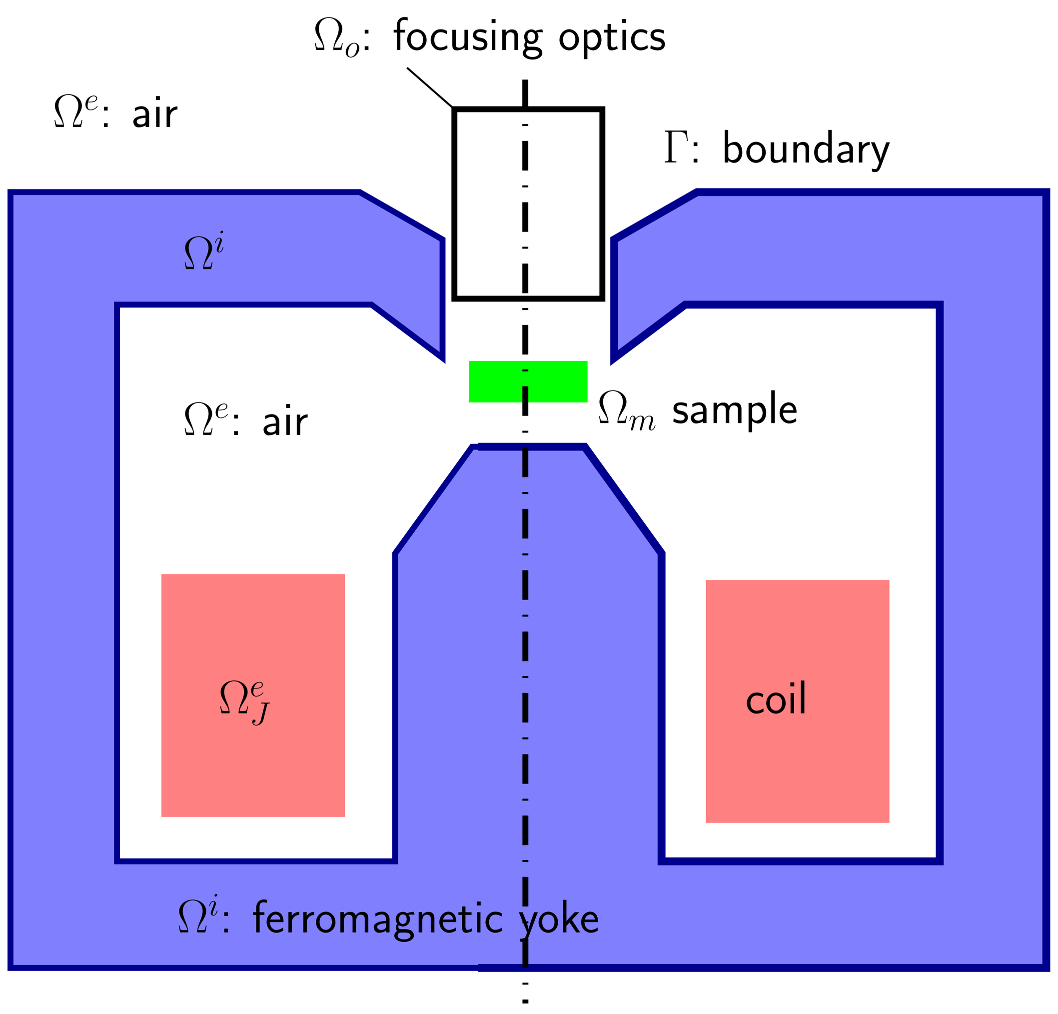
Shape Optimization for Nonlinear Axisymmetric Magnetostatics using a Coupling of FEM and BEM

D. Lukáš (Dep. of Applied Math.), K. Postava, and O. Životský (Dep. of Physics), VŠB–TU Ostrava

Microscopy

Mathematical model of magnetostatics

Axisymmetric electromagnet geometry



Maxwell's equations

$$\begin{aligned} \nabla\times\mathbf{H}^i &= 0 \text{ in } \Omega^i,\\ \nabla\times\mathbf{H}^e &= \mathbf{J} \text{ in } \Omega^e_J,\\ \nabla\times\mathbf{H}^e &= 0 \text{ in } \Omega^e\setminus\overline{\Omega^e_J},\\ \nabla\cdot\mathbf{B}^i &= 0 \text{ in } \Omega^i,\\ \nabla\cdot\mathbf{B}^e &= 0 \text{ in } \Omega^e \end{aligned}$$

Transmission conds.

