{x}))$

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using the lowest order Lagrange nodal elements on triangles

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Topology optimization for nonlinear magnetostatics

$\Omega \subset \mathbb{R}^3$ a fixed computational domain, $\Omega_\mathrm{d} \subset \Omega$ design domain,

$\mathcal{Q} := \{\rho : \Omega_\mathrm{d} \to \{0,1\}\}$ set of admissible material distributions,

Topology optimization for nonlinear magnetostatics

$\Omega \subset \mathbb{R}^3$ a fixed computational domain, $\Omega_\mathrm{d} \subset \Omega$ design domain,

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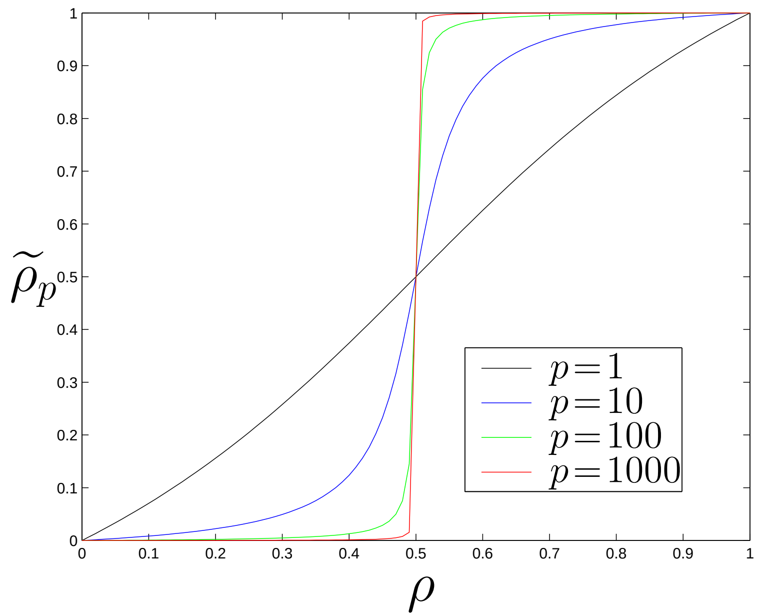
$\widetilde{\rho}_p(\rho) := \frac{1}{2}\left(1 + \frac{1}{\arctan p} \arctan\left(p(2\rho - 1)\right)\right)$ penalization of intermediate values,



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$\nu(\eta, \widetilde{\rho}) := \begin{cases} \nu_0 + (\nu(\eta) - \nu_0)\widetilde{\rho}, & \text{in } \Omega_{\mathrm{d}} \\ \nu_0, & \text{otherwise} \end{cases}$ nonlinearity,

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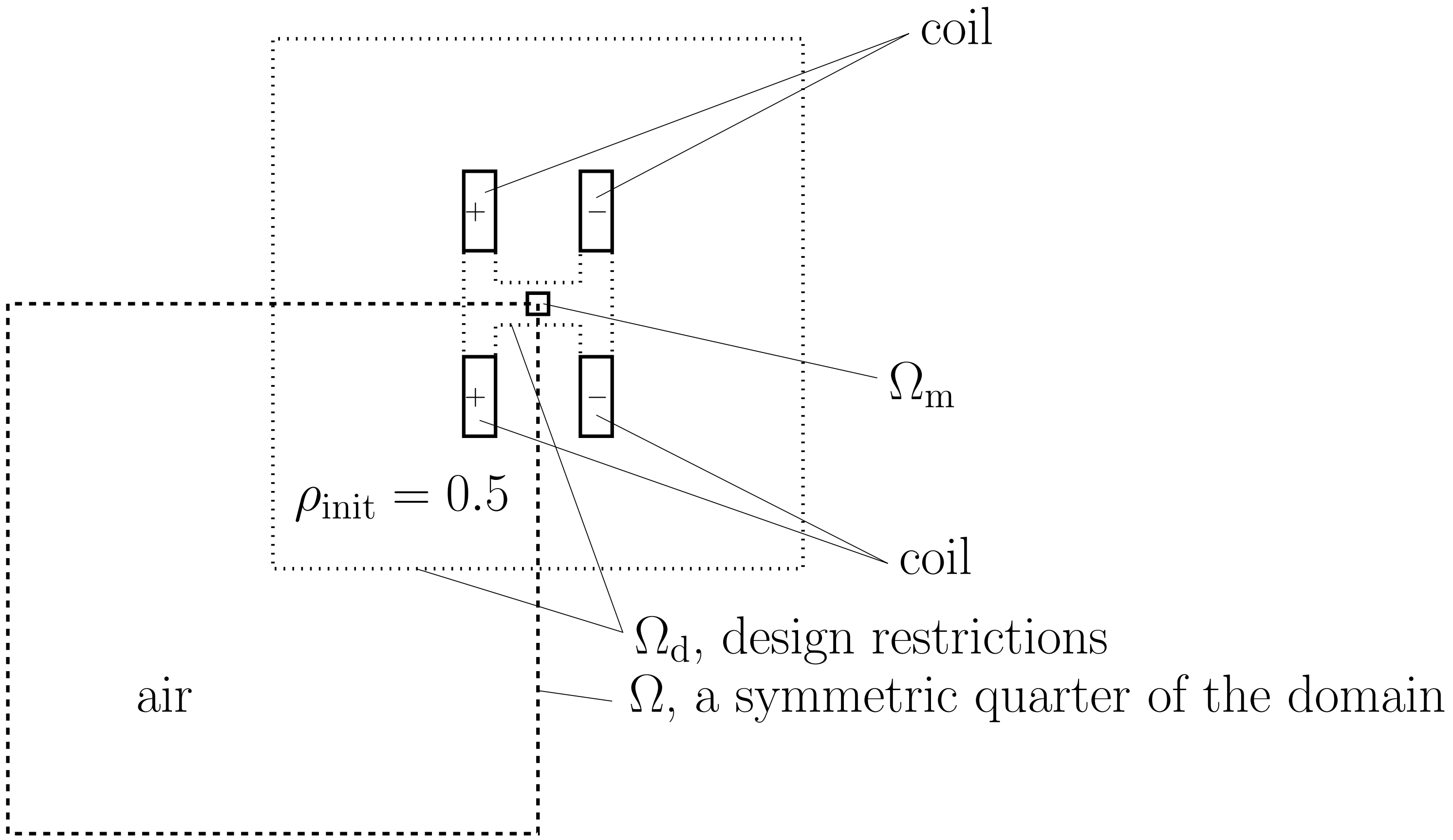
$\mathcal{I} : \mathbf{L}^2(\Omega) \times \mathcal{Q} \mapsto \mathbb{R}$ cost functional

$$\left\{ \begin{array}{lc} & \min_{\rho\in\mathcal{Q}} \mathcal{I}(\mathbf{curl}(\mathbf{u}), \widetilde{\rho}(\rho)) \\ \text{w.r.t.} & \\ & \int\limits_{\Omega_\mathrm{d}} \widetilde{\rho}(\rho)\, d\mathbf{x} \le V_\mathrm{max} \\ & \int\limits_{\Omega} \nu\left(\|\mathbf{curl}(\mathbf{u})\|, \widetilde{\rho}(\rho)\right) \mathbf{curl}(\mathbf{u}) \cdot \mathbf{curl}(\mathbf{v})\, d\mathbf{x} = \int\limits_{\Omega} \mathbf{J}\cdot\mathbf{v}\, d\mathbf{x} \text{ in } \mathbf{H}_{\mathbf{0},\perp}(\mathbf{curl};\Omega) \end{array} \right.$$

