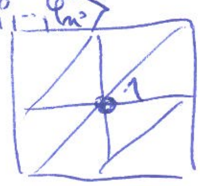
Metoda dvou sítí (two-grid)

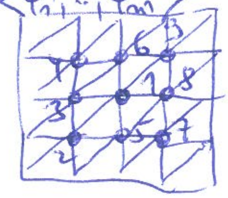
$$(P)\begin{cases} \text{Hledám } u(x)\in H_0^1(\Omega) \\ \int_\Omega k(x)\,\nabla u(x)\cdot\nabla v(x)\,dx = \int_\Omega f(x)\,v(x)\,dx \quad \forall v(x)\in H_0^1(\Omega) \end{cases}$$

z MKP diskretizace

$V^0,\ \mathcal{T}^H$ $\quad$ $V^1,\ \mathcal{T}^h$ $\quad$ $\boxed{V^0\subset V^1}$

$\langle \varphi_1^0,\dots,\varphi_{n^0}^0\rangle$ $\quad$ $\langle \varphi_1^1,\dots,\varphi_{n^1}^1\rangle$ $\quad$ $A^0, A^1 \dots$ spd





$A^0\bar u^0=\bar b^0$ $\quad$ $u^0(x)\in V^0$ $\quad$ $b_0\in (V^0)^*$

$A^1\bar u^1=\bar b^1$ $\quad$ $u^1(x)\in V^1$ $\quad$ $b_1\in (V^1)^*$

Interpolace z hrubé sítě

$I^0: V^0\mapsto V^1$ def. $\left(I^0(v^0(x))\right)(x) := v^0(x) \quad \forall v^0(x)\in V^0$

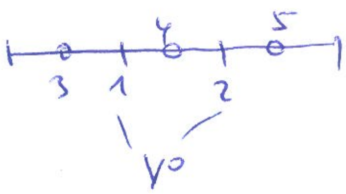
MKP-souřadnice: $\bar v^0, \bar v^1$

$v^1(x)=\sum_i (\bar v^1)_i\,\varphi_i^1(x)$

$v^0(x)=\sum_i (\bar v^0)_i\,\varphi_i^0(x)$

$$\boxed{(\mathbb{I}^0)_{ij}=\varphi_j^0(x_i^1)}\ ; \quad \forall\ \bar v^0 \longmapsto \mathbb{I}^0\cdot\bar v^0$$

např.: 1) $\mathbb{I}^0=\begin{bmatrix}1\\ 1/2\\ 1/2\\ \vdots\\ 1/2\end{bmatrix}$ pro 2d příklad

2) 

$$\mathbb{I}^0=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1/2 & 0\\ 1/2 & 1/2\\ 0 & 1/2\end{bmatrix}$$

Projekce rezidua (pravé strany) na hrubou síť

Lemma. $V^0 \overset{I^0}{\subset} V^1 \Rightarrow (V^0)^* \overset{(I^0)^T}{\supset} (V^1)^*$

Dk. $b^1 \in (V^1)^*$ je definováno pro V^1 a dědí i na $V^0 \subset V^1$

$$\|b^1\|_{(V^0)^*} = \sup_{\substack{v^0 \in V^0 \\ \neq 0}} \frac{\langle b^1, v^0\rangle}{\|v^0\|_V} \le \sup_{\substack{v^1 \in V^1 \\ \neq 0}} \frac{\langle b^1, v^1\rangle}{\|v^1\|_V} = \|b^1\|_{(V^1)^*} \quad \square$$

MKP-souřadnice:

$$(\bar b^0)_i := \langle b^0, \varphi_i^0(x)\rangle = \langle b^1, I^0(\varphi_i^0(x))(x)\rangle = \bar b^1 \cdot (I^0 \cdot \bar e_i) = ((I^0)^T \bar b^1)_i$$

$$\bar b^1 \mapsto \boxed{\bar b^0 := (I^0)^T \bar b^1}$$

Korekce na hrubé síti (coarse-grid correction)

