

# 2-Dimensional Primal Domain Decomposition Theory in Detail \*

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January 21, 2014

## Abstract

We give details of the theory of primal domain decomposition (DD) methods for a 2-dimensional second order elliptic equation with homogeneous Dirichlet boundary conditions and jumping coefficients. The problem is discretized by the finite element method. The computational domain is decomposed into triangular subdomains that align with the coefficients jumps. We prove that the condition number of the vertex-based DD preconditioner is  $O((1 + \log(H/h))^2)$ , independently of the coefficient jumps, where  $H$  and  $h$  denotes discretization parameters of the coarse and fine triangulations, respectively. Although this preconditioner and its analysis date back to the pioneering work by Bramble, Pasciak, and Schatz published in 1986, and it was revisited and extended by many authors including Dryja and Widlund in 1989, or Toselli and Widlund in 2005, the theory is hard to understand and some details, to our best knowledge, have never been published. In this paper we present all the proofs in detail by means of fundamental calculus.

**Keywords:** domain decomposition methods, finite element method, preconditioning

**MSC 2010:** 65N55, 65N30, 65F08

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\*This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070), by the Czech Ministry of Education under the project MSM6198910027, and by VŠB-TU Ostrava under the grant SGS SP2013/191.

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# 1 Introduction

We consider the homogeneous Dirichlet problem for the Poisson equation

$$\begin{aligned} -\operatorname{div}(\rho(x) \nabla u(x)) &= f(x), & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned}$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded polygonal domain with Lipschitz boundary,  $f \in L^2(\Omega)$ , and  $\rho \in L^\infty(\Omega)$  is a positive piecewise constant material function. The domain  $\Omega$  is decomposed into  $N$  nonoverlapping open triangular subdomains  $\Omega_i$  by means of a conforming finite element (FE) discretization  $\bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}_i$ . This is referred to as the coarse discretization or the domain decomposition (DD). The decomposition aligns with jumps of the material function so that  $\rho(x) = \rho_i > 0$  for  $x \in \Omega_i$ . We denote by  $\Gamma := \bigcup_{i=1}^M \bar{E}_i$ , the skeleton of the decomposition, where  $E_i$  is the interior of an edge apart from  $\partial\Omega$ , see Fig. 1. We denote the coarse discretization parameter by  $H := \max_{i=1, \dots, N} \operatorname{diam}(\Omega_i)$ .

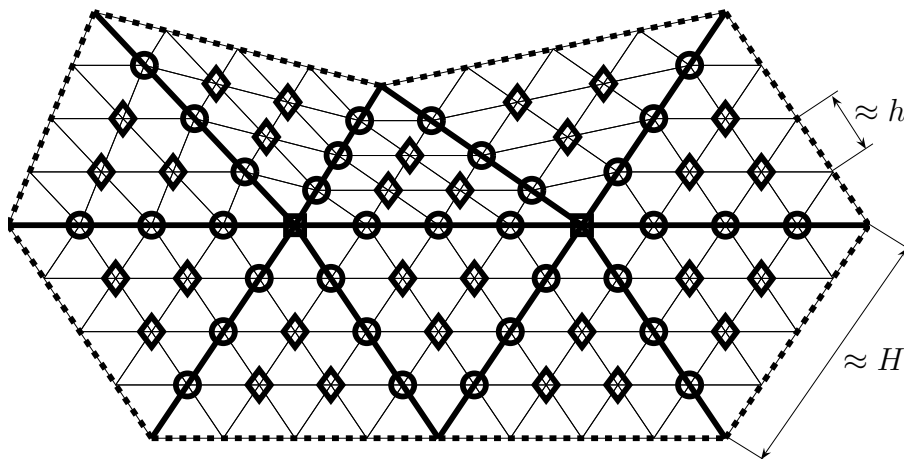


Figure 1: Decomposition of  $\Omega$  into  $N = 10$  subdomains with  $n^V = 2$  vertices  $x_i^V$  (marked by squares); dashed-line depicts  $\partial\Omega$ ; solid-bold-lines denote  $\Gamma$  decomposed into  $M = 11$  edges with edge nodes  $x_{i,j}^E$  (marked by circles); solid-thin-lines denote the fine triangulation with  $n = 65$  nodes; diamonds depict interior nodes  $x_{i,j}^I$ .

The related weak formulation

$$\text{find } u \in H_0^1(\Omega) : \underbrace{\sum_{i=1}^N \rho_i \int_{\Omega_i} \nabla u(x) \nabla v(x) dx}_{=:a(u,v)} = \underbrace{\int_{\Omega} f(x) v(x) dx}_{=:b(v)} \quad \forall v \in H_0^1(\Omega)$$

is discretized by the conforming finite element (FE) method on a subspace  $V := V^h := \langle \varphi_1(x), \dots, \varphi_n(x) \rangle \subset H_0^1(\Omega)$ , where  $(\varphi_i)_{i=1}^n$  denote the linear Lagrange basis functions related to the nodes depicted in Fig. 1. The underlying fine triangulation align with the domain decomposition. We arrive at the linear system

$$\mathbf{A} \mathbf{u} = \mathbf{b}, \quad (1)$$

where  $(\mathbf{A})_{i,j} := a(\varphi_i, \varphi_j)$ ,  $(\mathbf{b})_i := b(\varphi_i)$ , and  $u^h(x) := \sum_{j=1}^n (\mathbf{u})_j \varphi_j(x)$  approximates  $u(x)$ . By  $h$  we denote the fine discretization parameter, which is the maximal fine-triangle diameter.

Primal DD-methods rely on re-sorting the basis functions  $(\varphi_i)_{i=1}^n$  into  $N$  sets of functions  $(\varphi_{i,j}^I)_{j=1}^{n_i^I}$ ,  $i = 1, \dots, N$ , related to subdomain interior nodes  $x_{i,j}^I \in \Omega_i$ , see Fig. 1, and a set of functions  $(\varphi_k^\Gamma)_{k=1}^{n^\Gamma}$  related to skeleton nodes  $x_k^\Gamma \in \Gamma \setminus \partial\Omega$ , which either belongs to an edge  $E_i$ ,  $x_k^\Gamma = x_{i,j}^E$ , or it is a subdomain (coarse) vertex  $x_k^\Gamma = x_i^V$ , see Fig. 1. This translates (1) into the saddle-point system

$$\begin{pmatrix} \mathbf{A}_1^{I,I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_1^{I,\Gamma} \\ \mathbf{0} & \mathbf{A}_2^{I,I} & \dots & \mathbf{0} & \mathbf{A}_2^{I,\Gamma} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_N^{I,I} & \mathbf{A}_N^{I,\Gamma} \\ \mathbf{A}_1^{\Gamma,I} & \mathbf{A}_2^{\Gamma,I} & \dots & \mathbf{A}_N^{\Gamma,I} & \mathbf{A}^{\Gamma,\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^I \\ \mathbf{u}_2^I \\ \vdots \\ \mathbf{u}_N^I \\ \mathbf{u}^\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^I \\ \mathbf{b}_2^I \\ \vdots \\ \mathbf{b}_N^I \\ \mathbf{b}^\Gamma \end{pmatrix}, \quad (2)$$

where  $(\mathbf{A}_k^{I,I})_{i,j} := a(\varphi_{k,i}^I, \varphi_{k,j}^I)$ ,  $(\mathbf{A}_k^{I,\Gamma})_{i,j} = (\mathbf{A}_k^{\Gamma,I})_{j,i} := a(\varphi_{k,i}^I, \varphi_{k,j}^\Gamma)$ ,  $(\mathbf{b}_k^I)_i := b(\varphi_{k,i}^I)$ , and  $(\mathbf{b}^\Gamma)_i := b(\varphi_i^\Gamma)$ . Using a particular-solution approach, (2) can be solved in three steps:

1. Solve  $N$  independent systems  $\mathbf{A}_i^{I,I} \mathbf{v}_i^I = \mathbf{b}_i^I$ , being FE-counterparts of

$$\begin{aligned} -\rho_i \Delta v_i^I(x) &= f(x), & x \in \Omega_i, \\ v_i^I(x) &= 0, & x \in \partial\Omega_i. \end{aligned}$$

on subspaces  $V_i := V_i^h := \langle \varphi_{i,1}^I, \dots, \varphi_{i,n_i^I}^I \rangle$ .