Topology optimization for nonlinear magnetostatics

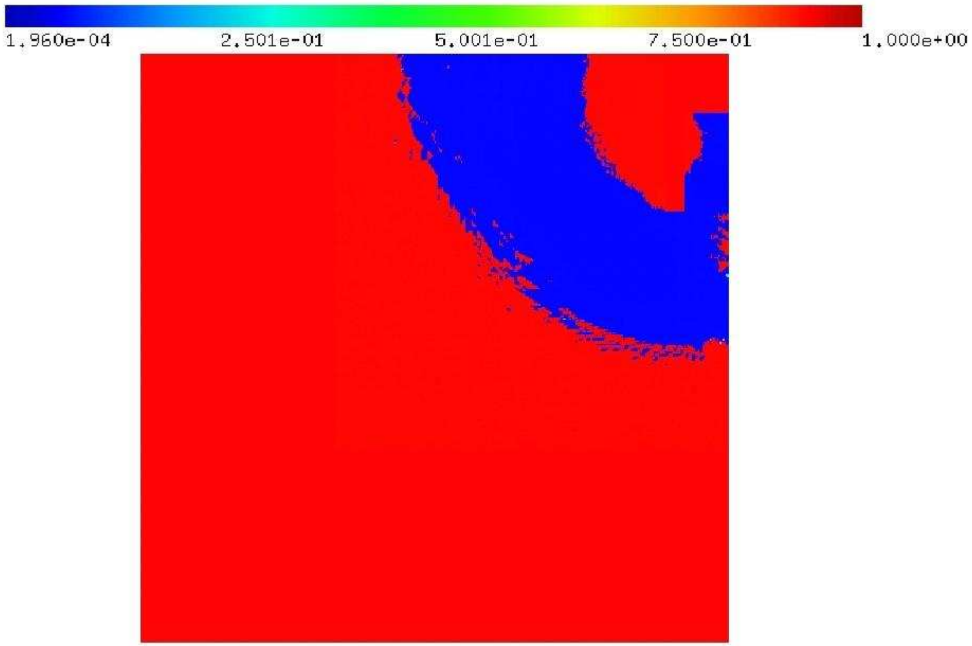
The model problem

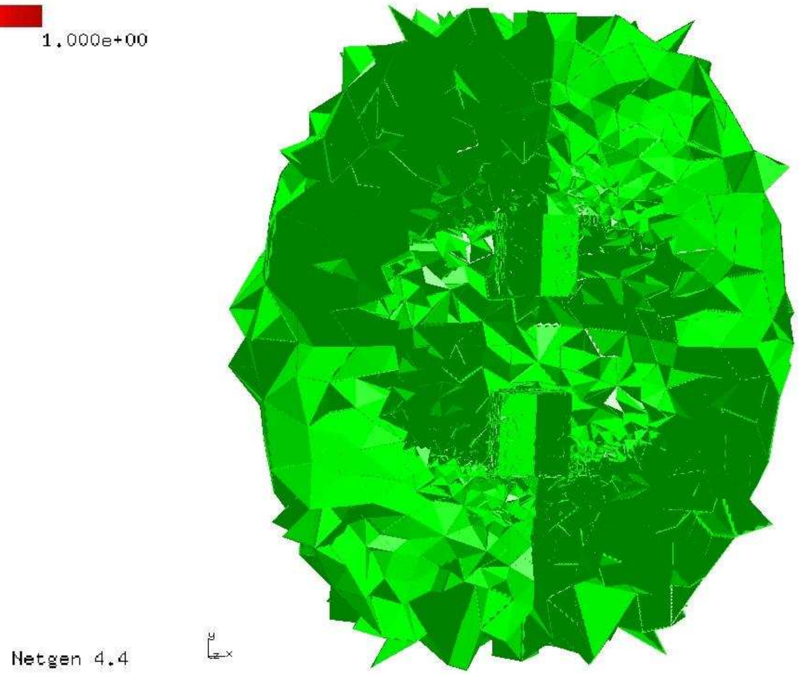
Let us consider only 2 coils and due to the symmetry the quarter of the domain.



Topology optimization for nonlinear magnetostatics

2D and 3D numerical results





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Nonlinear state sensitivity analysis

Newton method

Given $\boldsymbol{\rho}$

$i := 0$

Solve $\mathbf{A}(\mathbf{0}, \boldsymbol{\rho}) \cdot \mathbf{u}^0 = \mathbf{f}$

$\mathbf{f}^0 := \mathbf{f} - \mathbf{A}(\mathbf{u}^0, \boldsymbol{\rho}) \cdot \mathbf{u}^0$

while $\|\mathbf{f}^i\| / \|\mathbf{f}\| >$ prec **do**

$\quad i := i + 1$

$\quad$ Solve $\mathbf{A}'_{\mathbf{u}}(\mathbf{u}^{i-1}, \boldsymbol{\rho}) \cdot \mathbf{w}^i = \mathbf{f}^{i-1}$

$\quad$ Find $\tau^i : \|\mathbf{f}^i(\tau^i)\| < \|\mathbf{f}^{i-1}\|$

$\quad \mathbf{u}^i := \mathbf{u}^{i-1} + \tau^i \mathbf{w}^i$

$\quad \mathbf{f}^i := \mathbf{f} - \mathbf{A}(\mathbf{u}^i, \boldsymbol{\rho}) \cdot \mathbf{u}^i$

$\quad$ Store $\mathbf{w}^i$ and τ^i

end while

Store $\mathbf{u}^i$ and $k := i$

Calculate objective $J(\mathbf{u}^i, \boldsymbol{\rho})$

Adjoint Newton method

Given $\boldsymbol{\rho}$, k, $\mathbf{u}^k$, $\{\mathbf{w}^i\}_{i=1}^k$ and $\{\tau^i\}_{i=1}^k$

$\boldsymbol{\lambda} := J'_{\mathbf{u}}(\mathbf{u}^k, \boldsymbol{\rho})$

$\omega := \mathbf{0}$

for $i := k, \ldots, 1$ **do**

$\quad \mathbf{u}^{i-1} := \mathbf{u}^i - \tau^i \mathbf{w}^i$

$\quad$ Solve $\mathbf{A}'_{\mathbf{u}}(\mathbf{u}^{i-1}, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta} = \boldsymbol{\lambda}$

$\quad \boldsymbol{\omega} := \boldsymbol{\omega} + \tau^i \mathbf{G}_{\boldsymbol{\rho}}(\mathbf{u}^{i-1}, \mathbf{w}^i, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta}$

$\quad \boldsymbol{\lambda} := \boldsymbol{\lambda} + \tau^i \mathbf{G}_{\mathbf{u}}(\mathbf{u}^{i-1}, \mathbf{w}^i, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta}$

end for

Solve $\mathbf{A}(\mathbf{0}, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta} = \boldsymbol{\lambda}$

$\frac{dJ(\mathbf{u}^k(\boldsymbol{\rho}), \boldsymbol{\rho})}{d\boldsymbol{\rho}} := \boldsymbol{\omega} + \mathbf{H}_{\boldsymbol{\rho}}(\mathbf{u}^0, \boldsymbol{\rho})^T \cdot \boldsymbol{\eta} + J'_{\boldsymbol{\rho}}(\mathbf{u}^k, \boldsymbol{\rho})$

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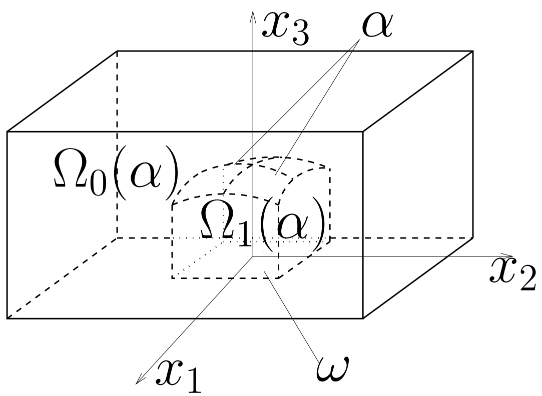
Shape optimization for nonlinear magnetostatics

