

2-Dimensional Primal Domain Decomposition Theory in Detail *

Dalibor Lukáš[†], Jiří Bouchala[†], Petr Vodstrčil[†], and Lukáš Malý[†]

November 4, 2013

Abstract

We give details of the theory of primal domain decomposition methods for a 2-dimensional Poisson equation with homogeneous Dirichlet boundary conditions and jumping coefficients. The problem is discretized by the finite element method. The computational domain is decomposed into triangular subdomains that align with the coefficients jumps. We prove that the condition number of the vertex-based domain decomposition preconditioner is $O((1 + \log(H/h))^2)$, independently of the coefficient jumps, where H and h denotes discretization parameters of the coarse and fine triangulations, respectively. Although this preconditioner and its analysis date back to the pioneering work by Bramble, Pasciak, and Schatz published in 1986, and it was revisited and extended by many authors including Dryja and Widlund in 1989, or Toselli and Widlund in 2005, the theory is hard to understand and some details, to our best knowledge, have never been published. In this paper we present all the proofs in detail by means of fundamental calculus.

Keywords: domain decomposition methods, finite element method, preconditioning

*This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070), by the Czech Ministry of Education under the project MSM6198910027, and by VŠB-TU Ostrava under the grant SGS SP2013/191.

[†]VŠB–Technical University of Ostrava, 17. listopadu 15, 708 33 Ostrava-Poruba, Czech Republic. Emails: {dalibor.lukas,jiri.bouchala,petr.vodstrcil,lukas.maly}@vsb.cz

1 Introduction

We consider the homogeneous Dirichlet problem for the Poisson equation

$$\begin{aligned} -\operatorname{div}(\rho(x) \nabla u(x)) &= f(x), & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, $f \in L^2(\Omega)$, and $\rho \in L^\infty(\Omega)$ is a piecewise constant positive material function. The domain Ω is decomposed into N nonoverlapping open triangular subdomains Ω_i by means of a conforming finite element (FE) discretization $\bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}_i$. This is referred to as the coarse discretization or the domain decomposition (DD). The decomposition aligns with jumps of the material function so that $\rho(x) = \rho_i > 0$ for $x \in \Omega_i$. We denote by $\Gamma := \bigcup_{i=1}^M \bar{E}_i$, the skeleton of the decomposition, where E_i is the interior of an edge apart from $\partial\Omega$, see Fig. 1. We denote the coarse discretization parameter by $H := \max_{i=1, \dots, N} \operatorname{diam}(\Omega_i)$.

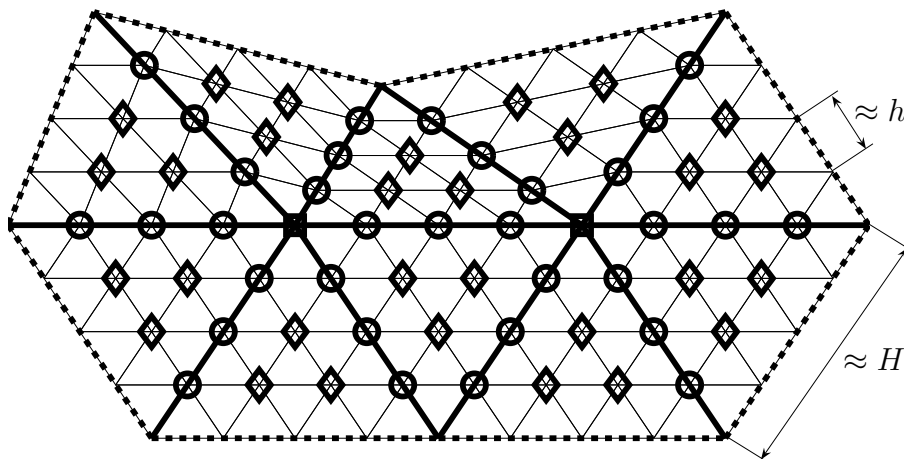


Figure 1: Decomposition of Ω into $N = 10$ subdomains with $n^V = 2$ vertices x_i^V (squares); dashed-line depicts $\partial\Omega$; solid-bold-lines denote Γ decomposed into $M = 11$ edges with edge nodes $x_{i,j}^E$ (circles); solid-thin-lines denote the fine triangulation with $n = 65$ nodes; diamonds depict interior nodes $x_{i,j}^I$.

The related weak formulation

$$\text{find } u \in H_0^1(\Omega) : \underbrace{\sum_{i=1}^N \rho_i \int_{\Omega_i} \nabla u(x) \nabla v(x) dx}_{=:a(u,v)} = \underbrace{\int_{\Omega} f(x) v(x) dx}_{=:b(v)} \quad \forall v \in H_0^1(\Omega)$$

is discretized by the conforming finite element (FE) method on a subspace $V := V^h := \langle \varphi_1(x), \dots, \varphi_n(x) \rangle \subset H_0^1(\Omega)$, where $(\varphi_i)_{i=1}^n$ denote the linear Lagrange basis functions related to the nodes depicted in Fig. 1. The underlying fine triangulation align with the domain decomposition. We arrive at the linear system

$$\mathbf{A} \mathbf{u} = \mathbf{b}, \quad (1)$$

where $(\mathbf{A})_{i,j} := a(\varphi_i, \varphi_j)$, $(\mathbf{b})_i := b(\varphi_i)$, and $u^h(x) := \sum_{j=1}^n (\mathbf{u})_j \varphi_j(x)$ approximates $u(x)$. By h we denote the fine discretization parameter, which is the maximal fine-triangle diameter.

Primal DD-methods rely on re-sorting the basis functions $(\varphi_i)_{i=1}^n$ into N sets of functions $(\varphi_{i,j}^I)_{j=1}^{n_i^I}$, $i = 1, \dots, N$, related to subdomain interior nodes $x_{i,j}^I \in \Omega_i$, see Fig. 1, and a set of functions $(\varphi_k^\Gamma)_{k=1}^{n^\Gamma}$ related to skeleton nodes $x_k^\Gamma \in \Gamma \setminus \partial\Omega$, which either belongs to an edge E_i , $x_k^\Gamma = x_{i,j}^E$, or it is a subdomain (coarse) vertex $x_k^\Gamma = x_i^V$, see Fig. 1. This translates (1) into the saddle-point system

$$\begin{pmatrix} \mathbf{A}_1^{I,I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_1^{I,\Gamma} \\ \mathbf{0} & \mathbf{A}_2^{I,I} & \dots & \mathbf{0} & \mathbf{A}_2^{I,\Gamma} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_N^{I,I} & \mathbf{A}_N^{I,\Gamma} \\ \mathbf{A}_1^{\Gamma,I} & \mathbf{A}_2^{\Gamma,I} & \dots & \mathbf{A}_N^{\Gamma,I} & \mathbf{A}^{\Gamma,\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^I \\ \mathbf{u}_2^I \\ \vdots \\ \mathbf{u}_N^I \\ \mathbf{u}^\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^I \\ \mathbf{b}_2^I \\ \vdots \\ \mathbf{b}_N^I \\ \mathbf{b}^\Gamma \end{pmatrix}, \quad (2)$$

where $(\mathbf{A}_k^{I,I})_{i,j} := a(\varphi_{k,i}^I, \varphi_{k,j}^I)$, $(\mathbf{A}_k^{I,\Gamma})_{i,j} = (\mathbf{A}_k^{\Gamma,I})_{j,i} := a(\varphi_{k,i}^I, \varphi_{k,j}^\Gamma)$, $(\mathbf{b}_k^I)_i := b(\varphi_{k,i}^I)$, and $(\mathbf{b}^\Gamma)_i := b(\varphi_i^\Gamma)$. Using a particular-solution approach, (2) can be solved in three steps:

1. Solve N independent systems $\mathbf{A}_i^{I,I} \mathbf{v}_i^I = \mathbf{b}_i^I$, being FE-counterparts of

$$\begin{aligned} -\rho_i \Delta v_i^I(x) &= f(x), & x \in \Omega_i, \\ v_i^I(x) &= 0, & x \in \partial\Omega_i. \end{aligned}$$

on subspaces $V_i := V_i^h := \langle \varphi_{i,1}^I, \dots, \varphi_{i,n_i^I}^I \rangle$.