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{B}^i-\mathbf{B}^e) &= 0 \text{ on } \Gamma,\\ \mathbf{n}\times(\mathbf{H}^i-\mathbf{H}^e) &= 0 \text{ on } \Gamma \end{aligned}$$

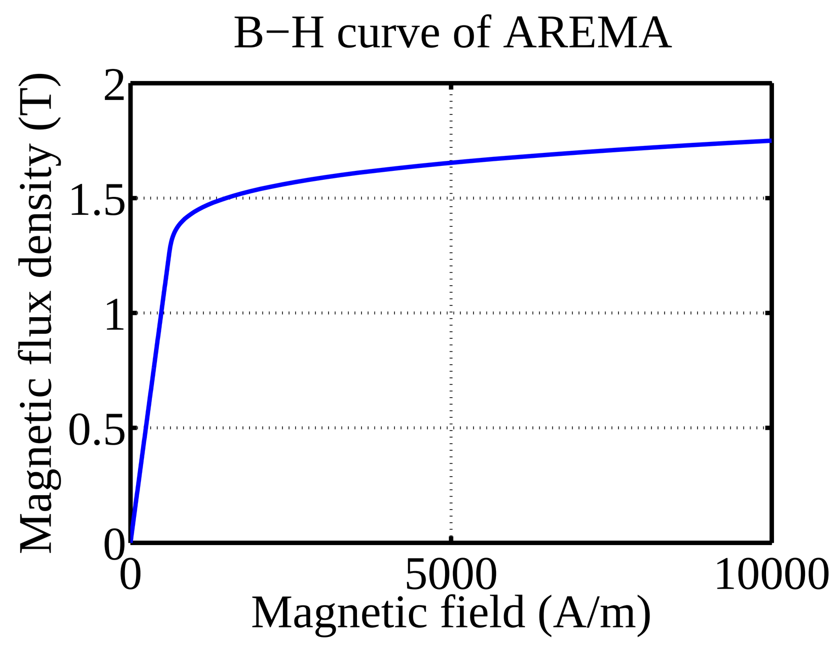
Decay at infinity

For $|\mathbf{x}|\to\infty$:

$$\begin{aligned} \mathbf{B}^e(\mathbf{x}) &= O(|\mathbf{x}|^{-2}),\\ \mathbf{H}^e(\mathbf{x}) &= O(|\mathbf{x}|^{-2}) \end{aligned}$$

Nonlinear B–H curve

$$\mathbf{B}^i = \mu(|\mathbf{B}^i|)\mathbf{H}^i \text{ in } \Omega^i,\quad \mathbf{B}^e = \mu_0\mathbf{H}^e \text{ in } \Omega^e$$



$\mathbf{B}$, $\mathbf{H}$ ... mag. flux density, intensity,
μ, μ_0 ... magnetic permeability,
$\mathbf{J}$ ... electric current density,
$\mathbf{n}$ ... outward unit normal to Ω^i

Coupling of finite and boundary elements

Finite elements method (FEM)

- Introduce the magnetic vector potential $\mathbf{A}$:

$$\mathbf{B}^i = \nabla\times\mathbf{A}^i \text{ in } \Omega^i,\quad \mathbf{B}^e = \nabla\times\mathbf{A}^e \text{ in } \Omega^e.$$

- Truncate the domain Ω^e to a bounded subdomain $\widetilde{\Omega^e}$.
- Minimize the following magnetostatic energy functional:

$$W(\mathbf{A}^i,\mathbf{A}^e) := \frac{1}{2}\int_{\Omega^i}\frac{1}{\mu(|\nabla\times\mathbf{A}^i|)}\left|\nabla\times\mathbf{A}^i\right|^2 dx + \frac{1}{2}\int_{\widetilde{\Omega^e}}\frac{1}{\mu_0}|\nabla\times\mathbf{A}^e|^2\,dx - \int_{\widetilde{\Omega^e_J}}\mathbf{J}\cdot\mathbf{A}^e\,dx$$

over the Sobolev space $H_0(\nabla\times;\Omega^i\cup\Omega^e_J\cup\widetilde{\Omega^e})$.

Pros and cons:

+ Enables to treat nonlinear B–H curve: Newton–FEM,
+ resulting algebraic systems are sparse, fast solvers,
– an additional error due to the domain truncation,
– discretization of the whole domain $\Omega^i\cup\Omega^e_J\cup\widetilde{\Omega^e}$.

Galerkin boundary elements method (BEM)

- Provided pcw. const. μ, Stratton–Chu representation holds:

$$\mathbf{B}^{i/e}(\mathbf{x}) = -\nabla\times\nabla\times\int_\Gamma(\mathbf{n}\times\mathbf{A}^{i/e})(\mathbf{y})E(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}) + {} - \nabla\times\int_\Gamma(\mathbf{n}\times\mathbf{B}^{i/e})(\mathbf{y})E(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}) + \nabla\times\int_{\Omega^e_J}\frac{\mathbf{J}(\mathbf{y})}{\mu_0}E(\mathbf{x},\mathbf{y})\,dy \quad \text{in } \Omega^{i/e},$$

where $E(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$ is the Lapl. fundamental solution.

- Transmission conditions tested over proper Sobolev spaces on Γ and jumping relations, see Hiptmair, SIAM J. Numer. Anal. 2002, then leads to a linear (Calderon projector) system solved for the unknown Cauchy data $\mathbf{n}\times\mathbf{A}$ and $\mathbf{n}\times\mathbf{B}$.

Pros and cons:

+ Discretization of the boundary Γ only, no additional error,
– cannot treat nonlinear materials,
– resulting linear system is dense, $\mathcal{H}$–matrix solvers.