Set of admissible shapes

$$\mathcal{U} := \{\alpha \in C(\overline{\omega}) \mid \alpha_{\mathrm{l}} \leq \alpha(\mathbf{x}) \leq \alpha_{\mathrm{u}} \text{ and } |\alpha(\mathbf{x}) - \alpha(\mathbf{y})| \leq C_{\mathrm{L}} \|\mathbf{x} - \mathbf{y}\|\},\ \alpha_n \rightrightarrows \alpha$$



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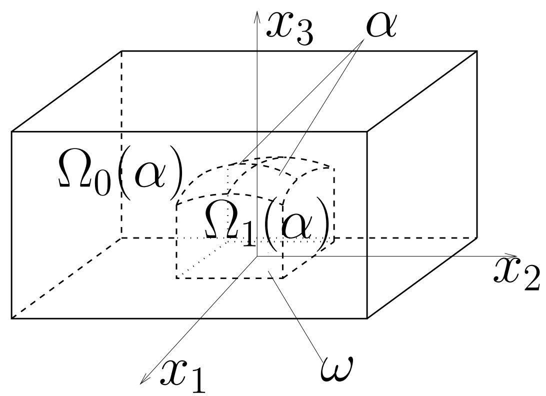
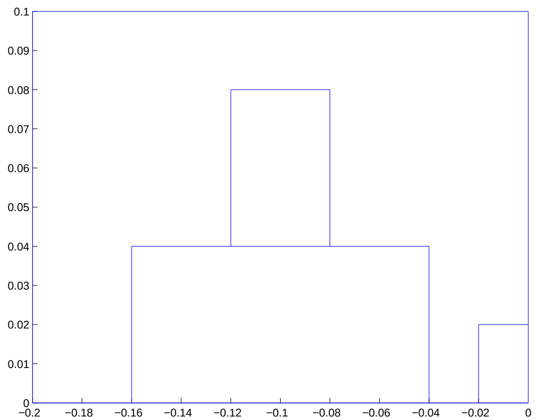


State problem

$$(W^v(\alpha)) \begin{cases} \text{Find } \mathbf{u}_\alpha \in \mathbf{H_0}(\mathbf{curl}; \Omega) : \\ \displaystyle\int_{\Omega_0(\alpha)} \nu_0 \mathbf{curl}(\mathbf{u}_\alpha) \cdot \mathbf{curl}(\mathbf{v})\, d\mathbf{x} + \int_{\Omega_1(\alpha)} \nu(\|\mathbf{curl}(\mathbf{u}_\alpha)\|)\mathbf{curl}(\mathbf{u}_\alpha) \cdot \mathbf{curl}(\mathbf{v})\, d\mathbf{x} + \\ \displaystyle + \varepsilon \int_\Omega \mathbf{u}_\alpha \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{J} \cdot \mathbf{v}\, d\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H_0}(\mathbf{curl}; \Omega) \end{cases}$$

Discretized shape optimization

$$(\widetilde{P}) \begin{cases} \min\limits_{\mathbf{p} \in \mathbb{R}^{n\Upsilon}} \widetilde{\mathcal{J}}(\mathbf{p}) \\ \text{subject to } \boldsymbol{v}(\mathbf{p}) \leq \mathbf{0} \end{cases}$$



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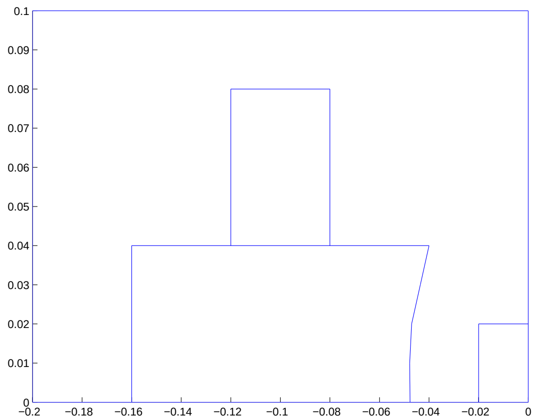
Structure of $\widetilde{\mathcal{J}}$

$$\mathbf{p} \xrightarrow{\pi_\omega^h \circ F} \boldsymbol{\alpha}^h$$



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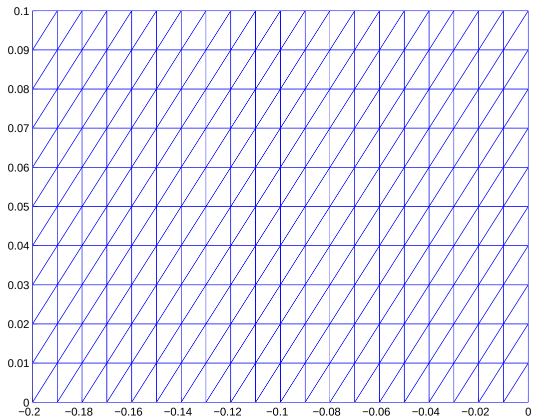

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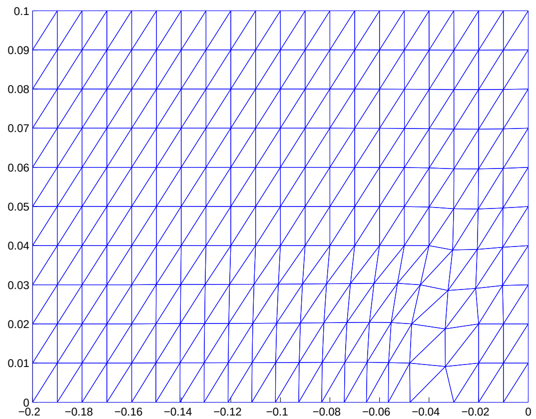

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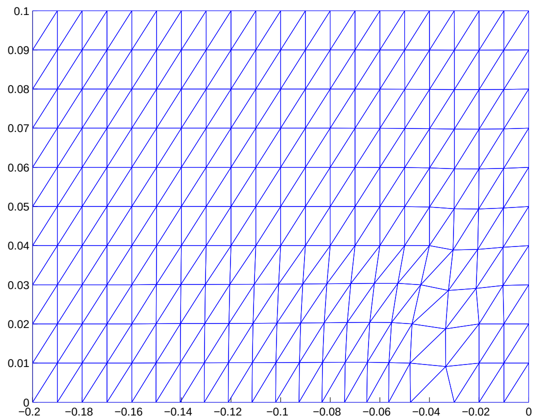

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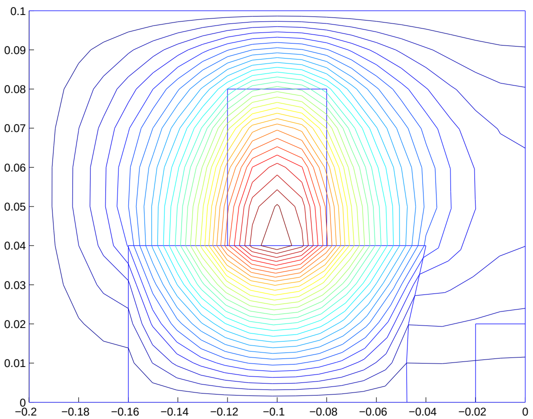

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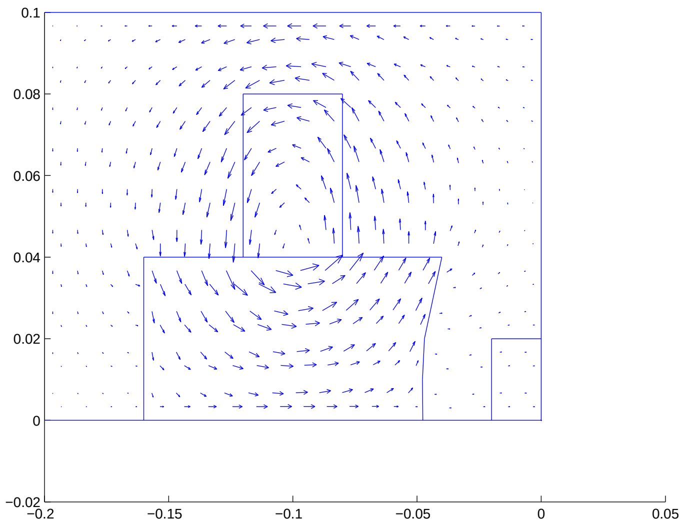