2. Solve $\mathbf{S} \mathbf{u}^\Gamma = \mathbf{b}^\Gamma - \sum_{i=1}^N \mathbf{A}_i^{\Gamma, \mathbf{I}} \mathbf{v}_i^\mathbf{I}$, where

$$\mathbf{S} := \mathbf{A}^{\Gamma, \Gamma} - \sum_{i=1}^N \mathbf{A}_i^{\Gamma, \mathbf{I}} \left(\mathbf{A}_i^{\mathbf{I}, \mathbf{I}} \right)^{-1} \mathbf{A}_i^{\mathbf{I}, \Gamma}. \quad (3)$$

3. Solve N concurrent systems $\mathbf{A}_i^{\mathbf{I}, \mathbf{I}} \mathbf{w}_i^\mathbf{I} = -\mathbf{A}_i^{\mathbf{I}, \Gamma} \mathbf{u}^\Gamma$, being FE-counterparts of

$$\begin{aligned} -\rho_i \Delta w_i^\mathbf{I}(x) &= 0, & x \in \Omega_i, \\ w_i^\mathbf{I}(x) &= u^\Gamma(x), & x \in \partial\Omega_i \cap \Gamma, \\ w_i^\mathbf{I}(x) &= 0, & x \in \partial\Omega_i \cap \partial\Omega, \end{aligned}$$

and set $\mathbf{u}_i^\mathbf{I} := \mathbf{v}_i^\mathbf{I} + \mathbf{w}_i^\mathbf{I}$.

The method can be also viewed in terms of the block LDL^T -factorization

$$\mathbf{A} = \begin{pmatrix} \mathbf{I}^\mathbf{I} & \mathbf{0} \\ \mathbf{A}^{\Gamma, \mathbf{I}} & (\mathbf{A}^{\mathbf{I}, \mathbf{I}})^{-1} \mathbf{A}^{\mathbf{I}, \Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{\mathbf{I}, \mathbf{I}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{I}^\mathbf{I} & (\mathbf{A}^{\mathbf{I}, \mathbf{I}})^{-1} \mathbf{A}^{\mathbf{I}, \Gamma} \\ \mathbf{0} & \mathbf{I}^\Gamma \end{pmatrix},$$

where $\mathbf{I}^\mathbf{I}$, \mathbf{I}^Γ denote the identity matrices, $\mathbf{A}^{\mathbf{I}, \mathbf{I}}$ and $\mathbf{A}^{\mathbf{I}, \Gamma} = (\mathbf{A}^{\Gamma, \mathbf{I}})^T$ is the upper-block-diagonal and off-diagonal part of \mathbf{A} , respectively.

The idea of primal DD-preconditioners is to replace the Schur complement \mathbf{S} in Step 2 by an approximation $\widehat{\mathbf{S}}$, which is cheap to invert, the condition number $\kappa(\widehat{\mathbf{S}}^{-1} \mathbf{S})$ increases modestly with H/h and it is independent of $(\rho_i)_{i=1}^N$.

The primal DD-methods can be thought as a block Gauss elimination combined with preconditioned Krylov space methods. The idea of re-ordering of the nodes date back to the nested-dissection sparse direct solver developed by George [Geo73]. The base for the analysis of DD-preconditioners was given in a famous series of papers by Bramble, Pasciak, and Schatz, cf. [BPS86]. Analysis in the Schwarz framework was presented by Dryja, Smith, and Widlund [DSW94]. Let us mention at least two other important DD-methods such as balancing DD proposed and analyzed by Mandel and Brezina [MB96], or finite element tearing and interconnecting proposed by Farhat and Roux [FR91] and analyzed by Mandel and Tezaur [MT96]. We refer to the monograph by Toselli and Widlund [TW05] for a more comprehensive overview.

The aim of this paper is to present a complete theory for the vertex-based DD-preconditioner in 2 dimensions by means of simple calculus. Although

many other DD-preconditioners rely on this theory, to our best knowledge it has never been presented in a single paper or a monograph without external references. Neither we have found a complete proof of the 2-dimensional counterpart of the edge lemma, a brief sketch of which is given in [DW89]. Moreover, we found and corrected an inaccuracy in the proof [BPS86] of a frequently-used discrete Sobolev inequality. We hope that our effort will be of some help to researchers, at a position similar to ours, who need to get a deeper understanding of the theory in order to develop their novel DD-methods.

The rest of the paper is organized as follows: In Section 2 we give the construction of the preconditioner. In Section 3 we present the analysis of the condition number of the DD-preconditioned algebraic system.

2 Vertex-based preconditioner

In Section 1 we re-ordered the basis functions $(\varphi_i)_{i=1}^n$ into N sets of interior functions and a set of skeleton functions, which arrived at (2). Similarly we shall now re-order the set of skeleton basis functions $(\varphi_i^\Gamma)_{i=1}^{n^\Gamma}$ into M , being the number of edges, sets of functions $(\varphi_{i,j}^E)_{j=1}^{n_i^E}$, $i = 1, \dots, M$, related to nodes $x_{i,j}^E \in E_i$, see Fig. 1, and into a set of functions $(\varphi_i^V)_{i=1}^{n^V}$ related to subdomain vertices $x_i^V \in \Gamma$. This re-ordering induces a permutation of the Schur complement (3), still denoted by \mathbf{S} ,

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}^{E,E} & \mathbf{S}^{E,V} \\ \mathbf{S}^{V,E} & \mathbf{S}^{V,V} \end{pmatrix}, \quad (4)$$

where the E-blocks of rows or columns are associated to the edge functions $\varphi_{i,j}^E$ and the V-blocks are associated to the vertex functions φ_i^V . The matrix $\mathbf{S}^{E,E}$ admits the following block structure

$$\mathbf{S}^{E,E} = \begin{pmatrix} \mathbf{S}_{1,1}^{E,E} & \cdots & \mathbf{S}_{1,M}^{E,E} \\ \vdots & \ddots & \vdots \\ \mathbf{S}_{M,1}^{E,E} & \cdots & \mathbf{S}_{M,M}^{E,E} \end{pmatrix}, \quad (5)$$

where $(\mathbf{S}_{i,j}^{E,E})_{k,l}$ is related to interaction of the basis functions $\varphi_{i,k}^E$ and $\varphi_{j,l}^E$. From (3) we can see that the block structure is sparse, since $\mathbf{S}_{i,j}^{E,E}$ is zero if there is no subdomain adjacent to both E_i and E_j .

Denote the overall number of interior edge nodes by $n^E := \sum_{i=1}^M n_i^E$. We introduce a matrix

$$\mathbf{R}^E = (\mathbf{R}_1^E, \dots, \mathbf{R}_M^E) \in \mathbb{R}^{n^V \times n^E}, \quad \mathbf{R}_i^E \in \mathbb{R}^{n^V \times n_i^E},$$

the transpose of which linearly interpolates function values from the coarse vertices x_k^V into interior nodes $x_{i,j}^E$ of an associated edge E_i . That means the entries of \mathbf{R}^E are given by values of the coarse-space basis functions

$$(\mathbf{R}_i^E)_{k,j} = \varphi_k^H(x_{i,j}^E), \quad (6)$$

where $(\varphi_i^H)_{i=1}^{n^V}$ are linear Lagrange FE-functions defined at vertices x_i^V of the domain decomposition. We change the base $(\varphi_i^V)_{i=1}^{n^V}$ to $(\varphi_i^H)_{i=1}^{n^V}$ so that

$$\mathbf{S} = \begin{pmatrix} \mathbf{I}^E & \mathbf{0} \\ -\mathbf{R}^E & \mathbf{I}^V \end{pmatrix} \begin{pmatrix} \mathbf{S}^{E,E} & \tilde{\mathbf{S}}^{E,V} \\ \tilde{\mathbf{S}}^{V,E} & \tilde{\mathbf{S}}^{V,V} \end{pmatrix} \begin{pmatrix} \mathbf{I}^E & -(\mathbf{R}^E)^T \\ \mathbf{0} & \mathbf{I}^V \end{pmatrix}, \quad (7)$$

where $\mathbf{I}^E, \mathbf{I}^V$ are identity matrices. Now the block $\mathbf{A}^H := \tilde{\mathbf{S}}^{V,V}$ is the FE-discretization of the bilinear form $a(u, v)$ in the coarse base.