2. Solve  $\mathbf{S} \mathbf{u}^\Gamma = \mathbf{b}^\Gamma - \sum_{i=1}^N \mathbf{A}_i^{\Gamma, \mathbf{I}} \mathbf{v}_i^{\mathbf{I}}$ , where

$$\mathbf{S} := \mathbf{A}^{\Gamma, \Gamma} - \sum_{i=1}^N \mathbf{A}_i^{\Gamma, \mathbf{I}} \left( \mathbf{A}_i^{\mathbf{I}, \mathbf{I}} \right)^{-1} \mathbf{A}_i^{\mathbf{I}, \Gamma}. \quad (3)$$

3. Solve  $N$  concurrent systems  $\mathbf{A}_i^{\mathbf{I}, \mathbf{I}} \mathbf{w}_i^{\mathbf{I}} = -\mathbf{A}_i^{\mathbf{I}, \Gamma} \mathbf{u}^\Gamma$ , being FE-counterparts of

$$\begin{aligned} -\rho_i \Delta w_i^{\mathbf{I}}(x) &= 0, & x \in \Omega_i, \\ w_i^{\mathbf{I}}(x) &= u^\Gamma(x), & x \in \partial\Omega_i \cap \Gamma, \\ w_i^{\mathbf{I}}(x) &= 0, & x \in \partial\Omega_i \cap \partial\Omega, \end{aligned}$$

and set  $\mathbf{u}_i^{\mathbf{I}} := \mathbf{v}_i^{\mathbf{I}} + \mathbf{w}_i^{\mathbf{I}}$ .

The method can be also viewed in terms of the block  $LDL^T$ -factorization

$$\mathbf{A} = \begin{pmatrix} \mathbf{I}^{\mathbf{I}} & \mathbf{0} \\ \mathbf{A}^{\Gamma, \mathbf{I}} & (\mathbf{A}^{\mathbf{I}, \mathbf{I}})^{-1} \mathbf{A}^{\mathbf{I}, \Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{\mathbf{I}, \mathbf{I}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{I}^{\mathbf{I}} & (\mathbf{A}^{\mathbf{I}, \mathbf{I}})^{-1} \mathbf{A}^{\mathbf{I}, \Gamma} \\ \mathbf{0} & \mathbf{I}^\Gamma \end{pmatrix},$$

where  $\mathbf{I}^{\mathbf{I}}$ ,  $\mathbf{I}^\Gamma$  denote the identity matrices,  $\mathbf{A}^{\mathbf{I}, \mathbf{I}}$  and  $\mathbf{A}^{\mathbf{I}, \Gamma} = (\mathbf{A}^{\Gamma, \mathbf{I}})^T$  is the upper-block-diagonal and off-diagonal part of  $\mathbf{A}$ , respectively.

The idea of primal DD-preconditioners is to replace the Schur complement  $\mathbf{S}$  in Step 2 by an approximation  $\widehat{\mathbf{S}}$ , which is cheap to invert, the condition number  $\kappa(\widehat{\mathbf{S}}^{-1} \mathbf{S})$  increases modestly with  $H/h$  and it is independent of  $(\rho_i)_{i=1}^N$ .

The primal DD-methods can be thought as a block Gauss elimination combined with preconditioned Krylov space methods. The idea of re-ordering of the nodes date back to the nested-dissection sparse direct solver developed by George [5]. The base for the analysis of DD-preconditioners was given in a famous series of papers by Bramble, Pasciak, and Schatz, cf. [1]. Analysis in the Schwarz framework was presented by Dryja, Smith, and Widlund [3]. Let us mention at least two other important DD-methods such as balancing DD proposed and analyzed by Mandel and Brezina [6], or finite element tearing and interconnecting proposed by Farhat and Roux [4] and analyzed by Mandel and Tezaur [7]. We refer to the monograph by Toselli and Widlund [9] for a more comprehensive overview.

The aim of this paper is to present a complete theory for the vertex-based DD-preconditioner in 2 dimensions by means of simple calculus. Although many other DD-preconditioners rely on this theory, to our best knowledge it

has never been presented in a single paper or a monograph without external references. Neither we have found a complete proof of the 2-dimensional counterpart of the edge lemma, a brief sketch of which is given in [2]. Moreover, we found and corrected an inaccuracy in the proof [1] of a frequently-used discrete Sobolev inequality. We hope that our effort will be of some help to researchers, at a position similar to ours, who need to get a deeper understanding of the theory in order to develop their novel DD-methods.

The rest of the paper is organized as follows: In Section 2 we give the construction of the preconditioner. In Section 3 we present the analysis of the condition number of the DD-preconditioned algebraic system.

## 2 Vertex-based preconditioner

In Section 1 we re-ordered the basis functions  $(\varphi_i)_{i=1}^n$  into  $N$  sets of interior functions and a set of skeleton functions, which arrived at (2). Similarly we shall now re-order the set of skeleton basis functions  $(\varphi_i^\Gamma)_{i=1}^{n^\Gamma}$  into  $M$ , being the number of edges, sets of functions  $(\varphi_{i,j}^E)_{j=1}^{n_i^E}$ ,  $i = 1, \dots, M$ , related to nodes  $x_{i,j}^E \in E_i$ , see Fig. 1, and into a set of functions  $(\varphi_i^V)_{i=1}^{n^V}$  related to subdomain vertices  $x_i^V \in \Gamma$ . This re-ordering induces a permutation of the Schur complement (3), still denoted by  $\mathbf{S}$ ,

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}^{\text{E,E}} & \mathbf{S}^{\text{E,V}} \\ \mathbf{S}^{\text{V,E}} & \mathbf{S}^{\text{V,V}} \end{pmatrix}, \quad (4)$$

where the E-blocks of rows or columns are associated to the edge functions  $\varphi_{i,j}^E$  and the V-blocks are associated to the vertex functions  $\varphi_i^V$ . The matrix  $\mathbf{S}^{\text{E,E}}$  admits the following block structure

$$\mathbf{S}^{\text{E,E}} = \begin{pmatrix} \mathbf{S}_{1,1}^{\text{E,E}} & \cdots & \mathbf{S}_{1,M}^{\text{E,E}} \\ \vdots & \ddots & \vdots \\ \mathbf{S}_{M,1}^{\text{E,E}} & \cdots & \mathbf{S}_{M,M}^{\text{E,E}} \end{pmatrix}, \quad (5)$$

where  $(\mathbf{S}_{i,j}^{\text{E,E}})_{k,l}$  is related to interaction of the basis functions  $\varphi_{i,k}^E$  and  $\varphi_{j,l}^E$ . From (3) we can see that the block structure is sparse, since  $\mathbf{S}_{i,j}^{\text{E,E}}$  is zero if there is no subdomain adjacent to both  $E_i$  and  $E_j$ .

Denote the overall number of interior edge nodes by  $n^E := \sum_{i=1}^M n_i^E$ . We introduce the matrix

$$\mathbf{R}^E = (\mathbf{R}_1^E, \dots, \mathbf{R}_M^E) \in \mathbb{R}^{n^V \times n^E}, \quad \mathbf{R}_i^E \in \mathbb{R}^{n^V \times n_i^E},$$

the transpose of which linearly interpolates function values from the coarse vertices  $x_k^V$  into interior nodes  $x_{i,j}^E$  of an associated edge  $E_i$ . That means the entries of  $\mathbf{R}^E$  are given by values of the coarse-space basis functions

$$(\mathbf{R}_i^E)_{k,j} = \varphi_k^H(x_{i,j}^E), \quad (6)$$

where  $(\varphi_i^H)_{i=1}^{n^V}$  are FE-functions uniquely defined by the values at vertices  $x_i^V$  of the domain decomposition. We change the base  $(\varphi_i^V)_{i=1}^{n^V}$  to  $(\varphi_i^H)_{i=1}^{n^V}$  so that

$$\mathbf{S} = \begin{pmatrix} \mathbf{I}^E & \mathbf{0} \\ -\mathbf{R}^E & \mathbf{I}^V \end{pmatrix} \begin{pmatrix} \mathbf{S}^{E,E} & \tilde{\mathbf{S}}^{E,V} \\ \tilde{\mathbf{S}}^{V,E} & \tilde{\mathbf{S}}^{V,V} \end{pmatrix} \begin{pmatrix} \mathbf{I}^E & -(\mathbf{R}^E)^T \\ \mathbf{0} & \mathbf{I}^V \end{pmatrix}, \quad (7)$$

where  $\mathbf{I}^E, \mathbf{I}^V$  are identity matrices. Now the block  $\mathbf{A}^H := \tilde{\mathbf{S}}^{V,V}$  is the FE-discretization of the bilinear form  $a(u, v)$  in the coarse base.