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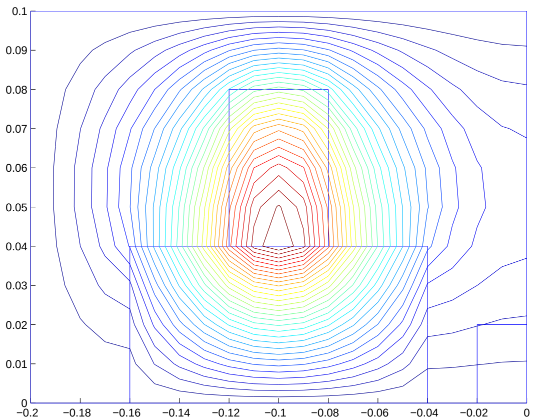
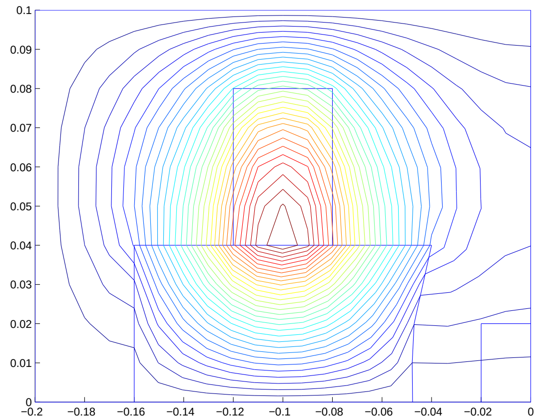
$$\xrightarrow{\textbf{curl}} \boldsymbol{B}^n \xrightarrow{\mathcal{I}^h(\boldsymbol{B}^n)} \widetilde{\mathcal{J}}^h(\mathbf{p})$$

$\mathcal{J}^h(\mathbf{p}_{\text{init}}) = 0.0077$



$\mathcal{J}^h(\mathbf{p}_{\text{disturbed}}) = 0.0143$



Discretized shape optimization

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$$\xrightarrow{\mathbf{curl}} \boldsymbol{B}^n \xrightarrow{\mathcal{I}^h(\boldsymbol{B}^n)} \widetilde{\mathcal{J}}^h(\mathbf{p})$$

Bottleneck

For fine discretizations it is hard to find a continuous shape–to–mesh mapping!

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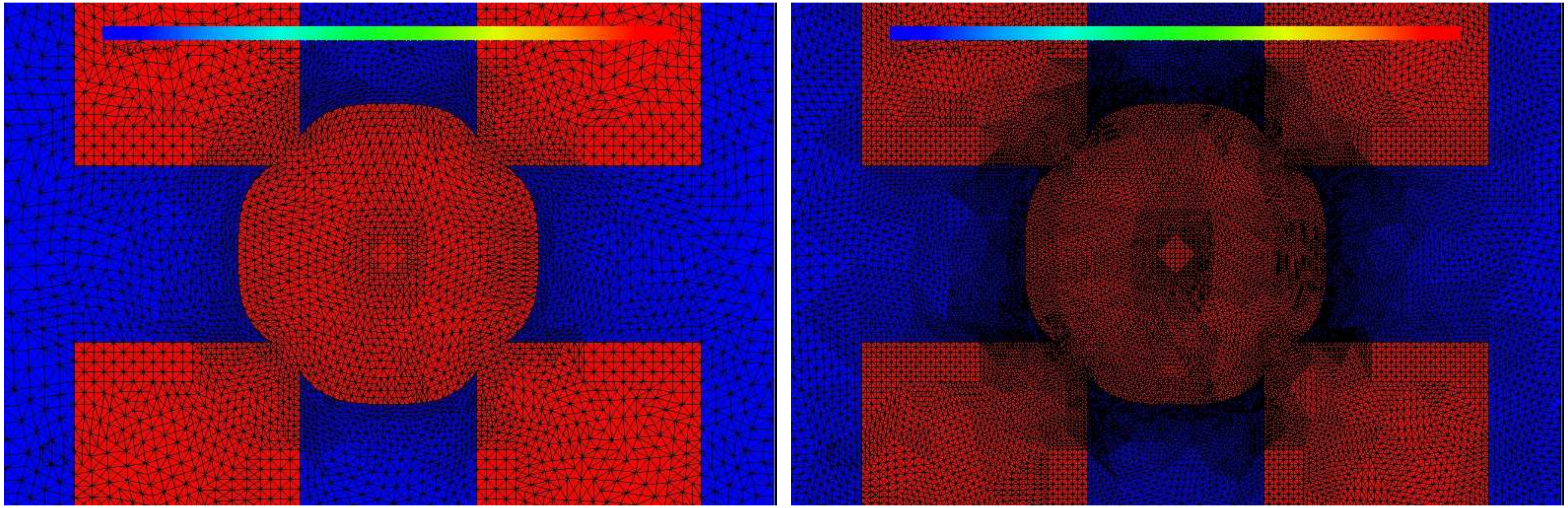
Multigrid OO SQP method, linear state problem
(following R. Stainko, C. Pechstein)

Discretize at the first level $\rightsquigarrow h^1, \alpha^1_{\text{init}}, \boldsymbol{A}^1(\alpha^1_{\text{init}})$
Solve by the SQP method and the nested direct solver $\rightsquigarrow \alpha^1_{\text{opt}}$
Store the first level preconditioner $\boldsymbol{C}^1_{\text{opt}} := \boldsymbol{A}^1(\alpha^1_{\text{opt}})^{-1}$
for $l = 2, 3, \ldots$
  Refine $h^{l-1} \rightsquigarrow h^l$
  Prolong $\alpha^{l-1}_{\text{opt}} \rightsquigarrow \alpha^l_{\text{init}}$
  Solve by the BFGS–SQP method and the nested multigrid solver $\rightsquigarrow \alpha^l_{\text{opt}}$
  Store the l–th level preconditioner $\boldsymbol{C}^l_{\text{opt}} \approx \boldsymbol{A}^l(\alpha^l_{\text{opt}})^{-1}$
end for

