A Comparison of TFETI and TBETI for Numerical Solution of Engineering Problems of Contact Mechanics

D. Lukáš[1], M. Sadowská[1], T. Kozubek[1], A. Markopoulos[2], and T. Brzobohatý[2]

Abstract Since the introduction of Finite Element Tearing and Interconnecting (FETI) by Farhat and Roux in 1991, the method has been recognized to be an efficient parallel technique for the solution of partial differential equations. In 2003 Langer and Steinbach formulated its boundary element counterpart (BETI), which reduces the problem dimension to subdomain boundaries. Recently, we have applied both FETI and BETI to contact problems of mechanics. In this paper we numerically compare their variants bearing the prefix Total (TFETI/TBETI) on a frictionless Hertz contact problem and on a realistic problem with a given friction.

1 Introduction

One of the leading representatives of domain decomposition methods is the Finite Element Tearing and Interconnecting (FETI) proposed by Farhat and Roux [1991]. It relies on a finite element discretization of a linear elliptic boundary value problem and a nonoverlapping decomposition of the related geometric computational domain into subdomains. Resulting local subproblems are glued by means of Lagrange multipliers. The dual coarse problem is solved for the Lagrange multipliers by the method of conjugate gradients. Farhat et al. [1994] proved that the condition number of the Schur complement, which arises from the elimination of the interior degrees of freedom, preconditioned by a projector orthogonal to the kernel is proportional to H/h, where H denotes the maximal subdomain diameter and h is the finite element discretization parameter. Moreover, Mandel and Tezaur [1996] proved a polylogarithmic bound on the condition number of the Schur com-

FEECS, VŠB - TU Ostrava, Czech Republic, dalibor.lukas@vsb.cz, marie.sadowska@vsb.cz, tomas.kozubek@vsb.cz · FME, VŠB - TU Ostrava, Czech Republic, alexandros.markopoulos@vsb.cz, tomas.brzobohaty@vsb.cz

plement preconditioned by the Dirichlet preconditioner. This result was extended by Klawonn and Widlund [2001] to the case of a redundant set of Lagrange multipliers and the correct (multiplicity or stiffness) scaling.

As the Lagrange multipliers live on the skeleton of the decomposition, it is very natural to employ a boundary integral representation of solutions to the local subproblems. This is the Boundary Element Tearing and Interconnecting (BETI) method, which was formulated and analyzed by Langer and Steinbach [2003]. The resulting discretized Steklov-Poincaré operators, which relate the local Cauchy data, are proved to be spectrally equivalent to the finite element Schur complements which eliminate interior degrees of freedom. An application of fully populated boundary element (BE) matrices can be sparsified to a linear complexity (up to a logarithmic factor), cf. Of et al. [2005]. Steinbach and Wendland [1998] proposed a preconditioning of the BE matrices by related opposite order BE operators. The latter two accelaration techniques were exploited by Langer et al. [2007] within the BETI method formulated in a twofold saddle-point system. It turned to be natural to impose additional Lagrange multipliers along the Dirichlet boundary, which was independently introduced as Total FETI (TFETI) by Dostál et al. [2006] and as All-Floating BETI by Of [2008], see also Of and Steinbach [2009].

An extension of FETI and BETI methods to contact problems is a challenging task due to the strong nonlinearity of the variational inequality under consideration. To name a few of many research groups attacking this problem, see Avery and Farhat [2009], Schöberl [1998], Kornhuber and Krause [2001], Wolmuth and Krause [2003]. The base for our development is a theoretically supported scalable algorithm for both coercive and semicoercive contact problems presented by Dostál et al. [2010] and in the monograph by Dostál [2009]. The first scalability results using TBETI for the scalar variational inequalities and the coercive contact problems were presented only recently by Bouchala et al. [2008] and Bouchala et al. [2009], respectively. We also refer to Sadowská et al. [2011].

The aim of this paper is to numerically compare TFETI and TBETI for two realistic problems. In Section 2 we recall the algebraic formulation of the TFETI and TBETI methods for contact problems. In Section 3 we describe different representations of the Schur complement. In Section 4 we compare the methods for the 3-dimensional (3d) Hertz contact problem without a friction and for a 3d contact problem of a ball bearing with a given friction. In Section 5 we conclude.