The primal, so-called vertex-based DD-preconditioner is constructed by neglecting  $\tilde{\mathbf{S}}^{E,V}, \tilde{\mathbf{S}}^{V,E}$ , and by skipping the off-diagonal blocks in (5), i.e.

$$\hat{\mathbf{S}} = \begin{pmatrix} \mathbf{I}^E & \mathbf{0} \\ -\mathbf{R}^E & \mathbf{I}^V \end{pmatrix} \begin{pmatrix} \bar{\mathbf{S}}^{E,E} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^H \end{pmatrix} \begin{pmatrix} \mathbf{I}^E & -(\mathbf{R}^E)^T \\ \mathbf{0} & \mathbf{I}^V \end{pmatrix},$$

where  $\bar{\mathbf{S}}^{E,E} := \text{diag}(\mathbf{S}_{1,1}^{E,E}, \dots, \mathbf{S}_{M,M}^{E,E})$ .

In each iteration of, e.g., the preconditioned conjugate gradient method an action of  $\hat{\mathbf{S}}^{-1}$  is required. We have the formula

$$\begin{aligned} \hat{\mathbf{S}}^{-1} &= \begin{pmatrix} \mathbf{I}^E & (\mathbf{R}^E)^T \\ \mathbf{0} & \mathbf{I}^V \end{pmatrix} \begin{pmatrix} (\bar{\mathbf{S}}^{E,E})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A}^H)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}^E & \mathbf{0} \\ \mathbf{R}^E & \mathbf{I}^V \end{pmatrix} = \\ &= \sum_{i=1}^M \begin{pmatrix} \mathbf{I}_i^E \\ \mathbf{0} \end{pmatrix} (\mathbf{S}_{i,i}^{E,E})^{-1} (\mathbf{I}_i^E, \mathbf{0}) + \begin{pmatrix} (\mathbf{R}^E)^T \\ \mathbf{I}^V \end{pmatrix} (\mathbf{A}^H)^{-1} \underbrace{(\mathbf{R}^E, \mathbf{I}^V)}_{=: \mathbf{R}^H}. \quad (8) \end{aligned}$$

This results in a modification of Step 2 of the three-steps method.

2a. Set

$$\mathbf{c}^\Gamma := \begin{pmatrix} \mathbf{c}^E \\ \mathbf{c}^V \end{pmatrix} := \mathbf{b}^\Gamma - \mathbf{A}^{\Gamma,I} \mathbf{v}^I.$$

2b. Solve  $M$  independent local systems  $\mathbf{S}_{i,i}^{E,E} \mathbf{w}_i^E = \mathbf{c}_i^E$ .

2c. Solve the global coarse system  $\mathbf{A}^H \mathbf{w}^H = \mathbf{c}^V + \mathbf{R}^E \mathbf{c}^E$ .

2d. Set

$$\widehat{\mathbf{u}}^\Gamma := \begin{pmatrix} \mathbf{w}^E + (\mathbf{R}^E)^T \mathbf{w}^H \\ \mathbf{w}^H \end{pmatrix}$$

The action of  $\widehat{\mathbf{S}}^{-1}$  comprises solution to a global system with the coarse matrix  $\mathbf{A}^H$  and solution to  $M$  local edge problems with matrices  $\mathbf{S}_{i,i}^{E,E}$ , which are local Schur complements related to the following systems:

$$\begin{pmatrix} \mathbf{A}_j^{I,I} & \mathbf{0} & \mathbf{A}_{j,i}^{I,E} \\ \mathbf{0} & \mathbf{A}_k^{I,I} & \mathbf{A}_{k,i}^{I,E} \\ \mathbf{A}_{i,j}^{E,I} & \mathbf{A}_{i,k}^{E,I} & \mathbf{A}_i^{E,E} \end{pmatrix} \begin{pmatrix} \mathbf{w}_j^I \\ \mathbf{w}_k^I \\ \mathbf{w}_i^E \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{c}_i^E \end{pmatrix},$$

where the relation of  $i$ ,  $j$ , and  $k$  is such that the domains  $\Omega_j$  and  $\Omega_k$  are connected via the edge  $E_i$ . We solve the system for  $\mathbf{w}_i^E$ . It is an FE-discretization on the space  $V_j + V_k + V_i^E$ , where  $V_i^E := \langle \varphi_{i,l}^E \rangle_{l=1}^{n_i^E}$ , of the following problem solved over the patch  $\Omega_j \cup \Omega_k$ :

$$\begin{aligned} -\rho_j \Delta w_j^I(x) &= 0, & x \in \Omega_j, \\ w_j^I(x) &= 0, & x \in \partial\Omega_j \setminus E_i, \\ -\rho_k \Delta w_k^I(x) &= 0, & x \in \Omega_k, \\ w_k^I(x) &= 0, & x \in \partial\Omega_k \setminus E_i, \\ w_i^E(x) := w_j^I(x) &= w_k^I(x), & x \in E_i, \\ \rho_j \frac{dw_j^I}{dn_j}(x) + \rho_k \frac{dw_k^I}{dn_k}(x) &= c_i^E(x), & x \in E_i, \end{aligned}$$

where  $n_j$  and  $n_k$  denote the outward unit normals to  $\Omega_j$  and  $\Omega_k$ , respectively.

The resulting preconditioner admits the following factorization

$$\widehat{\mathbf{A}} = \begin{pmatrix} \mathbf{I}^I & \mathbf{0} \\ \mathbf{A}^{\Gamma,I} & (\mathbf{A}^{I,I})^{-1} \mathbf{I}^\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{A}^{I,I} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{S}} \end{pmatrix} \begin{pmatrix} \mathbf{I}^I & (\mathbf{A}^{I,I})^{-1} \mathbf{A}^{I,\Gamma} \\ \mathbf{0} & \mathbf{I}^\Gamma \end{pmatrix}. \quad (9)$$

### 3 Analysis of the condition number

We shall analyze the condition number  $\kappa(\widehat{\mathbf{A}}^{-1}\mathbf{A})$  by means of finding spectral bounds  $\lambda_{\min} > 0$  and  $\lambda_{\max} > 0$  such that

$$\forall u \in V : \quad \lambda_{\min} \widehat{a}(u, u) \leq a(u, u) \leq \lambda_{\max} \widehat{a}(u, u),$$

where  $\widehat{a}(u, u)$  is the quadratic form related to  $\widehat{\mathbf{A}}$ . It will turn out that under shape-regularity and quasi-uniformity of both the coarse and fine discretizations the condition number  $\kappa$  is bounded by  $C(1 + \ln(H/h))^2$  from above. The constant  $C$  as well as all the other generic constants that appear in the theory below are independent of  $H$ ,  $h$ , and  $(\rho_i)_{i=1}^N$ .

#### 3.1 Orthogonal space splitting

Let us re-visit the algebraic construction of  $\widehat{\mathbf{A}}$ . First we re-sorted the basis functions according to the interior and skeleton nodes. This leads to

$$V = (V_1 \oplus_a \cdots \oplus_a V_N) + V^\Gamma,$$

where  $V^\Gamma := \langle \varphi_1^\Gamma, \dots, \varphi_{n_\Gamma}^\Gamma \rangle$  and where the  $a$ -orthogonality of  $V_i$  and  $V_j$ , for  $i \neq j$ , follows from  $\Omega_i \cap \Omega_j = \emptyset$ .