Multigrid OO SQP method, 2D linear state problem



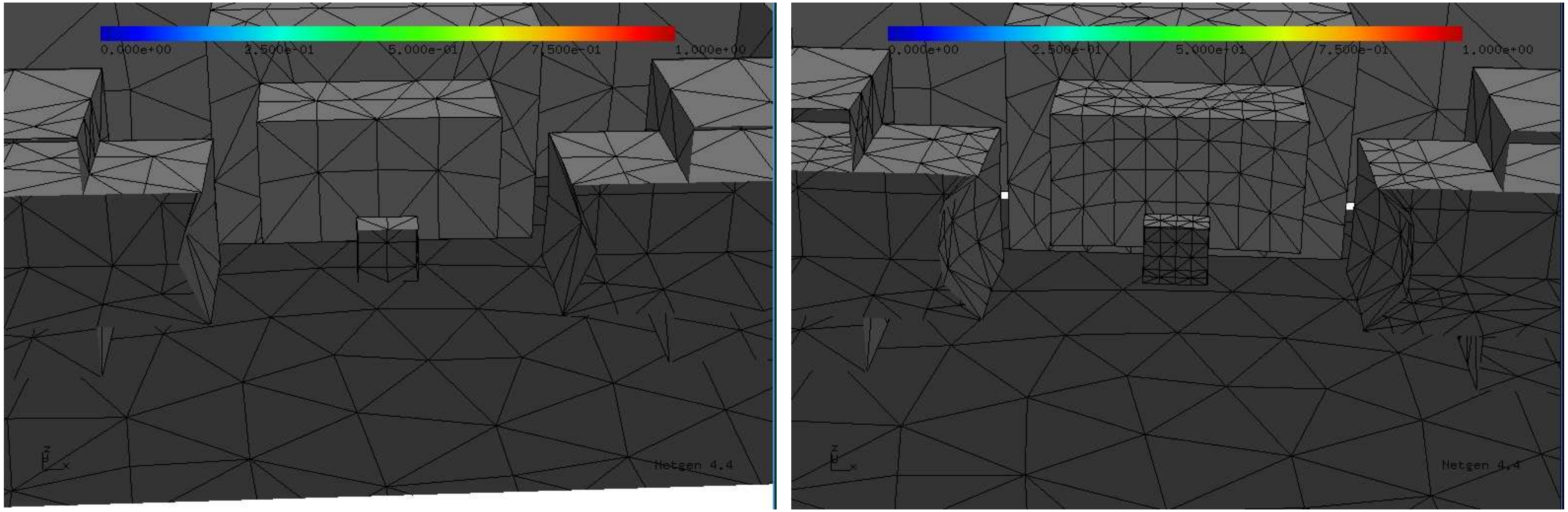
Design variables/SQP iterations: 4/14, 8/9, 16/9

State variables: 2905, 11489, 45697

CG iterations: 2–3, independent of the level

Total CPU times: 2min 19s, 12min 52s, 1h 30min

Multigrid OO SQP method, 3D linear state problem



Design variables/SQP iterations: 4/4, 16/36

State variables: 12431, 29017

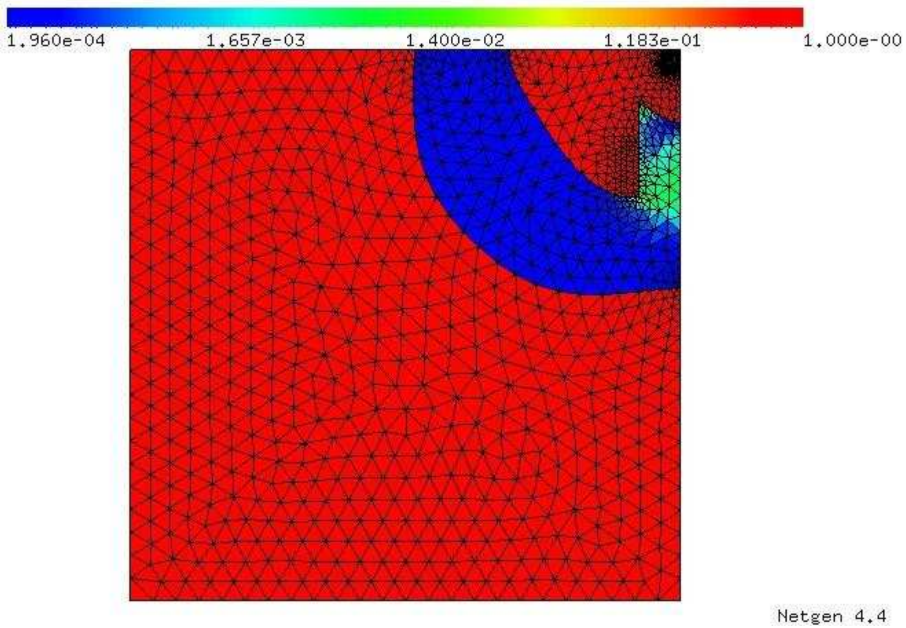
CG iterations: 3, independent of the level

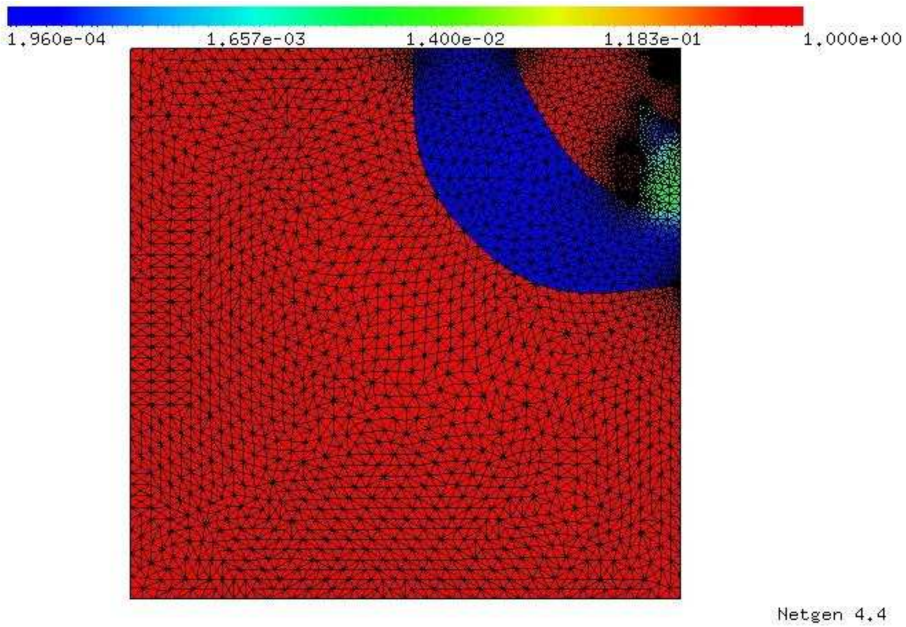
Total CPU times: 4min 19s, 2h 52min

Multigrid Œ SQP method, 2D nonlinear state problem

Discretize at the first level $\rightsquigarrow h^1, \alpha^1_{\text{init}}$
Solve by the SQP method and the nested NewtonŒdirect solver $\rightsquigarrow \alpha^1_{\text{opt}}$
Store the first level preconditioner $\boldsymbol{C}^1_{\text{opt}} := \boldsymbol{A}^{\text{linear},1}(\alpha^1_{\text{opt}})^{-1}$
for $l = 2, 3, \ldots$
    Refine $h^{l-1} \rightsquigarrow h^l$
    Prolong $\alpha^{l-1}_{\text{opt}} \rightsquigarrow \alpha^l_{\text{init}}$
    Solve by the BFGS–SQP method and the nested NewtonŒmultigrid solver $\rightsquigarrow \alpha^l_{\text{opt}}$
    Store the l–th level preconditioner $\boldsymbol{C}^l_{\text{opt}} \approx \boldsymbol{A}^{\text{linear},l}(\alpha^l_{\text{opt}})^{-1}$
end for

Multigrid OO SQP method, 2D nonlinear state problem





Design variables/SQP iterations: 19/6, 40/11, 82/3
State variables: 1098, 4240, 16659
typically 3–4 Newton nested iterations
typical CG iterations for linear/nonlinear step: –, 3/15, 4/40
Total CPU times: 1min 5s, 15min 53s, 38min 37s

Outline

- Benchmark problem
  - 2D/3D linear/nonlinear magnetostatics
- Topology optimization for nonlinear magnetostatics
  - Nonlinear state sensitivity analysis
- Shape optimization for nonlinear magnetostatics
  - Multilevel solver
- Sequential 2D topology–shape optimization
- Outlook

Sequential 2D topology–shape optimization

Topology or shape optimization?

Sequential 2D topology–shape optimization

Topology or shape optimization?