The primal, so-called vertex-based DD-preconditioner is constructed by neglecting $\tilde{\mathbf{S}}^{E,V}, \tilde{\mathbf{S}}^{V,E}$, and by skipping the off-diagonal blocks in (5), i.e.

$$\hat{\mathbf{S}} = \begin{pmatrix} \mathbf{I}^E & \mathbf{0} \\ -\mathbf{R}^E & \mathbf{I}^V \end{pmatrix} \begin{pmatrix} \bar{\mathbf{S}}^{E,E} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^H \end{pmatrix} \begin{pmatrix} \mathbf{I}^E & -(\mathbf{R}^E)^T \\ \mathbf{0} & \mathbf{I}^V \end{pmatrix},$$

where $\bar{\mathbf{S}}^{E,E} := \text{diag}(\mathbf{S}_{1,1}^{E,E}, \dots, \mathbf{S}_{M,M}^{E,E})$.

In each iteration of, e.g., the preconditioned conjugate gradient method an action of $\hat{\mathbf{S}}^{-1}$ is required. We have the formula

$$\begin{aligned} \hat{\mathbf{S}}^{-1} &= \begin{pmatrix} \mathbf{I}^E & (\mathbf{R}^E)^T \\ \mathbf{0} & \mathbf{I}^V \end{pmatrix} \begin{pmatrix} (\bar{\mathbf{S}}^{E,E})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A}^H)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}^E & \mathbf{0} \\ \mathbf{R}^E & \mathbf{I}^V \end{pmatrix} = \\ &= \sum_{i=1}^M \begin{pmatrix} \mathbf{I}_i^E \\ \mathbf{0} \end{pmatrix} (\mathbf{S}_{i,i}^{E,E})^{-1} (\mathbf{I}_i^E, \mathbf{0}) + \begin{pmatrix} (\mathbf{R}^E)^T \\ \mathbf{I}^V \end{pmatrix} (\mathbf{A}^H)^{-1} \underbrace{(\mathbf{R}^E, \mathbf{I}^V)}_{=: \mathbf{R}^H}. \quad (8) \end{aligned}$$

This results in a modification of Step 2 of the three-steps method.

2a. Set

$$\mathbf{c}^\Gamma := \begin{pmatrix} \mathbf{c}^E \\ \mathbf{c}^V \end{pmatrix} := \mathbf{b}^\Gamma - \mathbf{A}^{\Gamma,I} \mathbf{v}^I.$$

2b. Solve M independent local systems $\mathbf{S}_{i,i}^{E,E} \mathbf{w}_i^E = \mathbf{c}_i^E$.

2c. Solve the global coarse system $\mathbf{A}^H \mathbf{w}^H = \mathbf{c}^V + \mathbf{R}^E \mathbf{c}^E$.

2d. Set

$$\widehat{\mathbf{u}}^\Gamma := \begin{pmatrix} \mathbf{w}^E + (\mathbf{R}^E)^T \mathbf{w}^H \\ \mathbf{w}^H \end{pmatrix}$$

The action of $\widehat{\mathbf{S}}^{-1}$ comprises solution to a global system with the coarse matrix \mathbf{A}^H and solution to M local edge problems with matrices $\mathbf{S}_{i,i}^{E,E}$, which are local Schur complements related to the following systems:

$$\begin{pmatrix} \mathbf{A}_j^{I,I} & \mathbf{0} & \mathbf{A}_{j,i}^{I,E} \\ \mathbf{0} & \mathbf{A}_k^{I,I} & \mathbf{A}_{k,i}^{I,E} \\ \mathbf{A}_{i,j}^{E,I} & \mathbf{A}_{i,k}^{E,I} & \mathbf{A}_i^{E,E} \end{pmatrix} \begin{pmatrix} \mathbf{w}_j^I \\ \mathbf{w}_k^I \\ \mathbf{w}_i^E \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{c}_i^E \end{pmatrix},$$

where the relation of i , j , and k is such that the domains Ω_j and Ω_k are connected via the edge E_i . We solve the system for \mathbf{w}_i^E . It is an FE-discretization on the space $V_j + V_k + V_i^E$, where $V_i^E := \langle \varphi_{i,l}^E \rangle_{l=1}^{n_i^E}$, of the following problem solved over the patch $\Omega_j \cup \Omega_k$:

$$w_i^E(x) := \begin{cases} -\rho_j \Delta w_j^I(x) = 0, & x \in \Omega_j, \\ w_j^I(x) = 0, & x \in \partial\Omega_j \setminus E_i, \\ -\rho_k \Delta w_k^I(x) = 0, & x \in \Omega_k, \\ w_k^I(x) = 0, & x \in \partial\Omega_k \setminus E_i, \\ w_j^I(x) = w_k^I(x), & x \in E_i, \\ \rho_j \frac{dw_j^I}{dn_j}(x) + \rho_k \frac{dw_k^I}{dn_k}(x) = c_i^E(x), & x \in E_i. \end{cases}$$

The resulting preconditioner admits the following factorization

$$\widehat{\mathbf{A}} = \begin{pmatrix} \mathbf{I}^I & \mathbf{0} \\ \mathbf{A}^{\Gamma,I} & (\mathbf{A}^{I,I})^{-1} \mathbf{I}^\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{A}^{I,I} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{S}} \end{pmatrix} \begin{pmatrix} \mathbf{I}^I & (\mathbf{A}^{I,I})^{-1} \mathbf{A}^{I,\Gamma} \\ \mathbf{0} & \mathbf{I}^\Gamma \end{pmatrix}. \quad (9)$$

3 Analysis of the condition number

We shall analyze the condition number $\kappa(\widehat{\mathbf{A}}^{-1}\mathbf{A})$. It is equivalent to finding spectral bounds $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ such that

$$\forall u \in V : \quad \lambda_{\min} \widehat{a}(u, u) \leq a(u, u) \leq \lambda_{\max} \widehat{a}(u, u),$$

where $\widehat{a}(u, u)$ is the quadratic form related to $\widehat{\mathbf{A}}$. It will turn out that under shape-regularity and quasi-uniformity of both the coarse and fine discretizations the condition number estimate $\kappa \leq \lambda_{\max}/\lambda_{\min}$ is proportional to $(1 + \ln(H/h))^2$. The bounds are independent of $(\rho_i)_{i=1}^N$. Moreover, all the generic constants that appear in the theory below are independent of H , h , and $(\rho_i)_{i=1}^N$.

3.1 Orthogonal space splitting

Let us re-visit the algebraic construction of $\widehat{\mathbf{A}}$. First we re-sorted the basis functions according to the interior and skeleton nodes. This leads to

$$V = (V_1 \oplus_a \cdots \oplus_a V_N) + V^\Gamma,$$

where $V^\Gamma := \langle \varphi_1^\Gamma, \dots, \varphi_{n_\Gamma}^\Gamma \rangle$ and where the a -orthogonality of V_i and V_j , for $i \neq j$, follows from $\Omega_i \cap \Omega_j = \emptyset$.