Symmetric FEM–BEM coupling

- Makes use of both FEM and BEM advantages.
- Hiptmair proposes to solve for $\mathbf{A}^i$ and $\boldsymbol{\lambda}^e := \mathbf{n}\times\mathbf{B}^e$:

$$\begin{pmatrix} A_{\mathrm{FEM}}(|\nabla\times\mathbf{A}^i|) - A_{\mathrm{BEM}}, & K_{\mathrm{BEM}} \\ -K_{\mathrm{BEM}}^T, & -D_{\mathrm{BEM}}\end{pmatrix}\begin{pmatrix}\mathbf{A}^i\\ \boldsymbol{\lambda}^e\end{pmatrix} = \begin{pmatrix} b_{\mathrm{BEM}}\\ c_{\mathrm{BEM}}\end{pmatrix},$$

where $A_{FEM}(|\nabla\times\mathbf{A}^i|)$ is the FEM nonlinear operator related to the first term in $W(\mathbf{A}^i,\mathbf{A}^e)$, A_{BEM} is the FEM–BEM coupling term, K_{BEM} is the BEM double layer operator, D_{BEM} is the BEM hypersingular operator, and b_{BEM}, c_{BEM} denotes the BEM Newton terms.

- Axisymmetric setting leads to triangulation of Ω^i in the (r,z)–plane, where $\mathbf{x} := (r\cos t, r\sin t, z)$. $\mathbf{A}^i(\mathbf{x}) = A^i(r,z)(-\sin t,\cos t,0)$ is discretized by nodal FE–elements and $\boldsymbol{\lambda}^e(\mathbf{x}(p)) = \lambda^e(r(p),z(p))(-\sin t,\cos t,0)$ is discretized by segment–wise constant BE–elements, where p denotes the parameterization of the boundary Γ in the (r,z)–plane.
- Duffy transform enables assembling the BEM–matrices using a modest–order Gaussian tensor–product quadrature.

Optimal shape design

Formulation

- A multi–criterion goal: minimize inhomogeneities of the magnetic field in Ω_m and maximize its strength at the same time.
- Minimization of the inhomogeneity

$$\min_{\text{shapes }\alpha}\ \kappa^2 \quad \text{s.t. } B_z^{\mathrm{avg}} \ge B^{\mathrm{req}},$$

$$\kappa^2 := \frac{1}{|\Omega_m||B_z^{\mathrm{avg}}|^2}\int_{\Omega_m}\left[(B_r^e)^2 + (B_z^e - B_z^{\mathrm{avg}})^2\right] dx.$$

- Maximization of the magnetic strength

$$\max_{\text{shapes }\alpha}\ B_z^{\mathrm{avg}} \quad \text{s.t. } \kappa\le\kappa^{\mathrm{req}},\quad B_z^{\mathrm{avg}} := \frac{1}{|\Omega_m|}\int_{\Omega_m}B_z^e(\mathbf{x})\,dx.$$

- 16 design variables control the Bézier shape of the pole head as well as the cover. We employ shape nonpenetrating conditions by means of linear inequality constraints.
- The coil is completed by 3281 turns with the wire of diameter 0.8 mm and excited with the DC current of 1 A.

Numerical method

- We employ a steepest–descent active–set optimization method with projections onto the linear geometric as well as linearized field constraints.
- The shape derivatives are computed by a semi–analytical sensitivity analysis by differentiating the following maps:

$$\begin{tikzcd} \boldsymbol{p} \arrow[r, "\text{Bézier}", "\text{param.}"'] & \boldsymbol{\alpha} \arrow[r, "\text{elasticity}", "\text{grid deform.}"'] & \mathbf{z} \arrow[r, "\text{FEM–BEM}", "\text{Duffy–Gauss}"'] \arrow[d] & \boldsymbol{\mathcal{A}},\mathbf{b} \arrow[d, "\text{Newton}"', "\text{Gauss}"] \\ \kappa^2(\boldsymbol{p}), B_z^{\mathrm{avg}}(\boldsymbol{p}) & & \mathbf{B}^e \arrow[ll, "\text{cost func.}", "\text{constraints}"'] & \mathbf{A}^i,\boldsymbol{\lambda}^e \arrow[l, "\text{Stratton–Chu}", "\text{Gauss quad.}"'] \end{tikzcd}$$

- Discretized and solved first for $n := 262$ FEM nodes, $m := 125$ BEM segments and then for $n := 902$, $m := 250$:

problem, n	optim. iters. (stopping)	Newton iters. (typical)	evolution of κ [%]	evolution of B^{avg} [T]
$\min\kappa$, 262	54 (KKT)	5–13 (5)	2.6 → 0.5	0.153 → 0.119
$\max B^{\mathrm{avg}}$, 262	5 (step size)	5–13 (8)	0.5 → 6	0.119 → 0.231
$\min\kappa$, 902	35 (KKT)	7–14 (7)	2.7 → 0.5	0.150 → 0.116
$\max B^{\mathrm{avg}}$, 902	15 (step size)	7–19 (12)	0.5 → 6	0.116 → 0.216

Numerical results, manufactured shape

minimized homogeneity vers. maximized magnitude