Now we take into account the transformation of base determined by the right factor of (9). It transforms the basis functions  $\varphi_i^\Gamma$  to their discrete harmonic extensions  $\widetilde{\varphi}_i^\Gamma := \mathcal{H}(\varphi_i^\Gamma)$ . Recall that the discrete harmonic extension  $\widetilde{u}^\Gamma$  of  $u^\Gamma \in V^\Gamma$  is the solution to the following problem:

$$\text{find } \widetilde{u}^\Gamma \in V : \quad \widetilde{u}^\Gamma(x) = u^\Gamma(x) \text{ on } \Gamma \quad \text{and} \quad \forall j \forall v \in V_j : a(\widetilde{u}^\Gamma, v) = 0.$$

Note that  $\widetilde{u}^\Gamma|_{\Omega_j}$ ,  $j = 1, \dots, N$ , is an FE-counterpart of

$$\begin{aligned} -\Delta \widetilde{u}^\Gamma(x) &= 0, & x \in \Omega_j, \\ \widetilde{u}^\Gamma(x) &= u^\Gamma(x), & x \in \Gamma \cap \partial\Omega_j, \\ \widetilde{u}^\Gamma(x) &= 0, & x \in \partial\Omega \cap \partial\Omega_j. \end{aligned}$$

Denoting by  $\widetilde{V}^\Gamma := \mathcal{H}(V^\Gamma)$  we arrive at the  $a$ -orthogonal decomposition

$$V = V_1 \oplus_a \cdots \oplus_a V_N \oplus_a \widetilde{V}^\Gamma.$$



The Schur complement  $\mathbf{S}$  is the FE-discretization of the bilinear form

$$s(u^\Gamma, v^\Gamma) := a(\mathcal{H}(u^\Gamma), \mathcal{H}(v^\Gamma)), \quad u^\Gamma, v^\Gamma \in V^\Gamma,$$

in the base  $(\varphi_i^\Gamma)_{i=1}^{n^\Gamma}$ . The latter can be deduced from

$$\mathbf{S} = \begin{pmatrix} -\mathbf{A}^{\Gamma, \mathbf{I}} (\mathbf{A}^{\mathbf{I}, \mathbf{I}})^{-1} & \mathbf{I}^\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{A}^{\mathbf{I}, \mathbf{I}} & \mathbf{A}^{\mathbf{I}, \Gamma} \\ \mathbf{A}^{\Gamma, \mathbf{I}} & \mathbf{A}^{\Gamma, \Gamma} \end{pmatrix} \begin{pmatrix} -(\mathbf{A}^{\mathbf{I}, \mathbf{I}})^{-1} \mathbf{A}^{\mathbf{I}, \Gamma} \\ \mathbf{I}^\Gamma \end{pmatrix},$$

where the transformation factors consist of nodal coordinates of  $(\tilde{\varphi}_i^\Gamma)_{i=1}^{n^\Gamma}$ .

Finally, we take a closer look at the last transformation determined by the factor  $\mathbf{R}^H$  in (8). It transforms functions to the linear interpolation from its vertex values along all the skeleton edges. We denote this interpolation operator by  $I^H : C_0(\Omega) \rightarrow C_0(\Omega)$ ,  $I^H(v)(x) := \sum_{i=1}^{n^V} v(x_i^V) \varphi_i^H(x)$ . In particular,  $I^H(\varphi_i^V) = \varphi_i^H$ , see (6). Since the latter are discrete harmonic, we end up with the following decomposition

$$V = \underbrace{V_1 \oplus_a \cdots \oplus_a V_N}_{=: V^{\mathbf{I}}} \oplus_a \underbrace{(\tilde{V}^E + V^H)}_{=: \tilde{V}^\Gamma}, \quad (10)$$

where  $V^H := I^H(V)$ ,  $\tilde{V}^E := \mathcal{H}(V - V^H) = \mathcal{H}(\sum_{i=1}^M V_i^E)$ . Therefore, every  $u = u^{\mathbf{I}} + u^E + u^V \in V$  admits the unique decomposition

$$u = \tilde{u}^{\mathbf{I}} \oplus_a (\tilde{u}^E + u^H),$$

where  $u^H := I^H(u)$ ,  $\tilde{u}^E = \mathcal{H}(u - u^H)$ , and  $\tilde{u}^{\mathbf{I}} := u - \tilde{u}^E - u^H$ . The quadratic forms now read as follows:

$$a(u, u) = \underbrace{\sum_{i=1}^N a(\tilde{u}_i^{\mathbf{I}}, \tilde{u}_i^{\mathbf{I}})}_{=: a(\tilde{u}^{\mathbf{I}}, \tilde{u}^{\mathbf{I}})} + \underbrace{\sum_{i,j=1}^M a(\tilde{u}_i^E, \tilde{u}_j^E)}_{=: a(\tilde{u}^E, \tilde{u}^E)} + 2a(\tilde{u}^E, u^H) + a(u^H, u^H), \quad (11)$$

$$\hat{a}(u, u) = \sum_{i=1}^N a(\tilde{u}_i^{\mathbf{I}}, \tilde{u}_i^{\mathbf{I}}) + \sum_{i=1}^M a(\tilde{u}_i^E, \tilde{u}_i^E) + a(u^H, u^H) \quad (12)$$

with  $\tilde{u}^{\mathbf{I}} = \sum_{i=1}^N \tilde{u}_i^{\mathbf{I}}$ ,  $\tilde{u}_i^{\mathbf{I}} \in V_i$  and  $\tilde{u}^E = \sum_{i=1}^M \tilde{u}_i^E$ ,  $\tilde{u}_i^E \in \mathcal{H}(V_i^E)$ .

### 3.2 Upper bound

**Theorem 1.** *For all  $u \in V$  we have*

$$a(u, u) \leq 10 \widehat{a}(u, u),$$

*i.e.*,  $\lambda_{\max} := 10$ .

*Proof.* Let us take an arbitrary  $u \in V$  and its unique splitting  $u = u^I \oplus_a (\widetilde{u}^E + u^H)$ . For each skeleton edge  $E_i$  we define its edge-neighbourhood

$$N_i := \{j \in \{1, \dots, M\} : i \neq j \text{ and } \exists k \in \{1, \dots, N\} : E_i, E_j \subset \partial\Omega_k\}.$$

Since  $|N_i| \leq 4$ , as each skeleton edge  $E_i$  is associated to at most four other edges via two subdomains, and  $j \in N_i \Leftrightarrow i \in N_j$ , using  $2a(v, w) \leq a(v, v) + a(w, w)$ ,

$$\begin{aligned} a(\widetilde{u}^E, \widetilde{u}^E) &= \sum_{i=1}^M \sum_{j=1}^M a(\widetilde{u}_i^E, \widetilde{u}_j^E) = \sum_{i=1}^M \left\{ a(\widetilde{u}_i^E, \widetilde{u}_i^E) + \sum_{j \in N_i} a(\widetilde{u}_i^E, \widetilde{u}_j^E) \right\} \\ &\leq \sum_{i=1}^M \left\{ a(\widetilde{u}_i^E, \widetilde{u}_i^E) + \sum_{j \in N_i} \frac{1}{2} [a(\widetilde{u}_i^E, \widetilde{u}_i^E) + a(\widetilde{u}_j^E, \widetilde{u}_j^E)] \right\} \\ &\leq \left(1 + \frac{4}{2}\right) \sum_{i=1}^M a(\widetilde{u}_i^E, \widetilde{u}_i^E) + \frac{4}{2} \sum_{j=1}^M a(\widetilde{u}_j^E, \widetilde{u}_j^E) = 5 \sum_{i=1}^M a(\widetilde{u}_i^E, \widetilde{u}_i^E). \end{aligned}$$

Using the latter estimate the mixed term is estimated as follows:

$$a(\widetilde{u}^E, u^H) \leq \frac{1}{2} [a(\widetilde{u}^E, \widetilde{u}^E) + a(u^H, u^H)] \leq \frac{5}{2} \sum_{i=1}^M a(\widetilde{u}_i^E, \widetilde{u}_i^E) + \frac{1}{2} a(u^H, u^H).$$

Combining the estimates completes the proof with  $\lambda_{\max} := 10$ ,

$$a(u, u) \leq a(\widetilde{u}^I, \widetilde{u}^I) + 10 \sum_{i=1}^M a(\widetilde{u}_i^E, \widetilde{u}_i^E) + 2a(u^H, u^H) \leq 10 \widehat{a}(u, u).$$

□

### 3.3 Shape-regular quasi-uniform triangulations

**Assumption 1.** *Let us assume that the fine triangulation is from a family of shape-regular discretizations by which we mean that there exists  $\alpha_{\min} \in (0, \pi/3)$  independent of  $h$  such that every angle in the FE-triangulation, thus also in the domain decomposition, is bounded by  $\alpha_{\min}$  from below. Shape-regularity guarantees the angles to be bounded from above by  $\alpha_{\max} := \pi - 2\alpha_{\min}$ . From the law of sines we have a uniform upper bound on the ratio between the largest and shortest edge of a triangle  $T_i$  or a subdomain  $\Omega_i$ , i.e.,*

$$\frac{h_{\max}^i}{h_{\min}^i}, \frac{H_{\max}^i}{H_{\min}^i} \in \left\langle 1, \frac{1}{\sin \alpha_{\min}} \right\rangle. \quad (13)$$

*For the sake of simplicity we assume that to each  $x_k^V$  being a corner of  $\Omega_i$  there is exactly one adjacent triangle  $T$  such that  $T \subset \Omega_i$ .*

**Assumption 2.** *Let us further assume that both the fine and coarse triangulations are from families of quasi-uniform discretizations by which we mean that there exists a common constant  $C_{A2} \in (0, 1)$  independent of  $h$  and  $H$  such that for every triangle  $T_i$  and every subdomain  $\Omega_i$  the diameter  $h_{\max}^i$  and  $H_{\max}^i$ , respectively, is bounded*

$$h_{\max}^i \geq C_{A2} h, \quad H_{\max}^i \geq C_{A2} H. \quad (14)$$

*For the sake of simplicity we assume that  $H \geq 2h$ .*

We will need a discrete Sobolev inequality for the FE-functions.