Now we take into account the transformation of base determined by the right factor of (9). It transforms the basis functions φ_i^Γ to their discrete harmonic extensions $\widetilde{\varphi}_i^\Gamma := \mathcal{H}(\varphi_i^\Gamma)$. Recall that the discrete harmonic extension \widetilde{u}^Γ of $u^\Gamma \in V^\Gamma$ is the solution to the following problem:

$$\text{find } \widetilde{u}^\Gamma \in V : \quad \widetilde{u}^\Gamma(x) = u^\Gamma(x) \text{ on } \Gamma \quad \text{and} \quad \forall j \forall v \in V_j : a(\widetilde{u}^\Gamma, v) = 0.$$

Note that $\widetilde{u}^\Gamma|_{\Omega_j}$, $j = 1, \dots, N$, is an FE-counterpart of

$$\begin{aligned} -\Delta \widetilde{u}^\Gamma(x) &= 0, & x \in \Omega_j, \\ \widetilde{u}^\Gamma(x) &= u^\Gamma(x), & x \in \Gamma \cap \partial\Omega_j, \\ \widetilde{u}^\Gamma(x) &= 0, & x \in \partial\Omega \cap \partial\Omega_j. \end{aligned}$$

Denoting by $\widetilde{V}^\Gamma := \mathcal{H}(V^\Gamma)$ we arrive at the a -orthogonal decomposition

$$V = V_1 \oplus_a \cdots \oplus_a V_N \oplus_a \widetilde{V}^\Gamma.$$

The Schur complement \mathbf{S} is the FE-discretization of the bilinear form

$$s(u^\Gamma, v^\Gamma) := a(\mathcal{H}(u^\Gamma), \mathcal{H}(v^\Gamma)), \quad u^\Gamma, v^\Gamma \in V^\Gamma,$$

in the base $(\varphi_i^\Gamma)_{i=1}^{n^\Gamma}$. The latter can be deduced from

$$\mathbf{S} = \begin{pmatrix} -\mathbf{A}^{\Gamma, \mathbb{I}} (\mathbf{A}^{\mathbb{I}, \mathbb{I}})^{-1} & \mathbf{I}^\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{A}^{\mathbb{I}, \mathbb{I}} & \mathbf{A}^{\mathbb{I}, \Gamma} \\ \mathbf{A}^{\Gamma, \mathbb{I}} & \mathbf{A}^{\Gamma, \Gamma} \end{pmatrix} \begin{pmatrix} -(\mathbf{A}^{\mathbb{I}, \mathbb{I}})^{-1} \mathbf{A}^{\mathbb{I}, \Gamma} \\ \mathbf{I}^\Gamma \end{pmatrix},$$

where the transformation factors consist of nodal coordinates of $(\tilde{\varphi}_i^\Gamma)_{i=1}^{n^\Gamma}$.

Finally, we take a closer look at the last transformation determined by the factor \mathbf{R}^H in (8). It transforms functions to the linear interpolation from its vertex values along all the skeleton edges. We denote this interpolation operator by $I^H : C_0(\Omega) \rightarrow C_0(\Omega)$, $I^H(v)(x) := \sum_{i=1}^{n^V} v(x_i^V) \varphi_i^H(x)$. In particular, $I^H(\varphi_i^V) = \varphi_i^H$, see (6). Since the latter are discrete harmonic, we end up with the following decomposition

$$V = \underbrace{V_1 \oplus_a \cdots \oplus_a V_N}_{=: V^{\mathbb{I}}} \oplus_a \underbrace{(\tilde{V}^E + V^H)}_{=: \tilde{V}^\Gamma}, \quad (10)$$

where $V^H := I^H(V)$, $\tilde{V}^E := \mathcal{H}(V - V^H) = \mathcal{H}(\sum_{i=1}^M V_i^E)$. Therefore, every $u = u^{\mathbb{I}} + u^E + u^V \in V$ admits the unique decomposition

$$u = \tilde{u}^{\mathbb{I}} \oplus_a (\tilde{u}^E + u^H),$$

where $u^H := I^H(u)$, $\tilde{u}^E = \mathcal{H}(u - u^H)$, and $\tilde{u}^{\mathbb{I}} := u - \tilde{u}^E - u^H$. The quadratic forms now read as follows:

$$a(u, u) = \underbrace{\sum_{i=1}^N a(\tilde{u}_i^{\mathbb{I}}, \tilde{u}_i^{\mathbb{I}})}_{=: a(\tilde{u}^{\mathbb{I}}, \tilde{u}^{\mathbb{I}})} + \underbrace{\sum_{i,j=1}^M a(\tilde{u}_i^E, \tilde{u}_j^E)}_{=: a(\tilde{u}^E, \tilde{u}^E)} + 2a(\tilde{u}^E, u^H) + a(u^H, u^H) \quad (11)$$

$$\hat{a}(u, u) = \sum_{i=1}^N a(\tilde{u}_i^{\mathbb{I}}, \tilde{u}_i^{\mathbb{I}}) + \sum_{i=1}^M a(\tilde{u}_i^E, \tilde{u}_i^E) + a(u^H, u^H) \quad (12)$$

with $\tilde{u}^{\mathbb{I}} = \sum_{i=1}^N \tilde{u}_i^{\mathbb{I}}$, $\tilde{u}_i^{\mathbb{I}} \in V_i$ and $\tilde{u}^E = \sum_{i=1}^M \tilde{u}_i^E$, $\tilde{u}_i^E \in \mathcal{H}(V_i^E)$.

3.2 Upper bound

Theorem 1. *For all $u \in V$ it holds that*

$$a(u, u) \leq 10 \widehat{a}(u, u),$$

i.e., $\lambda_{\max} := 10$.

Proof. Let us take an arbitrary $u \in V$ and its unique splitting $u = u^I \oplus_a (\widetilde{u}^E + u^H)$. For each skeleton edge E_i we define its edge-neighbourhood

$$N_i := \{j \in \{1, \dots, M\} : i \neq j \text{ and } \exists k \in \{1, \dots, N\} : E_i, E_j \subset \partial\Omega_k\}.$$

Since $|N_i| \leq 4$ and $j \in N_i \Leftrightarrow i \in N_j$, using $2a(v, w) \leq a(v, v) + a(w, w)$,

$$\begin{aligned} a(\widetilde{u}^E, \widetilde{u}^E) &= \sum_{i=1}^M \sum_{j=1}^M a(\widetilde{u}_i^E, \widetilde{u}_j^E) = \sum_{i=1}^M \left\{ a(\widetilde{u}_i^E, \widetilde{u}_i^E) + \sum_{j \in N_i} a(\widetilde{u}_i^E, \widetilde{u}_j^E) \right\} \\ &\leq \sum_{i=1}^M \left\{ a(\widetilde{u}_i^E, \widetilde{u}_i^E) + \sum_{j \in N_i} \frac{1}{2} [a(\widetilde{u}_i^E, \widetilde{u}_i^E) + a(\widetilde{u}_j^E, \widetilde{u}_j^E)] \right\} \\ &\leq \left(1 + \frac{4}{2}\right) \sum_{i=1}^M a(\widetilde{u}_i^E, \widetilde{u}_i^E) + \frac{4}{2} \sum_{j=1}^M a(\widetilde{u}_j^E, \widetilde{u}_j^E) = 5 \sum_{i=1}^M a(\widetilde{u}_i^E, \widetilde{u}_i^E). \end{aligned}$$

Using the latter estimate the mixed term is estimated as follows:

$$a(\widetilde{u}^E, u^H) \leq \frac{1}{2} [a(\widetilde{u}^E, \widetilde{u}^E) + a(u^H, u^H)] \leq \frac{5}{2} \sum_{i=1}^M a(\widetilde{u}_i^E, \widetilde{u}_i^E) + \frac{1}{2} a(u^H, u^H).$$