2 TFETI/TBETI formulations

Both TFETI and TBETI methods for contact problems of mechanics lead, after a discretization, to the following problem:

$$\min_u \frac{1}{2}\langle Su, u\rangle - \langle f, u\rangle \text{ subject to } B_{\mathcal{I}} u \leq c_{\mathcal{I}} \text{ and } B_{\mathcal{E}} u = c_{\mathcal{E}},$$

where we search for the local boundary displacement fields $u := (u_1, \ldots, u_p)$ with p being the number of subdomains. The Hessian $S := \mathrm{diag}(S_1, \ldots, S_p)$ consists of the Schur complements which are local Neumann finite element stiffness matrices eliminated to subdomain boundaries in the case of TFETI, and which are symmetric boundary element discretizations of local Steklov-Poincaré operators in the case of TBETI. Note that $\mathrm{Ker}\, S_i$ is the space spanned by six linearized local rigid body modes. In $f := (f_1, \ldots, f_p)$ we cummulate local boundary tractions. Further, $B_{\mathcal{E}}$ is a full rank sign matrix, the first part of which interconnects teared degrees of freedom with corresponding first part of $c_{\mathcal{E}}$ to be zero, while the second parts of $B_{\mathcal{E}}$ and $c_{\mathcal{E}}$ realize the Dirichlet boundary condition. Finally, the inequality with $B_{\mathcal{I}}$, $c_{\mathcal{I}}$ prescribes linearized non-penetration conditions.

Due to expensive projections onto the linear inequality constraints, we switch to the dual formulation with simple bound and equality constraints

$$\min_{\lambda_{\mathcal{I}} \geq 0} \frac{1}{2}\langle BS^+B^T \lambda, \lambda\rangle - \langle BS^+ f - c, \lambda\rangle \text{ s.t. } (B^T\lambda - f) \perp \mathrm{Ker}\, S,$$

where we introduce Lagrange multipliers $\lambda := (\lambda_{\mathcal{I}}, \lambda_{\mathcal{E}})$ with $\mathcal{I}$ and $\mathcal{E}$ referring to the inequality and equality constraints, respectively. Further, we cover $B_{\mathcal{I}}$, $B_{\mathcal{E}}$ by B and similarly $c := (c_{\mathcal{I}}, c_{\mathcal{E}})$. Let S^+ be a pseudoinverse of S, i.e., $S\,S^+\,g = g$ for any $g \perp \mathrm{Ker}\, S$. Let us denote by $R := \mathrm{diag}(R_1, \ldots, R_p)$ the column basis of $\mathrm{Ker}\, S$ consisting of local rigid body modes R_i and by P the orthogonal projector from $\mathrm{Im}\, B$ onto $\mathrm{Ker}\, R^T B^T = (\mathrm{Ker}\, S)^\perp$. To homogenize the linear (orthogonality) constraint, assume we are given a feasible λ_0 and search for $\lambda := \tilde{\lambda} + \lambda_0$. Returning to the old notation, we arrive at the following constrained quadratic programming problem preconditioned by the projector P and regularized by the complementary projector $Q := I - P$:

$$\min_{\lambda_{\mathcal{I}} \geq -(\lambda_0)_{\mathcal{I}}} \frac{1}{2}\left\langle \left(\frac{1}{\rho} PFP + Q\right)\lambda, \lambda \right\rangle - \left\langle \frac{1}{\rho} P(BS^+ f_0 - c), \lambda \right\rangle \text{ s.t. } R^T B^T \lambda = 0, \tag{1}$$

where $F := BS^+B^T$ and $f_0 := f - B^T\lambda_0$. Finally, we scale the cost function by $\rho \approx \|PFP\|$. Now from Theorem 3.2 of Farhat et al. [1994] and from the spectral equivalence of local boundary element and finite element Schur complements S_i, see Lemma 3.2 of Langer and Steinbach [2003], we have the following optimality result valid for both TFETI and TBETI.

Theorem 1. *Denote $\mathcal{H} := (1/\rho)PFP + Q$. There exist $c, C > 0$ independent of h, H so that*

$$\lambda_{\min}(\mathcal{H}|\mathrm{Im}\, P) \geq c\frac{h}{H} \quad \textit{and} \quad \lambda_{\max}(\mathcal{H}|\mathrm{Im}\, P) = \|\mathcal{H}\| \leq C.$$

We are now in the position to use the augmented Lagrangian algorithm developed by Dostál [2006], see also Dostál [2009], for the solution of our constraint minimization problem (1). We mention that this algorithm is in some sense optimal.