**Lemma 1.** *Given a linear function  $v$  on a triangle with vertices  $A, B, C$  and an angle  $\alpha$  at  $A$ , we have*

$$\|\nabla v\|^2 \leq \frac{2 [(v(B) - v(A))^2 + (v(C) - v(A))^2]}{\min\{\|B - A\|^2, \|C - A\|^2\} \sin^2 \alpha}.$$

*Proof.* We introduce the coordinate system such that  $A$  is in the origin and the line segment  $\overline{AB}$  is the  $x_1$ -axis, then

$$\frac{\partial v}{\partial x_1} = \frac{v(B) - v(A)}{\|B - A\|}, \quad \cos \alpha \frac{\partial v}{\partial x_1} + \sin \alpha \frac{\partial v}{\partial x_2} = \frac{v(C) - v(A)}{\|C - A\|} =: \frac{dv}{ds}.$$

The assertion follows from the following manipulations:

$$\begin{aligned}
\|\nabla v\|^2 &= \left(\frac{\partial v}{\partial x_1}\right)^2 + \frac{1}{\sin^2 \alpha} \left(\frac{dv}{ds} - \cos \alpha \frac{\partial v}{\partial x_1}\right)^2 \\
&\leq \frac{1}{\sin^2 \alpha} \left[ \sin^2 \alpha \left(\frac{\partial v}{\partial x_1}\right)^2 + 2 \left(\frac{dv}{ds}\right)^2 + 2 \cos^2 \alpha \left(\frac{\partial v}{\partial x_1}\right)^2 \right] \\
&\leq \frac{2}{\sin^2 \alpha} \left[ \left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{dv}{ds}\right)^2 \right].
\end{aligned}$$

□

**Corollary 1.** *Under Assumptions 1 and 2 there exists  $C_{C1} > 0$  such that*

$$\forall i \in \{1, \dots, N\} \forall u \in V : \quad h \|\nabla u\|_{L^\infty(\Omega_i)} \leq C_{C1} \|u\|_{L^\infty(\Omega_i)}. \quad (15)$$

*Proof.* For  $x \in T_j \subset \Omega_i$  with vertices  $A, B, C$  Assumption 1 and Lemma 1 yield

$$\|\nabla u(x)\| \leq \frac{\sqrt{2}}{h_{\min}^j \sin \alpha_{\min}} \sqrt{4u(A)^2 + 2u(B)^2 + 2u(C)^2} \leq \frac{4\|u\|_{L^\infty(\Omega_i)}}{h_{\min}^j \sin \alpha_{\min}}.$$

The assertion follows from (13) and (14)

$$\|\nabla u(x)\| \leq \frac{4\|u\|_{L^\infty(\Omega_i)}}{h_{\max}^j \sin^2 \alpha_{\min}} \leq \frac{1}{h} \underbrace{\frac{4}{C_{A2} \sin^2 \alpha_{\min}}}_{=: C_{C1}} \|u\|_{L^\infty(\Omega_i)}.$$

□

### 3.4 Stability of the coarse space

The next lemma is crucial for the stability of the coarse space in the energy norm. We are inspired by the proof of Bramble, Pasciak, and Schatz in [1, L.3.3].

**Lemma 2.** *Under Assumptions 1 and 2 there exists  $C_{L2} > 0$  such that for all  $i \in \{1, \dots, N\}$*

$$\forall u \in V : \quad \|u\|_{L^\infty(\Omega_i)}^2 \leq C_{L2} \left(1 + \ln \frac{H}{h}\right) \left(\frac{1}{H^2} \|u\|_{L^2(\Omega_i)}^2 + |u|_{H^1(\Omega_i)}^2\right).$$

*Proof.* Without a loss of generality, assume that  $\|u\|_{L^\infty(\Omega_i)} = |u(0)|$ . We shall find an open cone  $\Lambda_{0,KH,\gamma} \subset \Omega_i$  with the vertex at the origin 0, the radius  $KH$  and the angle  $\gamma := \alpha_{\min}$  with  $K$  independent of  $H$ . For the construction of  $\Lambda_{0,KH,\gamma}$  we refer to Fig. 2 and the following description. Denote by  $d_a$ ,  $d_b$ , and  $d_c$  the distances of the origin to the prolongations of the sides of  $\Omega_i$  with lengths  $a$ ,  $b$ , and  $c$ , respectively, and assume that  $d_a$  is the largest distance. We choose  $\tilde{K}H := d_a$ . We take the open cone  $\Lambda_{A,\tilde{K}H,\alpha} \subset \Omega_i$  at the vertex  $A$  of  $\Omega_i$  that is opposite to the side  $a$ , where  $\alpha$  denotes the angle at  $A$ . By moving  $\Lambda_{A,\tilde{K}H,\alpha}$  to the origin we get the cone  $\Lambda_{0,\tilde{K}H,\alpha} \subset \Omega_i$ . It remains to find  $K > 0$  independent of  $H$  such that  $K \leq \tilde{K}$ . The area  $|\Omega_i|$  can be estimated as follows

$$|\Omega_i| = \frac{a d_a + b d_b + c d_c}{2} \leq \frac{a + b + c}{2} \tilde{K} H.$$

By (14) and (13) we have an  $H$ -independent estimate for the constant  $\tilde{K}$

$$\tilde{K} \geq 2 \frac{\frac{1}{2} H_{\max}^i H_{\min}^i \sin \alpha_{\min}}{3 H_{\max}^i H} \geq \frac{C_{A2}}{3} \frac{H_{\min}^i \sin \alpha_{\min}}{H_{\max}^i} \geq \frac{C_{A2}}{3} \sin^2 \alpha_{\min} =: K.$$

The construction of  $\Lambda_{0,KH,\gamma}$  is completed by shortening the radius and the angle of  $\Lambda_{0,\tilde{K}H,\alpha}$ .

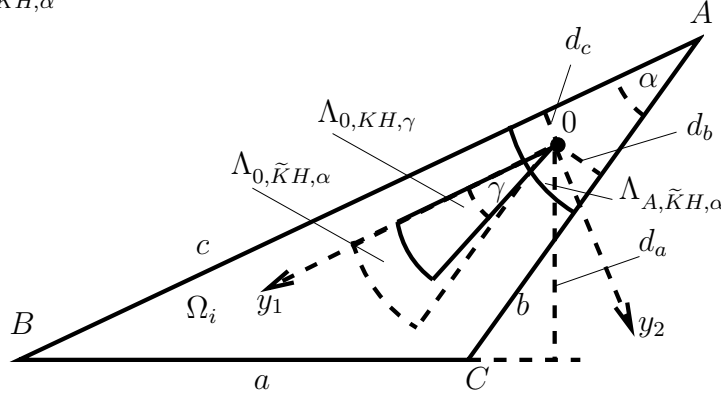


Figure 2: Construction of  $\Lambda_{0,KH,\gamma}$ .