Combining the estimates completes the proof with $\lambda_{\max} := 10$,

$$a(u, u) \leq a(\widetilde{u}^I, \widetilde{u}^I) + 10 \sum_{i=1}^M a(\widetilde{u}_i^E, \widetilde{u}_i^E) + 2a(u^H, u^H) \leq 10 \widehat{a}(u, u).$$

□

3.3 Shape-regular quasi-uniform triangulations

Assumption 1. *Let us assume that the fine triangulation is from a family of shape-regular discretizations by which we mean that there exists $\alpha_{\min} \in$*

$(0, \pi/3)$ independent of h such that every angle in the FE-triangulation, thus also in the domain decomposition, is bounded by α_{\min} from below. Shape-regularity guarantees the angles to be bounded from above by $\alpha_{\max} := \pi - 2\alpha_{\min}$. From the law of sines we have a uniform upper bound on the ratio between the largest and shortest edge of a triangle T_i or a subdomain Ω_i , i.e.,

$$\frac{h_{\max}^i}{h_{\min}^i}, \frac{H_{\max}^i}{H_{\min}^i} \in \left\langle 1, \frac{1}{\sin \alpha_{\min}} \right\rangle. \quad (13)$$

For the sake of simplicity we assume that to each x_k^V being a corner of Ω_i there is exactly one adjacent triangle T such that $T \subset \Omega_i$.

Assumption 2. Let us further assume that both the fine and coarse triangulations are from families of quasi-uniform discretizations by which we mean that there exists a common constant $C_{A2} \in (0, 1)$ independent of h and H such that for every triangle T_i and every subdomain Ω_i the diameter h_{\max}^i and H_{\max}^i , respectively, is bounded

$$h_{\max}^i \geq C_{A2} h, \quad H_{\max}^i \geq C_{A2} H. \quad (14)$$

For the sake of simplicity we assume that $H \geq 2h$.

We will need a discrete Sobolev inequality for the FE-functions.

Lemma 1. Given a linear function v on a triangle with vertices A, B, C and an angle α at A , it holds that

$$\|\nabla v\|^2 \leq \frac{2 [(v(B) - v(A))^2 + (v(C) - v(A))^2]}{\min\{\|B - A\|^2, \|C - A\|^2\} \sin^2 \alpha}.$$

Proof. We move the origin to A and the x_1 -axis to the line segment \overline{AB} , then

$$\frac{\partial v}{\partial x_1} = \frac{v(B) - v(A)}{\|B - A\|}, \quad \cos \alpha \frac{\partial v}{\partial x_1} + \sin \alpha \frac{\partial v}{\partial x_2} = \frac{v(C) - v(A)}{\|C - A\|} =: \frac{dv}{ds}.$$

The assertion follows from the following manipulations:

$$\begin{aligned} \|\nabla v\|^2 &= \left(\frac{\partial v}{\partial x_1} \right)^2 + \frac{1}{\sin^2 \alpha} \left(\frac{dv}{ds} - \cos \alpha \frac{\partial v}{\partial x_1} \right)^2 \\ &\leq \frac{1}{\sin^2 \alpha} \left[\sin^2 \alpha \left(\frac{\partial v}{\partial x_1} \right)^2 + 2 \left(\frac{dv}{ds} \right)^2 + 2 \cos^2 \alpha \left(\frac{\partial v}{\partial x_1} \right)^2 \right] \\ &\leq \frac{2}{\sin^2 \alpha} \left[\left(\frac{\partial v}{\partial x_1} \right)^2 + \left(\frac{dv}{ds} \right)^2 \right]. \end{aligned}$$

□

Corollary 1. *Under Assumptions 1 and 2 there exists $C_{C1} > 0$ such that*

$$\forall i \in \{1, \dots, N\} \forall u \in V : \quad h \|\nabla u\|_{L^\infty(\Omega_i)} \leq C_{C1} \|u\|_{L^\infty(\Omega_i)}. \quad (15)$$

Proof. For $x \in T_j \subset \Omega_i$ with vertices A, B, C Assumption 1 and Lemma 1 yield

$$\|\nabla u(x)\| \leq \frac{\sqrt{2}}{h_{\min}^j \sin \alpha_{\min}} \sqrt{4u(A)^2 + 2u(B)^2 + 2u(C)^2} \leq \frac{4\|u\|_{L^\infty(\Omega_i)}}{h_{\min}^j \sin \alpha_{\min}}.$$

The assertion follows from (13) and (14)

$$\|\nabla u(x)\| \leq \frac{4\|u\|_{L^\infty(\Omega_i)}}{h_{\max}^j \sin^2 \alpha_{\min}} \leq \frac{1}{h} \underbrace{\frac{4}{C_{A2} \sin^2 \alpha_{\min}}}_{=: C_{C1}} \|u\|_{L^\infty(\Omega_i)}.$$

□

3.4 Stability of the coarse space

The next lemma is crucial for the stability of the coarse space in the energy norm. We are inspired by the proof of Bramble, Pasciak, and Schatz in [BPS86, L.3.3].

Lemma 2. *Under Assumptions 1 and 2 there exists $C_{L2} > 0$ such that for all $i \in \{1, \dots, N\}$*

$$\forall u \in V : \quad \|u\|_{L^\infty(\Omega_i)}^2 \leq C_{L2} \left(1 + \ln \frac{H}{h} \right) \left(\frac{1}{H^2} \|u\|_{L^2(\Omega_i)}^2 + |u|_{H^1(\Omega_i)}^2 \right).$$

Proof. Without a loss of generality, assume that $\|u\|_{L^\infty(\Omega_i)} = |u(0)|$. We shall find an open cone $\Lambda_{0, KH, \gamma} \subset \Omega_i$ with the vertex at the origin 0, the radius KH and the angle $\gamma := \alpha_{\min}$ with K independent of H . For the construction of $\Lambda_{0, KH, \gamma}$ we refer to Fig. 2 and the following description. Denote by $d_a, d_b,$ and d_c the distances of the origin to the prolongations of the sides of Ω_i with lengths $a, b,$ and $c,$ respectively, and assume that d_a is the largest distance. We choose $\tilde{K}H := d_a$. We take the open cone $\Lambda_{A, \tilde{K}H, \alpha} \subset \Omega_i$ at the

Choose, independently of h and H , $\delta := \min\{(\sqrt{2} - 1)/(\sqrt{2}C_{C1}), K\}$, where C_{C1} is the constant in (15). We shall consider two cases. First, if $\delta h < \rho$, the Cauchy-Schwarz and triangle inequalities yield

$$\begin{aligned} \gamma |u(0)| \leq & \sqrt{\gamma} \sqrt{\int_0^\gamma u^2(y(\rho, \vartheta)) d\vartheta} + \left| \int_0^\gamma \int_0^{\delta h} u'_t(t, \vartheta) dt d\vartheta \right| \\ & + \left| \int_0^\gamma \int_{\delta h}^\rho u'_t(t, \vartheta) dt d\vartheta \right|. \end{aligned} \quad (17)$$

The second and third term in (17) can be estimated as follows:

$$\begin{aligned} \left| \int_0^\gamma \int_0^{\delta h} u'_t(t, \vartheta) dt d\vartheta \right| & \leq \gamma \delta h \|\nabla u\|_{L^\infty(\Omega_i)} \leq \gamma \delta C_{C1} |u(0)|, \\ \left| \int_0^\gamma \int_{\delta h}^\rho u'_t(t, \vartheta) dt d\vartheta \right| & = \left| \int_{\Lambda_{0,\rho,\gamma} \setminus \Lambda_{0,\delta h,\gamma}} \nabla u(y) \frac{y}{\|y\|^2} dy \right| \leq \|\nabla u\|_{L^2(\Omega_i)} \sqrt{\gamma} \sqrt{\ln \frac{\rho}{\delta h}}. \end{aligned}$$

