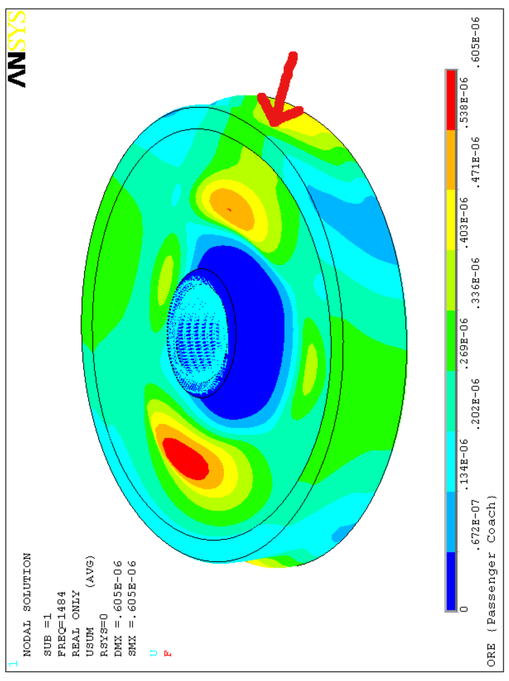
Multi–Frequency Acoustic Analysis of a Railway Wheel by a Fast BEM

D. Lukáš (Dep. of Applied Math.) and J. Szweda (Dep. of Mechanics), VŠB–TU Ostrava, Czech Rep.

Exterior Neumann problem for the Helmholtz equation

Mechanical vibrations of a rail wheel

ANSYS solution of a linear elasticity problem of a rail wheel loaded with a harmonic Dirac force at an eigen–frequency.



Exterior Helmholtz equation

Given an angular frequency $\omega := 2\pi f$ from the hearing range $f \in (10, 10^4)$ [Hz], the air density $\rho_0 := 1.2\,[\mathrm{kg\,m^{-3}}]$, the speed of sound $c := 340\,[\mathrm{m\,s^{-1}}]$, $\kappa := \omega/c$,
and the normal displacement speed $v_n(x,t) := \Re\{v_n(x)\mathrm{e}^{\imath\omega t}\}$, where $v_n(x) : \Gamma := \partial\Omega \to \mathbb{C}$, and where n denotes the unit outer normal to $\Omega \subset \mathbb{R}^3$.
Solve for the acoustical pressure $p(x,t) := \Re\{p(x)\mathrm{e}^{\imath\omega t}\}$:

$$\triangle p(x) + \kappa^2 p(x) = 0, \quad x \in \Omega^e := \mathbb{R}^3 \setminus \overline{\Omega},$$

Neumann boundary conditions

$$\frac{\partial p(x)}{\partial n} = g(x) := -\imath\omega\rho_0\, v_n(x), \quad x \in \Gamma,$$

Sommerfeld radiation condition

$$\lim_{r\to\infty} \int_{\partial B_r} \left| \frac{\partial p(x)}{\partial n} - i\kappa p(x) \right|^2 dS = 0.$$

Representation formula

$$p(x) = -(\widetilde{V}_\kappa \gamma_1^e p)(x) + (W_\kappa \gamma_0^e p)(x), \quad x \in \Omega^e,$$

with the Dirichlet and Neumann trace operators, respectively,

$$\gamma_0^e p(x) := \lim_{\Omega^e \ni \widetilde{x} \to x \in \Gamma} p(\widetilde{x}), \quad \gamma_1^e p(x) := \lim_{\Omega^e \ni \widetilde{x} \to x \in \Gamma} n(x) \cdot \nabla p(\widetilde{x}),$$

and with the single– and double–layer potentials, respectively,

$$(\widetilde{V}_\kappa w)(x) := \int_\Gamma U_\kappa(x,y) w(y)\, dS(y),$$

$$(W_\kappa v)(x) := \int_\Gamma \frac{\partial U_\kappa(x,y)}{\partial n(y)} v(y)\, dS(y),$$

where $U_\kappa(x,y) := \mathrm{e}^{i\kappa|x-y|}/(4\pi|x-y|)$.

Direct boundary integral equation

$$D_\kappa \gamma_0^e p(x) = (-(1/2)I + K'_\kappa)\, g(x), \quad x \in \Gamma,$$

$$D_\kappa v := -\gamma_1^e (W_\kappa v) \quad \text{and} \quad K'_\kappa g := \int_\Gamma \frac{\partial U_\kappa(x,y)}{\partial n(x)} g(y)\, dS(y)$$

are the hypersingular and adjoint double–layer operators.

Fast Galerkin boundary element method

Numerical quadrature of singular kernels

The idea goes back to Duffy, SIAM J. Numer. Anal. '82:
Remove singularity at the origin via $(x_1, x_2) =: (\eta_1, \eta_1\eta_2)$

$$\int_0^1 \int_0^{x_1} \frac{f(x_1,x_2)}{\sqrt{x_1^2 + x_2^2}}\, dx_2\, dx_1 = \int_0^1 \int_0^1 \frac{f(\eta_1, \eta_1\eta_2)}{\sqrt{1+\eta_2^2}}\, d\eta_2\, d\eta_1.$$

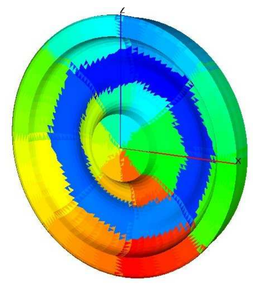
Extension to 3D Galerkin BEM integrals by Sauter, Schwab, et al., e.g. for the case of identical panels, via