We consider the coordinate system according to a side of  $\Lambda_{0,KH,\gamma}$ . For  $y(\rho, \vartheta) = \rho(\cos \vartheta, \sin \vartheta) \in \Lambda_{0,KH,\gamma}$  the fundamental theorem of calculus gives

$$u(0) = u(y(\rho, \vartheta)) - \int_0^\rho \underbrace{\nabla u(y(t, \vartheta)) (\cos \vartheta, \sin \vartheta)}_{=: u'_t(t, \vartheta)} dt.$$

Integrating  $\vartheta$  from 0 to  $\gamma$  and applying the triangle inequality yield

$$\gamma |u(0)| \leq \left| \int_0^\gamma u(y(\rho, \vartheta)) d\vartheta \right| + \left| \int_0^\gamma \int_0^\rho u'_t(t, \vartheta) dt d\vartheta \right|. \quad (16)$$

Choose, independently of  $h$  and  $H$ ,  $\delta := \min\{(\sqrt{2} - 1)/(\sqrt{2}C_{C1}), K\}$ , where  $C_{C1}$  is the constant in (15). We shall consider two cases. First, if  $\delta h < \rho$ , the Cauchy-Schwarz and triangle inequalities yield

$$\begin{aligned} \gamma |u(0)| \leq \sqrt{\gamma} \sqrt{\int_0^\gamma u^2(y(\rho, \vartheta)) d\vartheta} + \left| \int_0^\gamma \int_0^{\delta h} u'_t(t, \vartheta) dt d\vartheta \right| \\ + \left| \int_0^\gamma \int_{\delta h}^\rho u'_t(t, \vartheta) dt d\vartheta \right|. \end{aligned} \quad (17)$$

The second and third term in (17) can be estimated as follows:

$$\begin{aligned} \left| \int_0^\gamma \int_0^{\delta h} u'_t(t, \vartheta) dt d\vartheta \right| &\leq \gamma \delta h \|\nabla u\|_{L^\infty(\Omega_i)} \leq \gamma \delta C_{C1} |u(0)|, \\ \left| \int_0^\gamma \int_{\delta h}^\rho u'_t(t, \vartheta) dt d\vartheta \right| &= \left| \int_{\Lambda_{0,\rho,\gamma} \setminus \Lambda_{0,\delta h,\gamma}} \nabla u(y) \frac{y}{\|y\|^2} dy \right| \leq \|\nabla u\|_{L^2(\Omega_i)} \sqrt{\gamma} \sqrt{\ln \frac{\rho}{\delta h}}. \end{aligned}$$

Using the estimates, moving the second term from the right-hand side of (17) to the left, squaring the inequality and dividing by  $\gamma^2$  yield

$$\frac{1}{2} |u(0)|^2 \leq (1 - \delta C_{C1})^2 |u(0)|^2 \leq \frac{2}{\gamma} \left\{ \int_0^\gamma u^2(y(\rho, \vartheta)) d\vartheta + \ln \frac{\rho}{\delta h} \|\nabla u\|_{L^2(\Omega_i)}^2 \right\}. \quad (18)$$

In the second case,  $\delta h \geq \rho$ , we estimate the first term on the right-hand side of (16) by the Cauchy-Schwarz inequality and the second term by  $\gamma \delta C_{C1} |u(0)|$ , which leads to

$$\frac{1}{2} |u(0)|^2 \leq (1 - \delta C_{C1})^2 |u(0)|^2 \leq \frac{2}{\gamma} \int_0^\gamma u^2(y(\rho, \vartheta)) d\vartheta. \quad (19)$$

Multiplying (18) and (19) by  $2\rho$ , integrating  $\rho$  from  $\delta h$  to  $KH$  and from 0 to  $\delta h$ , respectively, and summing up the resulting inequalities yield

$$\begin{aligned} \frac{(KH)^2}{2} |u(0)|^2 \leq \frac{4}{\gamma} \left\{ \int_0^{KH} \int_0^\gamma \rho u^2(y(\rho, \vartheta)) d\vartheta d\rho \right. \\ \left. + \|\nabla u\|_{L^2(\Omega_i)}^2 \int_{\delta h}^{KH} \rho \ln \frac{\rho}{\delta h} d\rho \right\}. \end{aligned}$$

By estimating the second term on the right-hand side we complete the proof

$$\begin{aligned}
|u(0)|^2 &\leq \frac{4}{\gamma} \left\{ \frac{2}{(KH)^2} \|u\|_{L^2(\Omega_i)}^2 + \left( \ln \frac{K}{\delta} + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_i)}^2 \right\} \\
&\leq \frac{4}{\gamma} \underbrace{\max \left\{ 1, \frac{2}{K^2}, \ln \frac{K}{\delta} \right\}}_{=: C_{L2}} \left( 1 + \ln \frac{H}{h} \right) \left\{ \frac{1}{H^2} \|u\|_{L^2(\Omega_i)}^2 + |u|_{H^1(\Omega_i)}^2 \right\}.
\end{aligned}$$

□

**Corollary 2.** *Under Assumptions 1 and 2 there exists  $C_{C2} > 0$  such that for all  $i \in \{1, \dots, N\}$*

$$\forall u \in V : \|u - \bar{u}_i\|_{L^\infty(\Omega_i)}^2 \leq C_{C2} \left( 1 + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_i)}^2,$$

where  $\bar{u}_i := \frac{1}{|\Omega_i|} \int_{\Omega_i} u(x) dx$  with  $|\Omega_i|$  being the area of  $\Omega_i$ .

*Proof.* Combining the previous lemma and the Poincaré inequality [8]

$$\|u - \bar{u}_i\|_{L^2(\Omega_i)}^2 \leq C_P H^2 |u|_{H^1(\Omega_i)}^2,$$

where  $C_P := 1/\pi^2$ , the assertion follows with  $C_{C2} := C_{L2}(1 + C_P)$ . □

The next lemma gives stability of the coarse space. It can be found in [9, L.4.12].

**Lemma 3.** *Under Assumptions 1 and 2 there exists  $C_{L3} > 0$  such that for all  $i \in \{1, \dots, N\}$*

$$\forall u \in V : |I^H(u)|_{H^1(\Omega_i)}^2 \leq C_{L3} \left( 1 + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_i)}^2,$$

as a consequence of which

$$\forall u \in V : a(u^H, u^H) \leq C_{L3} \left( 1 + \ln \frac{H}{h} \right) a(u, u).$$

*Proof.* Denote by  $P_1, P_2$ , and  $P_3$  the vertices of a subdomain  $\Omega_i$ . We have

$$|I^H(u)|_{H^1(\Omega_i)}^2 = |I^H(u) - \bar{u}_i|_{H^1(\Omega_i)}^2 = \left| \sum_{j=1}^3 (u(P_j) - \bar{u}_i) \varphi_j^H \right|_{H^1(\Omega_i)}^2. \quad (20)$$

For  $j \in \{1, 2, 3\}$  and the remaining indices  $k$  and  $l$  we employ Lemma 1 with  $A := P_k$ ,  $\alpha := \alpha_k$  the angle at  $P_k$ ,  $B := P_j$ , and  $C := P_l$  and using (13) yield

$$\begin{aligned} |\varphi_j^H|_{H^1(\Omega_i)}^2 &= \|\nabla \varphi_j^H\|^2 |\Omega_i| \leq \frac{2 \cdot \frac{1}{2} \|P_j - P_k\| \|P_l - P_k\| \sin \alpha_k}{\min\{\|P_j - P_k\|^2, \|P_l - P_k\|^2\} \sin^2 \alpha_k} \\ &\leq \frac{H_{\max}^i}{H_{\min}^i \sin \alpha_k} \leq \frac{1}{\sin^2 \alpha_{\min}} =: \tilde{c}, \end{aligned}$$

where  $|\Omega_i|$  denotes the area of  $\Omega_i$ . By (20) and Corollary 2 we have

$$|I^H(u)|_{H^1(\Omega_i)}^2 \leq 3 \sum_{j=1}^3 |u(P_j) - \bar{u}_i|^2 |\varphi_j^H|_{H^1(\Omega_i)}^2 \leq 9 \tilde{c} C_{C2} \left(1 + \ln \frac{H}{h}\right) |u|_{H^1(\Omega_i)}^2,$$

which completes the proof with  $C_{L3} := 9 \tilde{c} C_{C2}$ .  $\square$

### 3.5 Stability of the edge space

To find  $\lambda_{\min}$  it remains to estimate the edge-term in (12) by (11) from above. Being inspired by [9, L.4.23] we introduce a system of edge-based functions  $(\theta_i(x))_{i=1}^M \subset C(\tilde{\Omega})$ , where  $\tilde{\Omega} := \bar{\Omega} \setminus \{x_j^V : j = 1, \dots, n^V\}$ . For the construction we refer to Fig. 3 and the following paragraph.