Using the estimates, moving the second term from the right-hand side of (17) to the left, squaring the inequality and dividing by γ^2 yield

$$\frac{1}{2} |u(0)|^2 \leq (1 - \delta C_{C1})^2 |u(0)|^2 \leq \frac{2}{\gamma} \left\{ \int_0^\gamma u^2(y(\rho, \vartheta)) d\vartheta + \ln \frac{\rho}{\delta h} \|\nabla u\|_{L^2(\Omega_i)}^2 \right\}. \quad (18)$$

In the second case, $\delta h \geq \rho$, we estimate the first term on the right-hand side of (16) by the Cauchy Schwarz inequality and the second term by $\gamma \delta C_{C1} |u(0)|$, which leads to

$$\frac{1}{2} |u(0)|^2 \leq (1 - \delta C_{C1})^2 |u(0)|^2 \leq \frac{2}{\gamma} \int_0^\gamma u^2(y(\rho, \vartheta)) d\vartheta. \quad (19)$$

Multiplying (18) and (19) by 2ρ , integrating ρ from δh to KH and from 0 to δh , respectively, and summing up the resulting inequalities yield

$$\begin{aligned} \frac{(KH)^2}{2} |u(0)|^2 \leq & \frac{4}{\gamma} \left\{ \int_0^{KH} \int_0^\gamma \rho u^2(y(\rho, \vartheta)) d\vartheta d\rho \right. \\ & \left. + \|\nabla u\|_{L^2(\Omega_i)}^2 \int_{\delta h}^{KH} \rho \ln \frac{\rho}{\delta h} d\rho \right\}. \end{aligned}$$

By estimating the second term on the right-hand side we complete the proof

$$\begin{aligned} |u(0)|^2 &\leq \frac{4}{\gamma} \left\{ \frac{2}{(KH)^2} \|u\|_{L^2(\Omega_i)}^2 + \left(\ln \frac{K}{\delta} + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_i)}^2 \right\} \\ &\leq \frac{4}{\gamma} \underbrace{\max \left\{ 1, \frac{2}{K^2}, \ln \frac{K}{\delta} \right\}}_{=: C_{L2}} \left(1 + \ln \frac{H}{h} \right) \left\{ \frac{1}{H^2} \|u\|_{L^2(\Omega_i)}^2 + |u|_{H^1(\Omega_i)}^2 \right\}. \end{aligned}$$

□

Corollary 2. *Under Assumptions 1 and 2 there exists $C_{C2} > 0$ such that for all $i \in \{1, \dots, N\}$*

$$\forall u \in V : \|u - \bar{u}_i\|_{L^\infty(\Omega_i)}^2 \leq C_{C2} \left(1 + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_i)}^2,$$

where $\bar{u}_i := \frac{1}{|\Omega_i|} \int_{\Omega_i} u(x) dx$ with $|\Omega_i|$ being the area of Ω_i .

Proof. Combining the previous lemma and the Poincaré inequality [PW60]

$$\|u - \bar{u}_i\|_{L^2(\Omega_i)}^2 \leq C_P H^2 |u|_{H^1(\Omega_i)}^2,$$

where $C_P := 1/\pi^2$, the assertion follows with $C_{C2} := C_{L2}(1 + C_P)$. □

The next lemma gives stability of the coarse space. It can be found in [TW05, L.4.12].

Lemma 3. *Under Assumptions 1 and 2 there exists $C_{L3} > 0$ such that for all $i \in \{1, \dots, N\}$*

$$\forall u \in V : |I^H(u)|_{H^1(\Omega_i)}^2 \leq C_{L3} \left(1 + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_i)}^2,$$

as a consequence of which

$$\forall u \in V : a(u^H, u^H) \leq C_{L3} \left(1 + \ln \frac{H}{h} \right) a(u, u).$$

Proof. Denote by P_1, P_2 , and P_3 the vertices of a subdomain Ω_i . We have

$$|I^H(u)|_{H^1(\Omega_i)}^2 = |I^H(u) - \bar{u}_i|_{H^1(\Omega_i)}^2 = \left| \sum_{j=1}^3 (u(P_j) - \bar{u}_i) \varphi_j^H \right|_{H^1(\Omega_i)}^2. \quad (20)$$

For $j \in \{1, 2, 3\}$ and the remaining indices k and l we employ Lemma 1 with $A := P_k$, $\alpha := \alpha_k$ the angle at P_k , $B := P_j$, and $C := P_l$ and using (13) yield

$$\begin{aligned} |\varphi_j^H|_{H^1(\Omega_i)}^2 &= \|\nabla \varphi_j^H\|^2 |\Omega_i| \leq \frac{2 \cdot \frac{1}{2} \|P_j - P_k\| \|P_l - P_k\| \sin \alpha_k}{\min\{\|P_j - P_k\|^2, \|P_l - P_k\|^2\} \sin^2 \alpha_k} \\ &\leq \frac{H_{\max}^i}{H_{\min}^i \sin \alpha_k} \leq \frac{1}{\sin^2 \alpha_{\min}} =: \tilde{c}, \end{aligned}$$

where $|\Omega_i|$ denotes the area of Ω_i . By (20) and Corollary 2 we have

$$|I^H(u)|_{H^1(\Omega_i)}^2 \leq 3 \sum_{j=1}^3 |u(P_j) - \bar{u}_i|^2 |\varphi_j^H|_{H^1(\Omega_i)}^2 \leq 9 \tilde{c} C_{C2} \left(1 + \ln \frac{H}{h}\right) |u|_{H^1(\Omega_i)}^2,$$

which completes the proof with $C_{L3} := 9 \tilde{c} C_{C2}$. \square

3.5 Stability of the edge space

To find λ_{\min} it remains to estimate the edge-term in (12) by (11) from above. Being inspired by [TW05, L.4.23] we introduce a system of edge-based functions $(\theta_i(x))_{i=1}^M \subset C(\tilde{\Omega})$, where $\tilde{\Omega} := \bar{\Omega} \setminus \{x_j^V : j = 1, \dots, n^V\}$. For the construction we refer to Fig. 3 and the following paragraph.

We decompose each subdomain Ω_j , $j \in \{1, 2, \dots, N\}$, with all three edges being a part of the skeleton, i.e., $E_{j_1}, E_{j_2}, E_{j_3} \subset \Gamma$, into six triangles ω_k . Without loss of generality we take $x \in \omega_1 \setminus \{P_1\}$ and introduce local coordinates $x = (x_1, x_2)$. We denote the angle at P_1 by α_1 and define the related edge functions θ_{j_1} , θ_{j_2} , and θ_{j_3} in ω_1 by

$$\theta_{j_1}(x) := 1 - \frac{2}{3 \tan \frac{\alpha_1}{2}} \frac{x_1}{x_2}, \quad \theta_{j_2}(x) = \theta_{j_3}(x) := \frac{1}{3 \tan \frac{\alpha_1}{2}} \frac{x_1}{x_2}. \quad (21)$$

The edge functions are analogously defined in $\omega_2, \dots, \omega_6$. For a subdomain Ω_j with only one or two edges assigned to the skeleton the construction of the related edge functions is similar. Note that the system completed by edge-functions assigned to $\partial\Omega$ forms a partition of unity on $\tilde{\Omega}$.

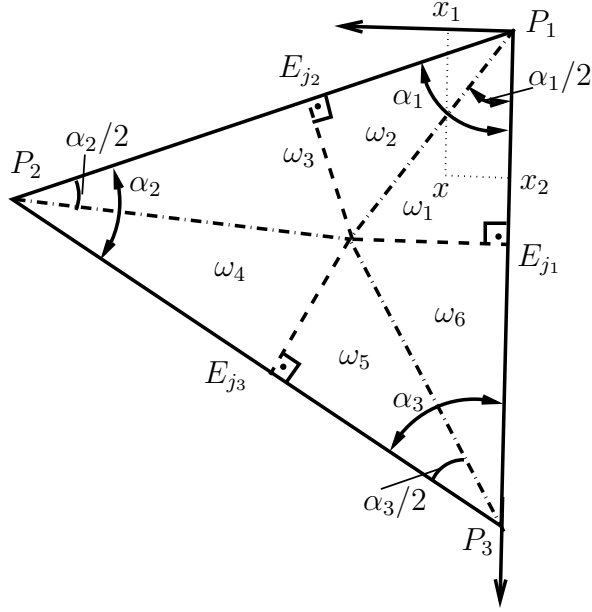


Figure 3: Decomposition of Ω_j used for the construction of θ_{j_1} , θ_{j_2} , and θ_{j_3} .