- Topology optimization: no design restrictions, time–consuming

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1. Coarse topology optimization with a moderate penalization of intermediate values

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Recently applied in structural mechanics

P.S. Tang and K.H. Chang, Struct. Multidisc. Optim. 22 (2001)

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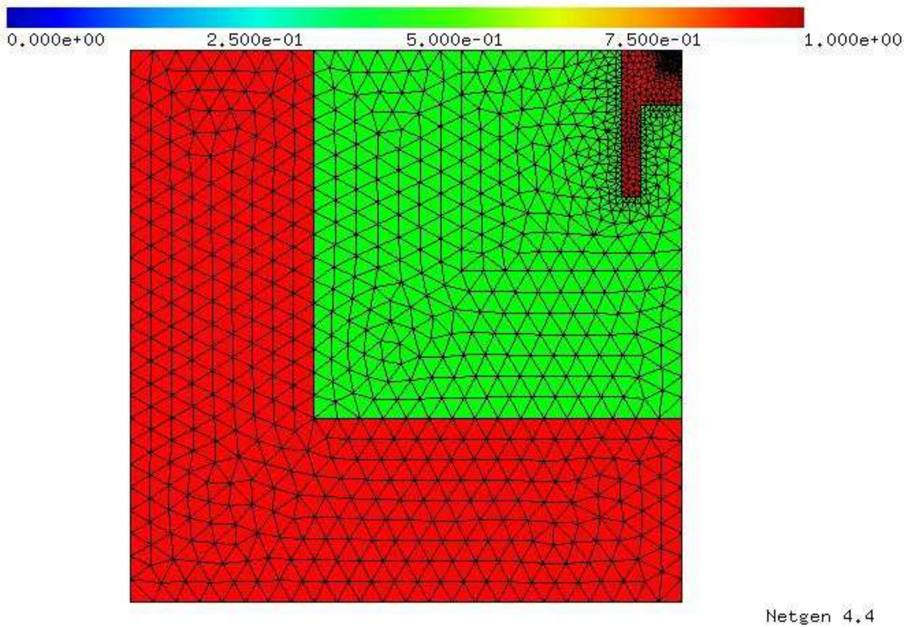
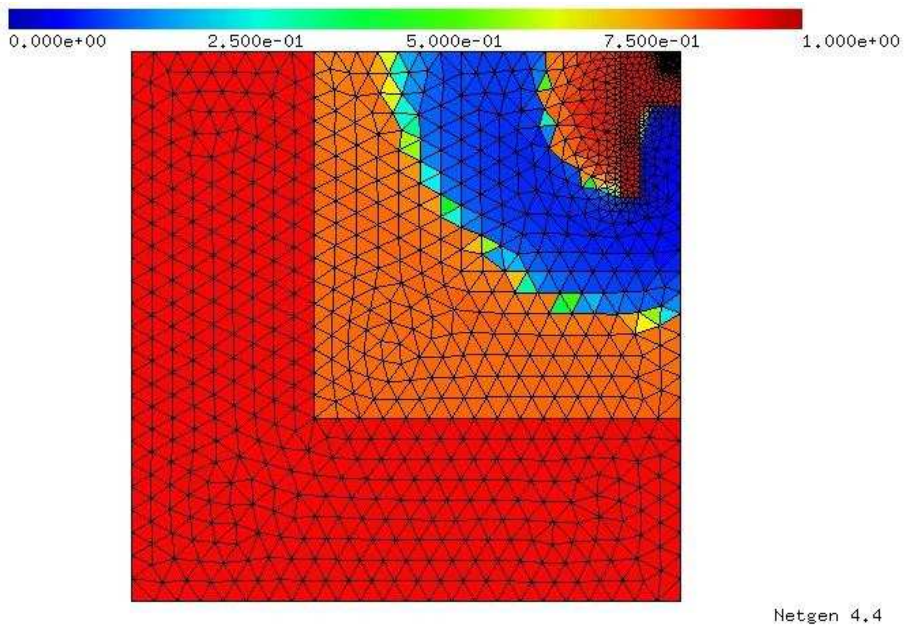
Coarse topology optimization with a moderate penalization

initial design



optimized design



Parameters of the computation

861 design variables, 7 steepest descent iterations, 1105 state variables, direct solver, total time: 2.5 sec

Sequential 2D topology–shape optimization

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Sequential 2D topology–shape optimization

Smooth shape approximation by least squares

$\mathcal{P} := \left\{ (\mathbf{p}_1, \ldots, \mathbf{p}_n) \mid \underline{\mathbf{p}_i} \le \mathbf{p}_i \le \overline{\mathbf{p}_i} \text{ for } i = 1, \ldots, n \right\}$ …set of admissible Bézier parameters

$$\min_{(\mathbf{p}_1,\ldots,\mathbf{p}_n)\in\mathcal{P}} \int_{\Omega_\mathrm{d}} \left(\rho^{\mathrm{opt}} - \chi(\Omega_1\left(\alpha_1(\mathbf{p}_1), \ldots, \alpha_n(\mathbf{p}_n)\right))\right)^2 \, d\mathbf{x},$$

where $\chi(\Omega_1)$ is the characteristic function of Ω_1

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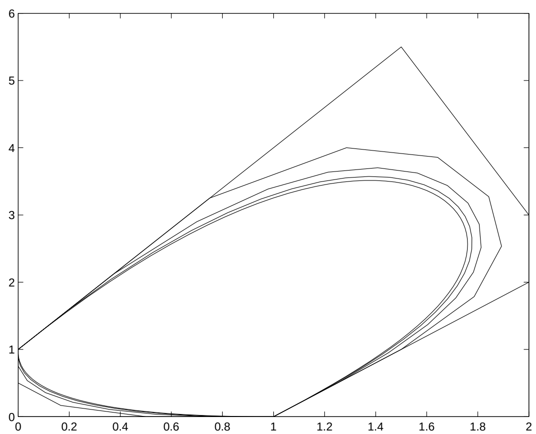
Polygonal approximation of Bézier shapes



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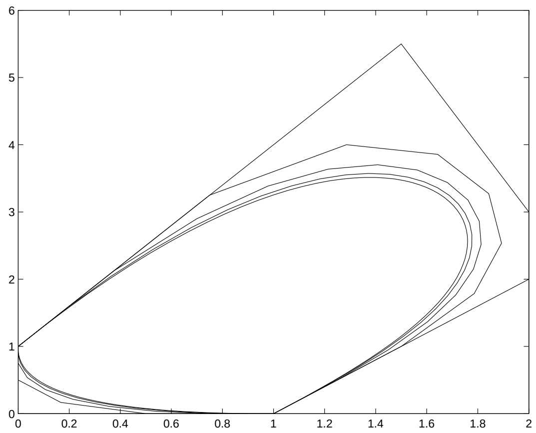


To avoid intersection of Bézier shapes with the grid:

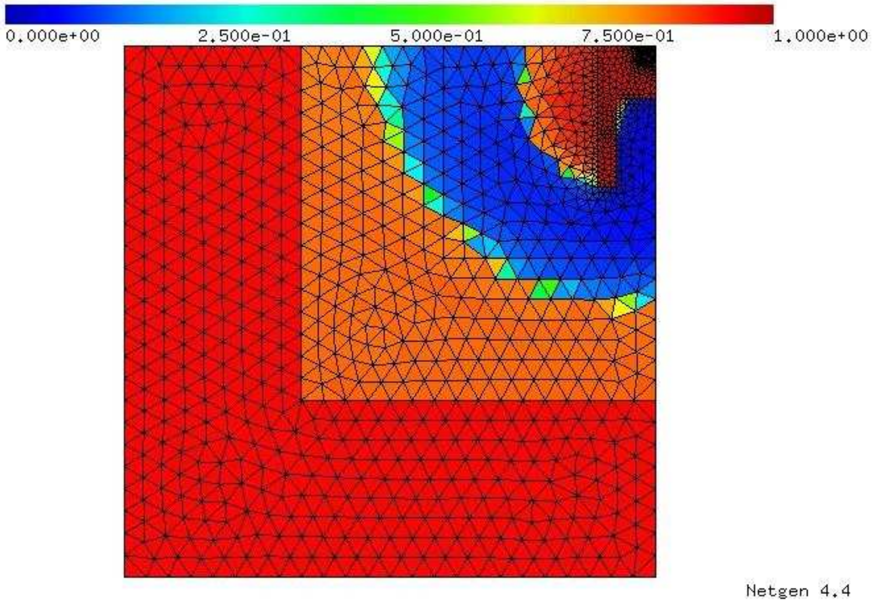
$$\begin{aligned}
\left[\mathbf{p}_i^{k+1}\right]_0 &:= \left[\mathbf{p}_i^k\right]_0 \\
\left[\mathbf{p}_i^{k+1}\right]_j &:= \tfrac{j-1}{m_i+1}\left[\mathbf{p}_i^k\right]_{j-1} + \tfrac{n-j}{m_i+1}\left[\mathbf{p}_i^k\right]_j, \; j = 2, \ldots, m_i \\
\left[\mathbf{p}_i^{k+1}\right]_{m_i+1} &:= \left[\mathbf{p}_i^k\right]_{m_i},
\end{aligned}$$