We decompose each subdomain  $\Omega_j$ ,  $j \in \{1, 2, \dots, N\}$ , with all three edges being a part of the skeleton, i.e.,  $E_{j_1}, E_{j_2}, E_{j_3} \subset \Gamma$ , into six triangles  $\omega_k$ . Without loss of generality we take  $x \in \omega_1 \setminus \{P_1\}$  and introduce local coordinates  $x = (x_1, x_2)$ . We denote the angle at  $P_1$  by  $\alpha_1$  and define the related edge functions  $\theta_{j_1}$ ,  $\theta_{j_2}$ , and  $\theta_{j_3}$  in  $\omega_1$  by

$$\theta_{j_1}(x) := 1 - \frac{2}{3 \tan \frac{\alpha_1}{2}} \frac{x_1}{x_2}, \quad \theta_{j_2}(x) = \theta_{j_3}(x) := \frac{1}{3 \tan \frac{\alpha_1}{2}} \frac{x_1}{x_2}. \quad (21)$$

The edge functions are analogously defined in  $\omega_2, \dots, \omega_6$ . For a subdomain  $\Omega_j$  with only one or two edges assigned to the skeleton the construction of the related edge functions is similar. Note that the system completed by edge-functions assigned to  $\partial\Omega$  forms a partition of unity on  $\tilde{\Omega}$ .



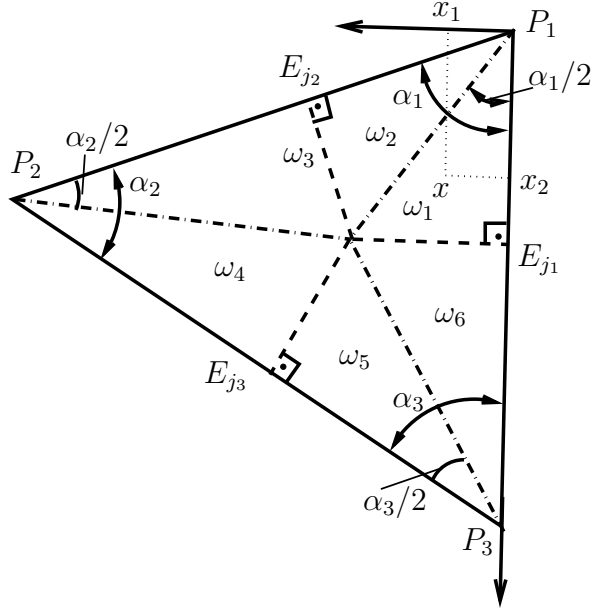


Figure 3: Decomposition of  $\Omega_j$  used for the construction of  $\theta_{j_1}$ ,  $\theta_{j_2}$ , and  $\theta_{j_3}$ .

**Lemma 4.** *Under Assumption 1 there exists  $C_{L4} > 0$  such that for all  $i \in \{1, \dots, M\}$*

$$\|\nabla\theta_i(x)\| \leq C_{L4}/r^H(x) \text{ almost everywhere in } \tilde{\Omega},$$

where, for  $x \in \Omega_j$  with the vertices  $P_1$ ,  $P_2$ , and  $P_3$ ,  $r^H(x) := \min_{k=1,2,3} \|x - P_k\|$ .

*Proof.* The assertion follows from the construction above. For  $x \in \omega_1$ :

$$\|\nabla\theta_{j_1}(x)\| = \frac{2}{3 \tan \frac{\alpha_1}{2}} \frac{\sqrt{(x_1)^2 + (x_2)^2}}{(x_2)^2} \leq \underbrace{\frac{2}{3 \tan \frac{\alpha_{\min}}{2} \cos^2 \frac{\alpha_{\max}}{2}}}_{=: C_{L4}} \underbrace{\frac{1}{\sqrt{(x_1)^2 + (x_2)^2}}}_{\leq 1/r^H(x)}$$

by Assumption 1. The estimate holds true for  $\|\nabla\theta_{j_2}(x)\|$  and  $\|\nabla\theta_{j_3}(x)\|$ . The other cases,  $x \in \omega_k$ , are analogous.  $\square$

Similarly to replacing the FE-projection by interpolation when estimating the FE-approximation error, we will estimate the energy of the FE-interpolation of  $\theta_i(u - u^H)$ , rather than the energy of  $\tilde{u}_i^E$ . We need the following, so-called edge lemma, the proof of which is sketched in [2].

**Lemma 5.** *Under Assumptions 1 and 2 there exists  $C_{L5} > 0$  such that for all edges  $E_i$ ,  $i \in \{1, \dots, M\}$ , both the adjacent domains  $\Omega_j$ ,  $E_i \subset \partial\Omega_j$ , and*

$$\forall u \in V : \quad |I^h(\theta_i w)|_{H^1(\Omega_j)}^2 \leq C_{L5} \left\{ \left(1 + \ln \frac{H}{h}\right) \|w\|_{L^\infty(\Omega_j)}^2 + |w|_{H^1(\Omega_j)}^2 \right\}, \quad (22)$$

where  $w := u - u^H$  and  $I^h : C_0(\Omega) \rightarrow V$  is the FE-interpolation operator, i.e.,  $I^h(v)(x) := \sum_{i=1}^n v(x_i) \varphi_i(x)$ , where  $x_i$  is the node related to  $\varphi_i$ .

*Proof.* Let us take an edge  $E_i$  and an adjacent domain  $\Omega_j$ . By Assumption 1 to each coarse vertex  $P_k$ ,  $k = 1, 2, 3$ , of  $\Omega_j$  an exactly one fine triangle  $T$  with vertices  $A = P_k$ ,  $B$ , and  $C$  is associated. In the case that none of  $B$  and  $C$  lies on  $E_i$ ,  $I^h(\theta_i w)$  vanishes on  $T$ . We are left to analyze the other two triangles, for both of which we can consider  $C \in E_i$ . The contribution of such a triangle to  $|I^h(\theta_i w)|_{H^1(\Omega_j)}^2$  is, using (13), as follows:

$$\begin{aligned} |I^h(\theta_i w)|_{H^1(T)}^2 &= \frac{w^2(C)}{\|C - A\|^2 \sin^2 \alpha} \frac{\|B - A\| \|C - A\| \sin \alpha}{2} \\ &\leq \frac{h_{\max}^T}{2 h_{\min}^T \sin \alpha_{\min}} w^2(C) \leq \underbrace{\frac{1}{2 \sin^2 \alpha_{\min}}}_{=: \tilde{k}_1} \|w\|_{L^\infty(\Omega_j)}^2. \end{aligned} \quad (23)$$

In case of a triangle  $T \subset \Omega_j$  that none of its vertices  $A$ ,  $B$ , and  $C$  is a vertex of  $\Omega_j$ , Lemma 1 yields

$$\|\nabla I^h(\theta_i w)\|^2 \leq \frac{2 \{[(\theta_i w)(B) - (\theta_i w)(A)]^2 + [(\theta_i w)(C) - (\theta_i w)(A)]^2\}}{\min\{\|B - A\|^2, \|C - A\|^2\} \sin^2 \alpha},$$

where  $\alpha$  denotes the angle at  $A$ . Since  $\theta_i w$  is piecewise differentiable along the line segments  $\overline{AB}$  and  $\overline{AC}$ , we can adopt the Lagrange mean value theorem. The latter combined with (13), the construction (21), and Lemma 4 yields

$$\begin{aligned} \|\nabla I^h(\theta_i w)\|^2 &\leq \frac{2 \|\nabla(\theta_i w)\|_{L^\infty(T)}^2}{\sin^2 \alpha} \underbrace{\frac{\|B - A\|^2 + \|C - A\|^2}{\min\{\|B - A\|^2, \|C - A\|^2\}}}_{\leq 1 + (h_{\max}^T/h_{\min}^T)^2} \\ &\leq \underbrace{4 \left(1 + \frac{1}{\sin^2 \alpha_{\min}}\right)}_{=: \tilde{k}_2} \left\{ \|\nabla \theta_i\|_{L^\infty(T)}^2 \|w\|_{L^\infty(T)}^2 + \|\theta_i\|_{L^\infty(T)}^2 \|\nabla w\|_{L^\infty(T)}^2 \right\} \\ &\leq \tilde{k}_2 \left\{ (C_{L4}/r^{H,h}(x))^2 \|w\|_{L^\infty(T)}^2 + \|\nabla w\|_{L^\infty(T)}^2 \right\}, \end{aligned}$$

where  $\|\nabla f\|_{L^\infty(T)} := \operatorname{ess\,sup}_{x \in T} \|\nabla f(x)\|$  and  $r^{H,h}(x) := \operatorname{dist}(T(x), \{P_1, P_2, P_3\})$ , where  $T(x)$  is the open triangle containing  $x$ , which is a well-defined function up to the interfaces between fine triangles. Denote by  $\tilde{\Omega}_j$  the union of such non-corner triangles. They contribute to  $|I^h(\theta_i w)|_{H^1(\Omega_j)}^2$  as follows:

$$|I^h(\theta_i w)|_{H^1(\tilde{\Omega}_j)}^2 \leq \tilde{k}_2 \left\{ \|w\|_{L^\infty(\Omega_j)}^2 (C_{L4})^2 \int_{\tilde{\Omega}_j} (1/r^{H,h}(x))^2 dx + |w|_{H^1(\tilde{\Omega}_j)}^2 \right\}. \quad (24)$$