Lemma 4. *Under Assumption 1 there exists $C_{L4} > 0$ such that for all $i \in \{1, \dots, M\}$*

$$\|\nabla\theta_i(x)\| \leq C_{L4}/r^H(x) \text{ almost everywhere in } \tilde{\Omega},$$

where, for $x \in \Omega_j$ with the vertices P_1 , P_2 , and P_3 , $r^H(x) := \min_{k=1,2,3} \|x - P_k\|$.

Proof. The assertion follows from the construction above. For $x \in \omega_1$:

$$\|\nabla\theta_{j_1}(x)\| = \frac{2}{3 \tan \frac{\alpha_1}{2}} \frac{\sqrt{(x_1)^2 + (x_2)^2}}{(x_2)^2} \leq \underbrace{\frac{2}{3 \tan \frac{\alpha_{\min}}{2} \cos^2 \frac{\alpha_{\max}}{2}}}_{=: C_{L4}} \underbrace{\frac{1}{\sqrt{(x_1)^2 + (x_2)^2}}}_{\leq 1/r^H(x)}$$

by Assumption 1. The estimate holds true for $\|\nabla\theta_{j_2}(x)\|$ and $\|\nabla\theta_{j_3}(x)\|$. The other cases, $x \in \omega_k$, are analogous. \square

Similarly to replacing the FE-projection by interpolation when estimating the FE-approximation error, we will estimate the energy of the FE-interpolation of $\theta_i(u - u^H)$, rather than the energy of \tilde{u}_i^E . We need the following, so-called edge lemma, the proof of which is sketched in [DW89].

Lemma 5. *Under Assumptions 1 and 2 there exists $C_{L5} > 0$ such that for all edges E_i , $i \in \{1, \dots, M\}$, both the adjacent domains Ω_j , $E_i \subset \partial\Omega_j$, and*

$$\forall u \in V : \quad |I^h(\theta_i w)|_{H^1(\Omega_j)}^2 \leq C_{L5} \left\{ \left(1 + \ln \frac{H}{h}\right) \|w\|_{L^\infty(\Omega_j)}^2 + |w|_{H^1(\Omega_j)}^2 \right\}, \quad (22)$$

where $w := u - u^H$ and $I^h : C_0(\Omega) \rightarrow V$ is the FE-interpolation operator, i.e., $I^h(v)(x) := \sum_{i=1}^n v(x_i) \varphi_i(x)$, where x_i is the node related to φ_i .

Proof. Let us take an edge E_i and an adjacent domain Ω_j . By Assumption 1 to each coarse vertex P_k , $k = 1, 2, 3$, of Ω_j an exactly one fine triangle T with vertices $A = P_k$, B , and C is associated. In the case that none of B and C lies on E_i , $I^h(\theta_i w)$ vanishes on T . We are left to analyze the other two triangles, for both of which we can consider $C \in E_i$. The contribution of such a triangle to $|I^h(\theta_i w)|_{H^1(\Omega_j)}^2$ is, using (13), as follows:

$$\begin{aligned} |I^h(\theta_i w)|_{H^1(T)}^2 &= \frac{w^2(C)}{\|C - A\|^2 \sin^2 \alpha} \frac{\|B - A\| \|C - A\| \sin \alpha}{2} \\ &\leq \frac{h_{\max}^T}{2 h_{\min}^T \sin \alpha_{\min}} w^2(C) \leq \underbrace{\frac{1}{2 \sin^2 \alpha_{\min}}}_{=: \tilde{k}_1} \|w\|_{L^\infty(\Omega_j)}^2. \end{aligned} \quad (23)$$

In case of a triangle $T \subset \Omega_j$ that none of its vertices A , B , and C is a vertex of Ω_j , Lemma 1 yields

$$\|\nabla I^h(\theta_i w)\|^2 \leq \frac{2 \{[(\theta_i w)(B) - (\theta_i w)(A)]^2 + [(\theta_i w)(C) - (\theta_i w)(A)]^2\}}{\min\{\|B - A\|^2, \|C - A\|^2\} \sin^2 \alpha},$$

where α denotes the angle at A . Since $\theta_i w$ is piecewise differentiable along the line segments \overline{AB} and \overline{AC} , we can adopt the Lagrange mean value theorem. The latter combined with (13), the construction (21), and Lemma 4 yields

$$\begin{aligned} \|\nabla I^h(\theta_i w)\|^2 &\leq \frac{2 \|\nabla(\theta_i w)\|_{L^\infty(T)}^2}{\sin^2 \alpha} \underbrace{\frac{\|B - A\|^2 + \|C - A\|^2}{\min\{\|B - A\|^2, \|C - A\|^2\}}}_{\leq 1 + (h_{\max}^T/h_{\min}^T)^2} \\ &\leq \underbrace{4 \left(1 + \frac{1}{\sin^2 \alpha_{\min}}\right)}_{=: \tilde{k}_2} \left\{ \|\nabla \theta_i\|_{L^\infty(T)}^2 \|w\|_{L^\infty(T)}^2 + \|\theta_i\|_{L^\infty(T)}^2 \|\nabla w\|_{L^\infty(T)}^2 \right\} \\ &\leq \tilde{k}_2 \left\{ (C_{L4}/r^{H,h}(x))^2 \|w\|_{L^\infty(T)}^2 + \|\nabla w\|_{L^\infty(T)}^2 \right\}, \end{aligned}$$

where $\|\nabla f\|_{L^\infty(T)} := \operatorname{ess\,sup}_{x \in T} \|\nabla f(x)\|$ and $r^{H,h}(x) := \operatorname{dist}(T(x), \{P_1, P_2, P_3\})$, where $T(x)$ is the open triangle containing x , which is a well-defined function up to the interfaces between fine triangles. Denote by $\tilde{\Omega}_j$ the union of such non-corner triangles. They contribute to $|I^h(\theta_i w)|_{H^1(\Omega_j)}^2$ as follows:

$$|I^h(\theta_i w)|_{H^1(\tilde{\Omega}_j)}^2 \leq \tilde{k}_2 \left\{ \|w\|_{L^\infty(\Omega_j)}^2 (C_{L4})^2 \int_{\tilde{\Omega}_j} (1/r^{H,h}(x))^2 dx + |w|_{H^1(\tilde{\Omega}_j)}^2 \right\}. \quad (24)$$

It remains to estimate the integral. We have

$$\int_{\tilde{\Omega}_j} \frac{1}{(r^{H,h}(x))^2} dx \leq \sum_{k=1}^3 \int_{\tilde{\Omega}_j} \frac{1}{\inf_{y \in T(x)} \|y - P_k\|^2} dx. \quad (25)$$

Let us introduce three systems of local polar coordinates so that, each of which has the respective origin at a coarse vertex P_k , the x_1 -axis coincides with an edge of Ω_j , and Ω_j lies in the upper half-space. We denote by v_{\min}^T the smallest height of a triangle T . The law of sines, Assumptions 1 and 2 yield

$$v_{\min}^T \geq h_{\min}^T \sin \alpha_{\min} \geq h_{\max}^T \sin^2 \alpha_{\min} \geq C_{A2} h \sin^2 \alpha_{\min}.$$