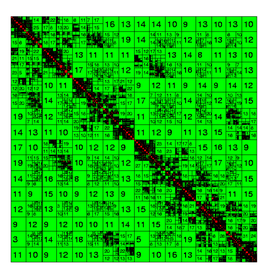
- move singularity to origin $z := x - y$,
- split the 4D integration domain into s simplices,
- employ the Duffy substitution,
- apply the Gauss tensor–product quadrature

$$\int_K \int_K k(x,y) \approx \sum_{j=1}^{s} \sum_{i_1=1}^{n_1} \cdots \sum_{i_4=1}^{n_4} w_{i_1} \ldots w_{i_4} k_j(\eta_{i_1}, \ldots, \eta_{i_4}).$$

Sparse approximation of BEM matrices

Hierarchical boundary clustering, hierarchical matrices





Admissible pairs of clusters,

$$\min\{\operatorname{diam} C_x, \operatorname{diam} C_y\} \le \eta \operatorname{dist}(C_x, C_y), \quad \eta \in (0,1)$$

Generate admissible blocks $C_x \times C_y$ of indices of the matrix entries. Store the nonadmissible blocks and replace the admissible ones by a low–rank approximation. The technique dates back to Hackbusch and Nowak, Numer. Math. '89.

Adaptive cross approximation (ACA)

Provided an asymptotically smooth kernel function

$$\exists c_1, c_2 > 0\ \exists g \le 0\ \forall \alpha \in \mathbb{N}_0^3 : \left| \partial_y^\alpha k(x,y) \right| \le c_1 p! (c_2)^p |x-y|^{g-p},$$

where $p := |\alpha|$, build succesive 1–rank cross–interpolations of admissible blocks

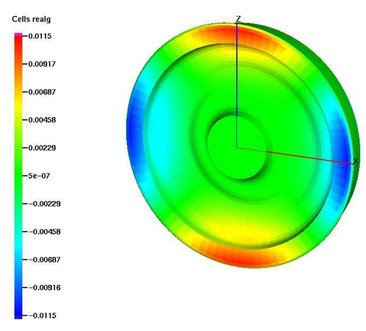
$$(\mathbf{A})_{C_x \times C_y} \approx \sum_{m=1}^{r(\varepsilon)} \mathbf{u}_m \mathbf{v}_m^T$$

such that $(\mathbf{u}_m)_i (\mathbf{v}_m)_j = (\mathbf{A})_{ij}$ for certain $(i,j) \in C_x \times C_y$.
The rank $r(\varepsilon)$ is adaptively controlled by the precision ε.
ACA has been developed by Rjasanow, Bebendorf, et al.

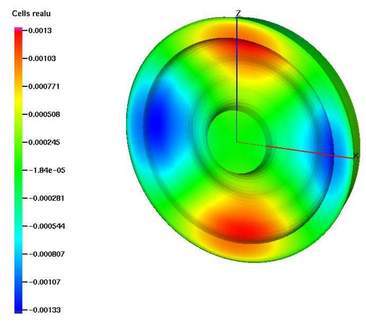
Numerical results

Low–frequency case (341 Hz)

Neumann data

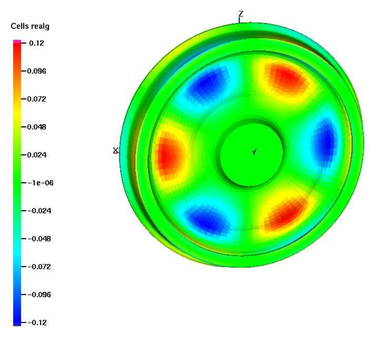


Dirichlet solution

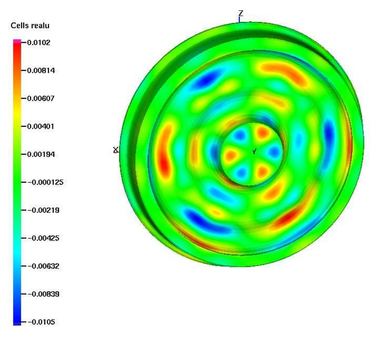


Medium–frequency case (2706 Hz)

Neumann data

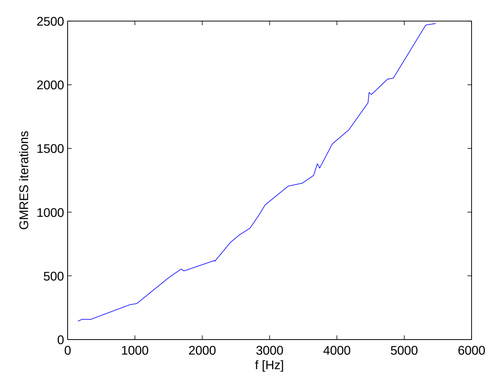


Dirichlet solution



ACA compression, GMRES iterations

15112 triangles, 22668 nodes, 16 Gauss points
Tested for 96 frequencies in the range 151 Hz – 5468 Hz
Compression of D_κ: 15–20%, compression of K'_κ: 12–13%
Assemble times of D_κ: 3.4–4.5 hours, K'_κ: 24–26 minutes
GMRES iterations: 142–2494 solved in 223 s – 2.78 hours



Outlook

- modified BEM by Engleder & Steinbach, Numer. Math. '08
- shape optimization to minimize the acoustic noise
- quadrature–ACA error analysis, Fast Multipoles
- Maxwell's equations, Schrödinger's equation