It remains to estimate the integral. We have

$$\int_{\tilde{\Omega}_j} \frac{1}{(r^{H,h}(x))^2} dx \leq \sum_{k=1}^3 \int_{\tilde{\Omega}_j} \frac{1}{\inf_{y \in T(x)} \|y - P_k\|^2} dx. \quad (25)$$

Let us introduce three systems of local polar coordinates so that, each of which has the respective origin at a coarse vertex  $P_k$ , the  $x_1$ -axis coincides with an edge of  $\Omega_j$ , and  $\Omega_j$  lies in the upper half-space. We denote by  $v_{\min}^T$  the smallest height of a triangle  $T$ . The law of sines, Assumptions 1 and 2 yield

$$v_{\min}^T \geq h_{\min}^T \sin \alpha_{\min} \geq h_{\max}^T \sin^2 \alpha_{\min} \geq C_{A2} h \sin^2 \alpha_{\min}.$$

Thus, by choosing  $c := C_{A2} \sin^2 \alpha_{\min}$  the domain

$$\Lambda := \{x = (x_1, x_2) = \rho(\cos \alpha, \sin \alpha) \in \mathbb{R}^2 : ch \leq \rho \leq H \text{ and } 0 \leq \alpha \leq \alpha_{\max}\}$$

covers  $\tilde{\Omega}_j$  with respect to each of the coordinate systems. Let us denote the respective counterparts of  $\Lambda$  associated to  $P_1, P_2$ , and  $P_3$  by  $\Lambda_1, \Lambda_2$ , and  $\Lambda_3$ . Let us adopt the  $k$ -th local polar coordinates  $x(\rho, \alpha)$  and note that

$$\inf_{y \in T(x(\rho, \alpha))} \|y - P_k\| \geq \max\{ch, \rho - h\}.$$

We have the estimate

$$\begin{aligned} \int_{\tilde{\Omega}_j} \frac{1}{\inf_{y \in T(x)} \|y - P_k\|^2} dx &\leq \int_0^{\alpha_{\max}} \int_{ch}^H \frac{\rho}{(\max\{ch, (\rho - h)\})^2} d\rho d\alpha \\ &= \alpha_{\max} \left( \frac{2c+1}{2c^2} + \ln \frac{H-h}{ch} + \frac{h}{ch} - \frac{h}{H-h} \right) \leq \alpha_{\max} \left( \underbrace{\frac{4c+1}{2c^2}}_{=: \tilde{c}} + \ln \frac{H}{ch} \right), \end{aligned}$$

where we used  $H \geq 2h$  from Assumption 2. Using the latter and (25), (24) is estimated by

$$|I^h(\theta_i w)|_{H^1(\tilde{\Omega}_j)}^2 \leq \tilde{k}_2 \left\{ \|w\|_{L^\infty(\Omega_j)}^2 (C_{L4})^2 3 \alpha_{\max} \left[ \tilde{c} + \ln \frac{H}{ch} \right] + |w|_{H^1(\tilde{\Omega}_j)}^2 \right\}.$$

After adding the two contributions (23), the assertion follows with

$$C_{L5} := \max \left\{ 3 \tilde{k}_2 (C_{L4})^2 \alpha_{\max} (\tilde{c} - \ln c) + 2\tilde{k}_1, 3 \tilde{k}_2 (C_{L4})^2 \alpha_{\max}, \tilde{k}_2 \right\}.$$

□

Now we can analyze the stability of the edge space. The following lemma is proven in [9].

**Lemma 6.** *Under Assumptions 1 and 2 there exists  $C_{L6} > 0$  such that*

$$\forall u \in V : \quad \sum_{i=1}^M a(\tilde{u}_i^E, \tilde{u}_i^E) \leq C_{L6} \left( 1 + \ln \frac{H}{h} \right)^2 a(u, u).$$

*Proof.* Denote by  $\Omega_{i_1}$  and  $\Omega_{i_2}$  domains adjacent to  $E_i$  and by  $\{E_{j_i}\}_{i=1}^{M_j}$ ,  $M_j \leq 3$ , edges adjacent to  $\Omega_j$ . Recall that  $w := u - I^H(u)$  and  $\tilde{u}_i^E := \mathcal{H}(w_i^E)$ , where  $w_i^E := w$  on  $E_i$  and  $w_i^E := 0$  elsewhere on  $\Gamma \cup \partial\Omega$ . The discrete harmonicity of  $\tilde{u}_i^E$  and Lemma 5 yield

$$\begin{aligned} \sum_{i=1}^M a(\tilde{u}_i^E, \tilde{u}_i^E) &= \sum_{i=1}^M \sum_{j=1}^2 \rho_{i_j} |\tilde{u}_i^E|_{H^1(\Omega_{i_j})}^2 \leq \sum_{i=1}^M \sum_{j=1}^2 \rho_{i_j} |I^h(\theta_i w)|_{H^1(\Omega_{i_j})}^2 \\ &= \sum_{j=1}^N \rho_j \sum_{i=1}^{M_j} |I^h(\theta_{j_i} w)|_{H^1(\Omega_j)}^2 \\ &\leq \sum_{j=1}^N \rho_j 3 C_{L5} \left\{ \left( 1 + \ln \frac{H}{h} \right) \|w\|_{L^\infty(\Omega_j)}^2 + |w|_{H^1(\Omega_j)}^2 \right\}. \end{aligned}$$

Now  $(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)$  and Corollary 2 give

$$\begin{aligned} \|w\|_{L^\infty(\Omega_j)}^2 &= \|(u - \bar{u}_j) - (I^H(u) - \bar{u}_j)\|_{L^\infty(\Omega_j)}^2 \\ &\leq 2 \left( \|u - \bar{u}_j\|_{L^\infty(\Omega_j)}^2 + \underbrace{\|I^H(u) - \bar{u}_j\|_{L^\infty(\Omega_j)}^2}_{\leq \|u - \bar{u}_j\|_{L^\infty(\Omega_j)}^2} \right) \leq 4 C_{C2} \left( 1 + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_j)}^2. \end{aligned}$$

Similarly, Lemma 3 gives

$$\begin{aligned} |w|_{H^1(\Omega_j)}^2 &= |u - I^H(u)|_{H^1(\Omega_j)}^2 \leq 2 \left( |u|_{H^1(\Omega_j)}^2 + |I^H(u)|_{H^1(\Omega_j)}^2 \right) \\ &\leq 2 (1 + C_{L3}) \left( 1 + \ln \frac{H}{h} \right) |u|_{H^1(\Omega_j)}^2. \end{aligned}$$

Combining the estimates yields  $C_{L6} := 3C_{L5} [4C_{C2} + 2(1 + C_{L3})]$ .  $\square$

### 3.6 Lower bound

**Theorem 2.** *Under Assumptions 1 and 2 there exists  $C > 0$  such that*

$$\forall u \in V : \quad \widehat{a}(u, u) \leq C \underbrace{\left( 1 + \ln \frac{H}{h} \right)^2}_{=1/\lambda_{\min}(H,h)} a(u, u).$$

*Proof.* Comparing (11) and (12), the assertion is a consequence of Lemma 3 and 6 with  $C := 1 + C_{L3} + C_{L6}$ .  $\square$

We conclude with an estimate of the condition number

$$\kappa \left( \widehat{\mathbf{A}}^{-1} \mathbf{A} \right) \leq 10 C \left( 1 + \ln \frac{H}{h} \right)^2$$

with  $C > 0$  independent of  $H$ ,  $h$ , and  $(\rho_i)_{i=1}^N$  in a family of shape-regular quasi-uniform triangulations.

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