where $\mathbf{p}_i^0 := \mathbf{p}_i$

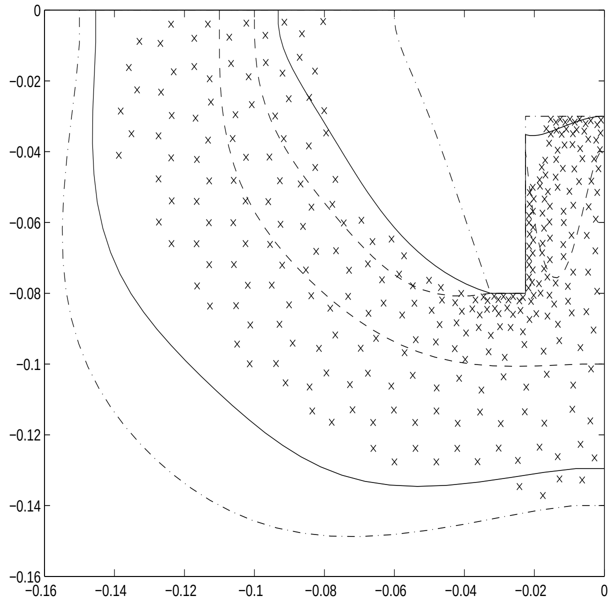
Sequential 2D topology–shape optimization

Smooth shape approximation by least squares

optimized topology design



smooth shape approximation



Parameters of the computation

19 Bézier control parameters, 8 SQP iterations,
total time: 26 sec

Sequential 2D topology–shape optimization

Topology or shape optimization?

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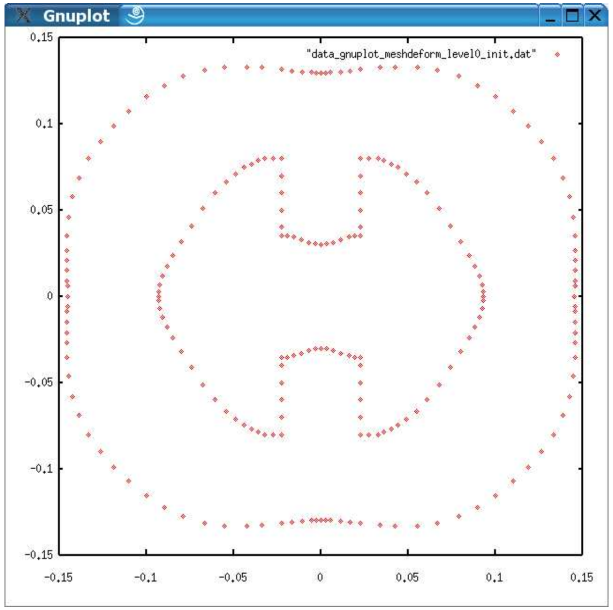
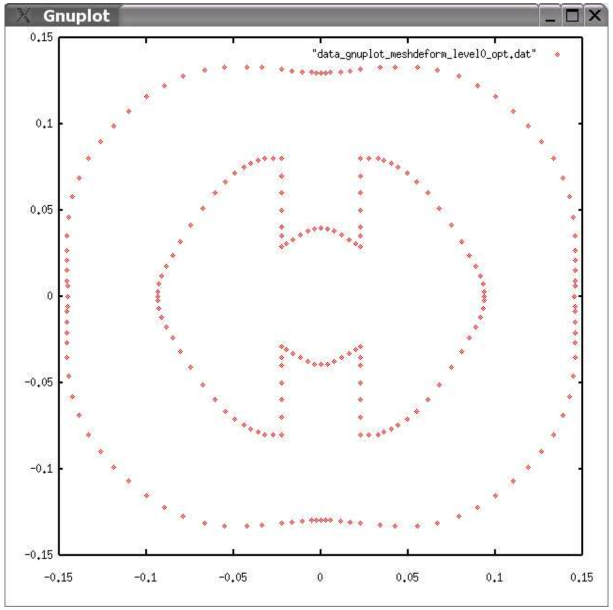
Sequential 2D topology–shape optimization

Multilevel shape optimization, mesh deformation approach

initial design



first–level optimized design



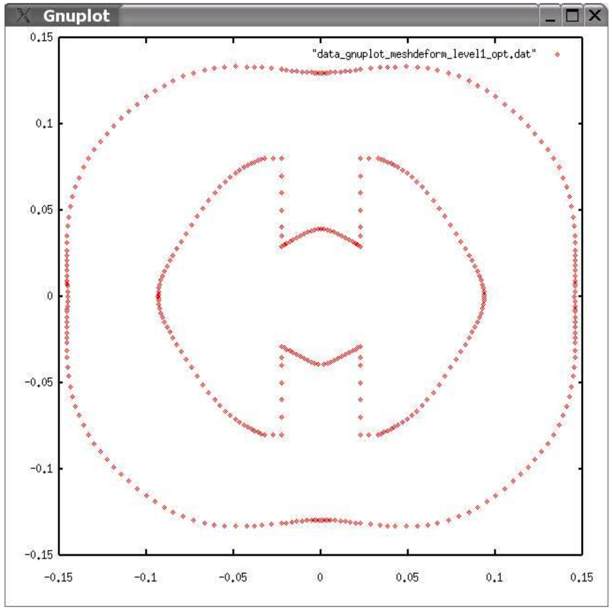
Parameters of the computation

19 Bézier control parameters, 10 SQP iterations, 1098 state variables, direct solver, 3 inner nonlinear iterations, total time: 32 sec

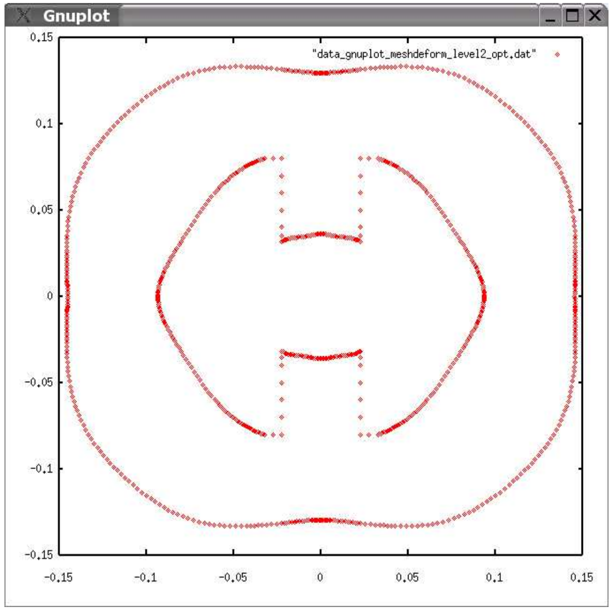
Sequential 2D topology–shape optimization