Thus, by choosing $c := C_{A2} \sin^2 \alpha_{\min}$ the domain

$$\Lambda := \{x = (x_1, x_2) = \rho(\cos \alpha, \sin \alpha) \in \mathbb{R}^2 : ch \leq \rho \leq H \text{ and } 0 \leq \alpha \leq \alpha_{\max}\}$$

covers $\tilde{\Omega}_j$ with respect to each of the coordinate systems. Let us denote the respective counterparts of Λ associated to P_1, P_2 , and P_3 by Λ_1, Λ_2 , and Λ_3 . Let us adopt the k -th local polar coordinates $x(\rho, \alpha)$ and note that

$$\inf_{y \in T(x(\rho, \alpha))} \|y - P_k\| \geq \max\{ch, \rho - h\}.$$

We have the estimate

$$\begin{aligned} \int_{\tilde{\Omega}_j} \frac{1}{\inf_{y \in T(x)} \|y - P_k\|^2} dx &\leq \int_0^{\alpha_{\max}} \int_{ch}^H \frac{\rho}{(\max\{ch, (\rho - h)\})^2} d\rho d\alpha \\ &= \alpha_{\max} \left(\frac{2c+1}{2c^2} + \ln \frac{H-h}{ch} + \frac{h}{ch} - \frac{h}{H-h} \right) \leq \alpha_{\max} \left(\underbrace{\frac{4c+1}{2c^2}}_{=: \tilde{c}} + \ln \frac{H}{ch} \right), \end{aligned}$$

where we used $H \geq 2h$ from Assumption 2. Using the latter and (25), (24) is estimated by

$$|I^h(\theta_i w)|_{H^1(\tilde{\Omega}_j)}^2 \leq \tilde{k}_2 \left\{ \|w\|_{L^\infty(\Omega_j)}^2 (C_{L4})^2 3 \alpha_{\max} \left[\tilde{c} + \ln \frac{H}{ch} \right] + |w|_{H^1(\tilde{\Omega}_j)}^2 \right\}.$$

After adding the two contributions (23), the assertion follows with

$$C_{L5} := \max \left\{ 3 \tilde{k}_2 (C_{L4})^2 \alpha_{\max} (\tilde{c} - \ln c) + 2\tilde{k}_1, 3 \tilde{k}_2 (C_{L4})^2 \alpha_{\max}, \tilde{k}_2 \right\}.$$

□

Now we can analyze the stability of the edge space. The following lemma is proven in [TW05].

Lemma 6. *Under Assumptions 1 and 2 there exists $C_{L6} > 0$ such that*

$$\forall u \in V : \quad \sum_{i=1}^M a(\tilde{u}_i^E, \tilde{u}_i^E) \leq C_{L6} \left(1 + \ln \frac{H}{h} \right)^2 a(u, u).$$

Proof. Denote by Ω_{i_1} and Ω_{i_2} domains adjacent to E_i and by $\{E_{j_i}\}_{i=1}^{M_j}$, $M_j \leq 3$, edges adjacent to Ω_j . Recall that $w := u - I^H(u)$ and $\tilde{u}_i^E := \mathcal{H}(w_i^E)$, where $w_i^E := w$ on E_i and $w_i^E := 0$ elsewhere on $\Gamma \cup \partial\Omega$. The discrete harmonicity of \tilde{u}_i^E and Lemma 5 yield

$$\begin{aligned} \sum_{i=1}^M a(\tilde{u}_i^E, \tilde{u}_i^E) &= \sum_{i=1}^M \sum_{j=1}^2 \rho_{i_j} |\tilde{u}_i^E|_{H^1(\Omega_{i_j})}^2 \leq \sum_{i=1}^M \sum_{j=1}^2 \rho_{i_j} |I^h(\theta_i w)|_{H^1(\Omega_{i_j})}^2 \\ &= \sum_{j=1}^N \rho_j \sum_{i=1}^{M_j} |I^h(\theta_{j_i} w)|_{H^1(\Omega_j)}^2 \\ &\leq \sum_{j=1}^N \rho_j 3 C_{L5} \left\{ \left(1 + \ln \frac{H}{h} \right) \|w\|_{L^\infty(\Omega_j)}^2 + |w|_{H^1(\Omega_j)}^2 \right\}. \end{aligned}$$

Now $(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)$ and Corollary 2 give

$$\begin{aligned} \|w\|_{L^\infty(\Omega_j)}^2 &= \|(u - \bar{u}_j) - (I^H(u) - \bar{u}_j)\|_{L^\infty(\Omega_j)}^2 \\ &\leq 2 \left(\|u - \bar{u}_j\|_{L^\infty(\Omega_j)}^2 + \underbrace{\|I^H(u) - \bar{u}_j\|_{L^\infty(\Omega_j)}^2}_{\leq \|u - \bar{u}_j\|_{L^\infty(\Omega_j)}^2} \right) \leq 4 C_{C2} \left(1 + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_j)}^2. \end{aligned}$$

Similarly, Lemma 3 gives

$$\begin{aligned} |w|_{H^1(\Omega_j)}^2 &= |u - I^H(u)|_{H^1(\Omega_j)}^2 \leq 2 \left(|u|_{H^1(\Omega_j)}^2 + |I^H(u)|_{H^1(\Omega_j)}^2 \right) \\ &\leq 2 (1 + C_{L3}) \left(1 + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_j)}^2. \end{aligned}$$

Combining the estimates yields $C_{L6} := 3C_{L5} [4C_{C2} + 2(1 + C_{L3})]$. \square

3.6 Lower bound

Theorem 2. *Under Assumptions 1 and 2 there exists $C > 0$ such that*

$$\forall u \in V : \quad \widehat{a}(u, u) \leq C \underbrace{\left(1 + \ln \frac{H}{h} \right)^2}_{=1/\lambda_{\min}(H,h)} a(u, u).$$

Proof. Comparing (11) and (12), the assertion is a consequence of Lemma 3 and 6 with $C := 1 + C_{L3} + C_{L6}$. \square

We conclude with an estimate of the condition number

$$\kappa \left(\widehat{\mathbf{A}}^{-1} \mathbf{A} \right) \leq 10 C \left(1 + \ln \frac{H}{h} \right)^2$$

with $C > 0$ independent of H , h , and $(\rho_i)_{i=1}^N$ in a family of shape-regular quasi-uniform triangulations.

References

- [BPS86] Bramble, J.H., Pasciak, J.E., and Schatz, A.H.: The construction of preconditioners for elliptic problems by substructuring, I. *Math. Comp.* **47**(175), 103–134 (1986)
- [DW89] Dryja, M. and Widlund, O.B.: Some domain decomposition algorithms for elliptic problems. In *Iterative Methods for Large Linear Systems*, 273–291. Academic Press (1989)

- [DSW94] Dryja, M., Smith, B.F., and Widlund, O.B.: Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions. *SIAM J. Numer. Anal.* **31**(6), 1662–1694 (1994)
- [FR91] Farhat, C. and Roux, F.-X.: A method of finite element tearing and interconnecting and its parallel solution algorithm. *Internat. J. Numer. Methods Engrg.* **32**, 1205–1227 (1991)
- [Geo73] George, A.: Nested dissection of a regular finite element mesh. *SIAM J. Numer. Anal.* **10**, 345–363 (1973)
- [MB96] Mandel, J. and Brezina, M.: Balancing domain decomposition for problems with large jumps in coefficients. *Math. Comp.* **65**, 1387–1401 (1996)
- [MT96] Mandel, J. and Tezaur, R.: Convergence of a substructuring method with Lagrange multipliers. *Numer. Math.* **73**, 473–487 (1996)
- [PW60] Payne, L.E. and Weinberger, H.F.: An optimal Poincaré inequality for convex domains. *Archive for Rational Mechanics and Analysis* **5**(1), 286—292 (1960)
- [TW05] Toselli, A. and Widlund, O.B.: *Domain Decomposition Methods — Algorithms and Theory*. Springer (2005)