3 Schur complements

The local Schur complements S_i represent symmetric discretizations of the Steklov-Poincaré operator $\tilde{S}_i$ mapping the Dirichlet data to the Neumann data. In particular, $\tilde{S}_i(u_i) := \sigma_i(\varepsilon(\tilde{u}_i)) \cdot n_i$ in the case of elastostatics, where n_i is the outward unit normal to the subdomain Ω_i, $\sigma_i(\varepsilon(\tilde{u}_i))$ denotes the elastostatic stress evaluated using the local linearized Hooke's law between the stress σ_i and the strain $\varepsilon(\tilde{u}_i)$, and where $\tilde{u}_i$ solves the following inhomogeneous Dirichlet boundary value problem:

$$\operatorname{div} \sigma_i(\varepsilon(\tilde{u}_i(x)) = 0 \text{ in } \Omega_i, \quad \tilde{u}_i(x) = u_i(x) \text{ on } \partial\Omega_i. \tag{2}$$

In the case of TFETI we solve (2) approximately by the finite element method. The approximation of $\tilde{S}_i$ is then as follows:

$$S_i := (A_i)_{BB} - (A_i)_{BI}(A_i)_{II}^{-1}(A_i)_{IB},$$

where $(A_i)_{jk} := \int_{\Omega_i} \sigma_i(\varepsilon(\varphi_j^{(i)}(x))) : \varepsilon(\varphi_k^{(i)}(x))\, dx$ is the Neumann finite element matrix assembled in the vector lowest order nodal basis functions $\varphi_j^{(i)}$, and where B and I are the sets of indices of boundary and interior degrees of freedom, respectively.

In the case of TBETI the interior degrees of freedom are already eliminated in the continuous formulation via a boundary integral representation of $\tilde{u}_i(x)$ while making use of the known elastostatic fundamental solution. After the lowest order Galerkin boundary element discretization, we arrive at the following relation between the approximated nodal based Dirichlet data, still denoted by u_i, and the element-based Neumann data, denoted by $t_i \approx \sigma_i(\varepsilon(\tilde{u}_i)) \cdot n_i$:

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} (1/2)M_i - K_i & V_i \\ D_i & ((1/2)M_i + K_i)^T \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix}$$

with fully populated boundary element matrices V_i, K_i, and D_i, which are referred to as single-layer, double-layer, and hypersingular matrix, respectively, and with the boundary mass matrix M_i. We then employ the following symmetric approximation of the Schur complement $\tilde{S}_i$:

$$S_i := D_i + ((1/2)M_i + K_i)^T\, V_i^{-1}\, ((1/2)M_i + K_i)\,.$$

4 Numerical comparison

All the presented simulations are performed using a parallel Matlab within our MatSol library, see Kozubek et al.. The implementations of TFETI and TBETI are consistent. The only point where they differ is assembling of FEM and BEM matrices and subsequent Cholesky factorizations. In the preprocessing phase times for the BEM matrices assembling dominate. Our simulations were run on a cluster of 48 cores with 2.5 GHz and the infiband interface, which are equipped with licences of Matlab parallel computing engine.

First we consider a frictionless 3–dimensional Hertz problem, as depicted in Fig. 1, with the Young modulus $2.1 \cdot 10^5$ MPa and the Poisson ratio 0.3, where the ball is loaded from top by the force 5000 N. ANSYS discretization of the two bodies is decomposed by METIS into 1024 subdomains. The comparison of TFETI and TBETI in terms of computational times and number of Hessian multiplications is given in Tab. 1. In Fig. 2 we can see a fine correspondence of contact pressures computed by TFETI and TBETI to the analytical solution. The convergence criterion was the decay of the dual error to 10^{-6} relatively to the initial dual residuum.

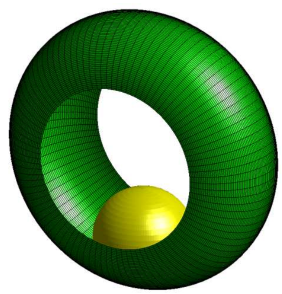

Fig. 1 Geometry of the Hertz problem

method	number of primal DOFs	number of dual DOFs	preprocessing time	solution time	number of Hessian applications
TFETI	4,088,832	926,435	21 min	1 h 49 min	593
TBETI	1,849,344	926,435	1h 33 min	1 h 30 min	667

Table 1 Numerical performance of TFETI and TBETI applied to the Hertz problem

In the second example we solve the contact problem of ball bearing, which consists of 10 bodies. We impose Dirichlet boundary condition along the outer perimeter and load the opposite part of the inner diameter with the force 4500 N as depicted in Fig. 3. The Young modulus and the Poisson ratio