Multilevel shape optimization, mesh deformation approach

2nd–level optimized design



3rd–level optimized design

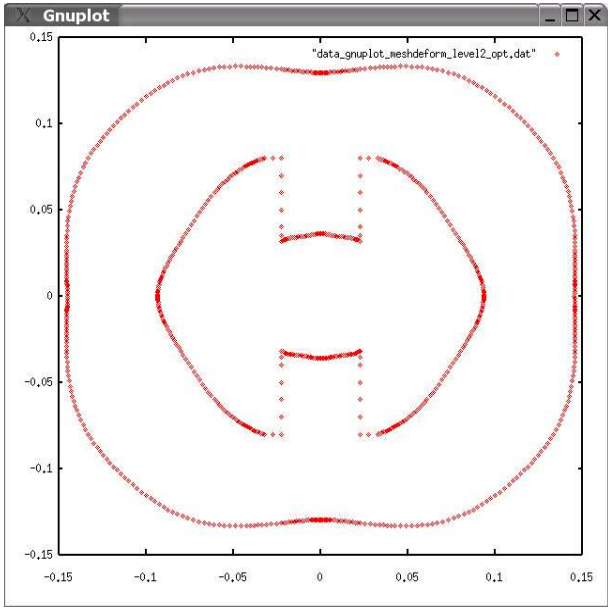


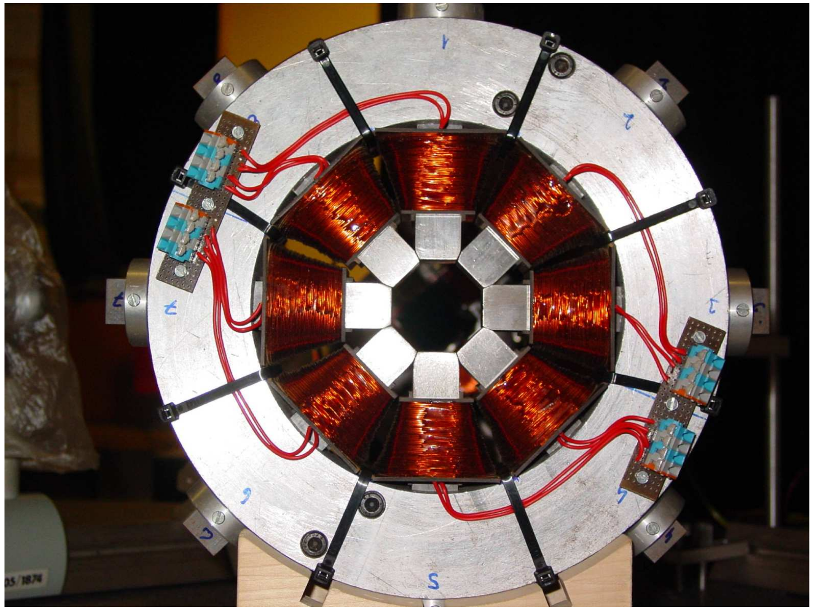
Parameters of the computation

40, 82, 166, 334 design parameters, 10–15 SQP iterations, 4k–262k state variables, 3–6(9–80) (non)linear CG iters. Total times: 3, 9, 49 min, 6.5 hours

Sequential 2D topology–shape optimization

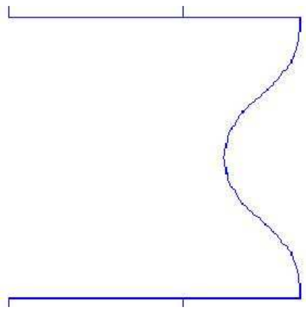
The optimized geometry

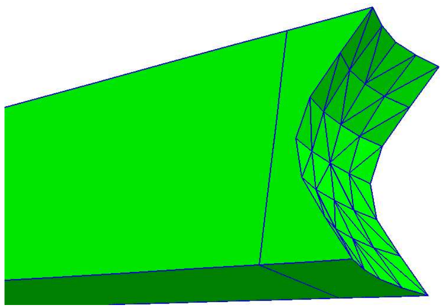




Maltese cross electromagnet

Optimized pole heads





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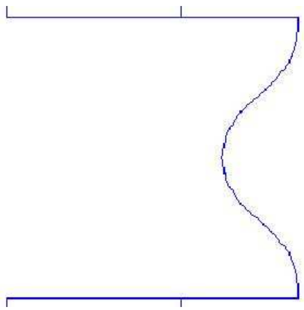
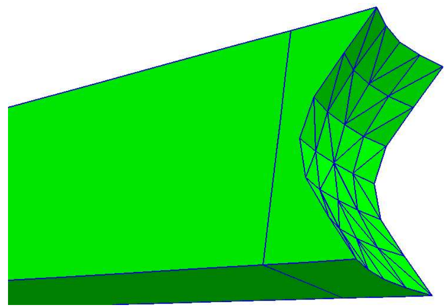




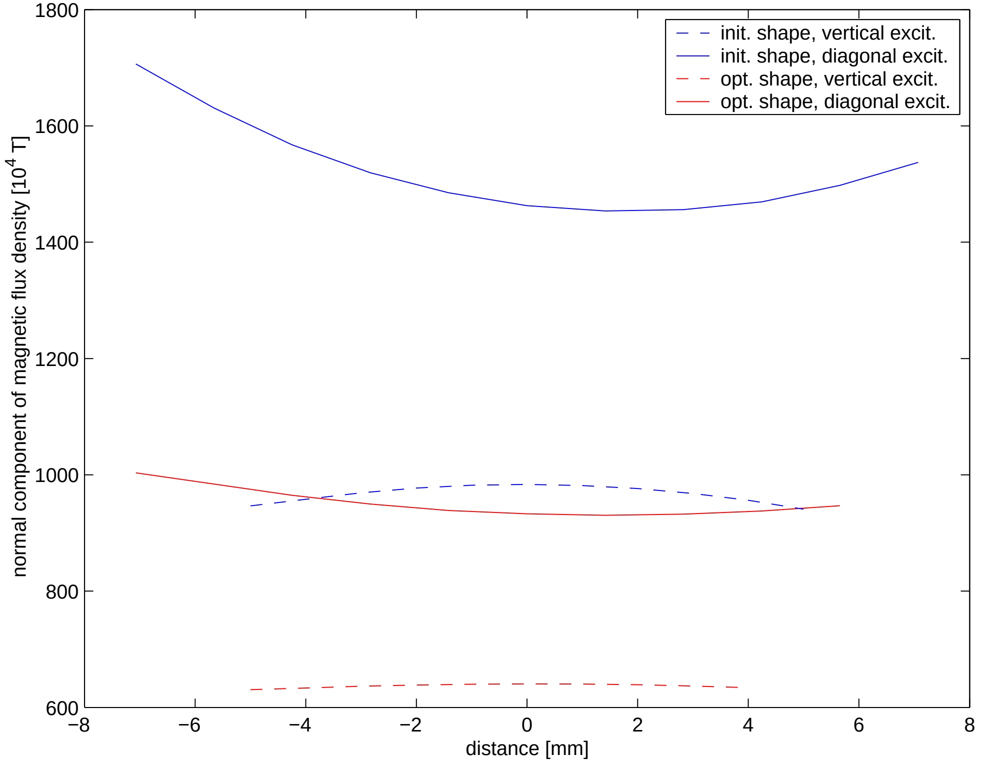
Parameters

design variables	7	12
deg. of freedom	12272	29541
SQP iterations	72	93
cost func. decrease	$1.97.10^{-6}$ to $1.49.10^{-6}$	$2.57.10^{-6}$ to $7.32.10^{-7}$
comput. time	2 hours	30 hours

Maltese cross electromagnet

Manufacture and measurements

The calculated cost functional has improved twice and the measured cost functional has improved even 4.5–times.



Outline

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  - Nonlinear state sensitivity analysis
- Shape optimization for nonlinear magnetostatics
  - Multilevel solver
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Outlook

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  - Multigrid analysis for the disturbed bilinear form

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- Application to optimal design for eddy currents
- Software development, industrial benchmarks