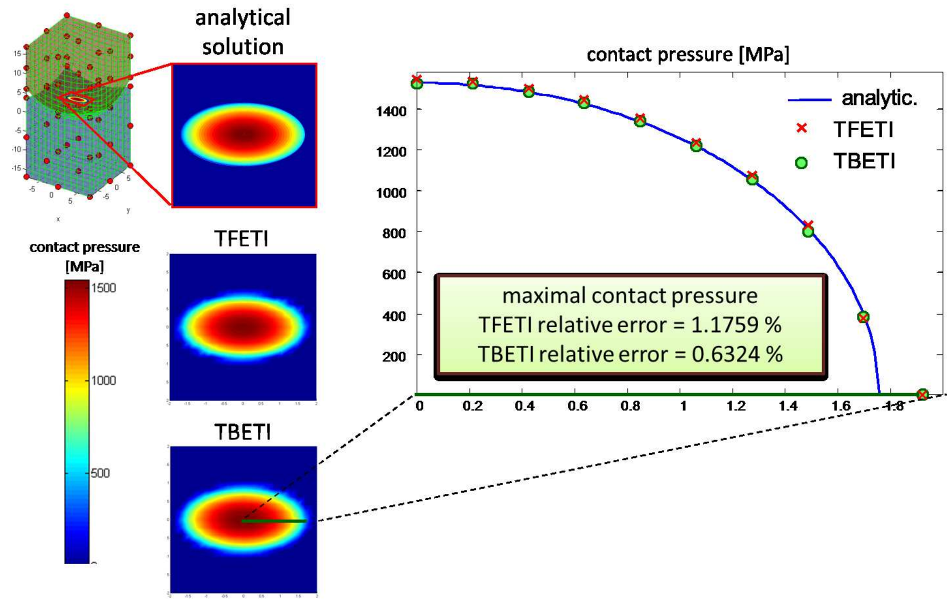

Fig. 2 Correspondence of numerical Hertz contact pressures to the analytic solution

of the balls and rings are $2.1 \cdot 10^5$ MPa and 0.3, respectively. Those of the cage are $2 \cdot 10^4$ MPa and 0.4, respectively. To get rid of the rigid body modes in the solution we introduce a small boundary gravitation term for each of the bodies. The discretized geometry was decomposed into 960 subdomains. Numerical comparison of TFETI and TBETI is shown in Tab. 2 and the resulting vertical displacement field is depicted in Fig. 4.

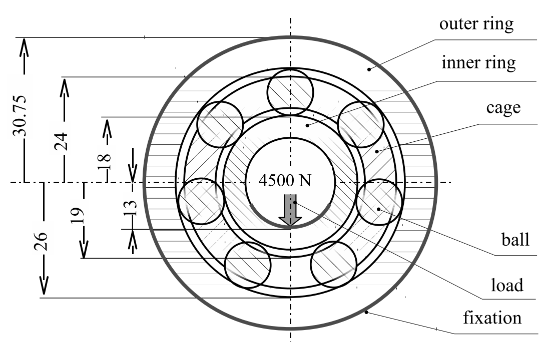

Fig. 3 Ball bearing: geometry, applied force and the Dirichlet boundary

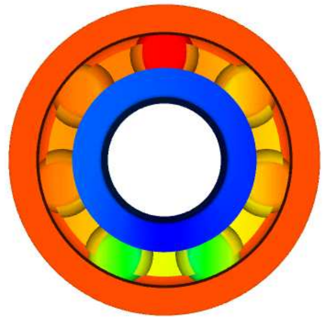

Fig. 4 Ball bearing: vertical component of the computed displacement field

method	number of primal DOFs	number of dual DOFs	preprocessing time	solution time	number of Hessian applications
TFETI	1,759,782	493,018	129 s	2 h 5 min	3203
TBETI	1,071,759	493,018	715 s	1 h 52 min	2757

Table 2 Numerical performance of TFETI and TBETI applied to the ball bearing problem

5 Conclusion

In the paper we compared TFETI and TBETI and numerically documented their performance for two engineering problems. Concerning timings and numbers of iterations it was shown that the methods are rather equal up to the assembling phase, which is more expensive in TBETI case. On the other hand, the accuracy of the boundary element discretization is usually much higher than the corresponding finite element discretization. This statement is supported by the theory provided that the solution is sufficiently regular. It can be also seen from Fig. 2, where one can guess that the TFETI relative error of 1.1759 % can be obtained with much less TBETI degrees of freedom.

Acknowledgement

This research has been financially supported by the grants GA CR 201/07/0294 and the Ministry of Education of the Czech Republic No. MSM6198910027. This work was also supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070).

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