A-ortonormální projekce na hrubou síť:

$$u^1 \mapsto \underbrace{P^0 u^1}_{=:u^0} \overset{def.}{\iff} a(P^0 u^1, w^0) = a(u^1, w^0) \quad \forall w^0 \in V^0$$

$$\boxed{A^0 \bar u^0 = (I^0)^T A^1 \bar u^1}$$

Chyba projekce:

$$\bar u^1 - I^0 P^0 \bar u^1 = \boxed{\left[I - I^0 (A^0)^{-1} (I^0)^T A^1\right] \bar u^1}$$

Vyhlazení na jemné síti (smoother)

$\hat A^1 \approx A^1$ tak, že $\boxed{(\hat A^1)^{-1}}$ je levné, např. $(\hat A^1)^{-1} = \frac{1}{\Lambda} I$

kde $\lambda_{max}(A) \le \Lambda$

Metoda dvou sítí (1 iterace ≈ předpodmínění)

$\hat u^1_{1+m} := T(\overset{A\bar e^1_0}{\overset{\|}{\bar r^1_0}})$:

$\hat u^1_1 := I^0 (A^0)^{-1} (I^0)^T \bar r^1_0$ (korekce)

$\hat r^1_1 := \bar r^1_0 - A^1 \hat u^1_1 = A^1 \left[I - I^0 (A^0)^{-1} (I^0)^T A^1\right] \bar e^1_0$

$\quad = A^1 \hat e^1_1$

for $i = 1:m$

$\quad \hat u^1_{1+i} := \hat u^1_{1+i-1} + \overbrace{(\hat A^1)^{-1} \hat r^1_{1+i-1}}^{\hat v^1_{1+i-1}}$

$\quad \hat r^1_{1+i} := \hat r^1_{1+i-1} - A^1 \hat v^1_{1+i-1}$

end for

Richardson + T:

$\bar u^1_0 = \bar 0$, $\bar r^1_0 = \bar b^1$

$k := 0$

while $\|\bar r^1_k\| / \|\bar b\| > \varepsilon$

$\quad \bar u^1_{k+1} := \bar u^1_k + T(\bar r^1_k)$

$\quad \bar r_{k+1} = \bar b^1 - A^1 \bar u^1_{k+1}$

endwhile

Metoda dvou sítí (pokrač.)

Věta.

- $\forall \bar{w}^1: \bar{w}^1 A^1 \bar{w}^1 \leq \bar{w}^1 \hat{A}^1 \bar{w}^1 \; (\Leftrightarrow \|\bar{w}^1\|_A \leq \|\bar{w}^1\|_{\hat{A}})$
  (smoothing property)
- $\exists K > 0 \; \forall \bar{w}^1:$
  $\|(I - P^0)\bar{w}^1\|_{\hat{A}} \leq K \cdot \|(I-P^0)\bar{w}^1\|_A$
  (approximation property)

$\Rightarrow \|\hat{u}^1 - \hat{u}^1_{1+m}\|_A \leq \sqrt{K \cdot \gamma(m)} \cdot \|\bar{u}^1 - \bar{u}^1_0\|_A$

kde

$$\gamma(m) = \frac{1}{2m+1}\left(1 - \frac{1}{2m+1}\right)^{2m}$$

Důkaz.

- Bi-ortogonální báze

$A := A^1, \hat{A} := \hat{A}^1$ .. spd $\Rightarrow \hat{A}^{-1/2} A \hat{A}^{-1/2}$ .. spd

$$\boxed{\hat{A}^{-1/2} A \underbrace{\hat{A}^{-1/2} \bar{v}_i}_{=: \bar{\varphi}_i} = \lambda_i \bar{v}_i}$$

$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

$\bar{v}_1 \perp \bar{v}_2 \perp \dots \perp \bar{v}_n$ .. ortonormální báze $\mathbb{R}^n$

Pleď: $\bar{v}_i \cdot \bar{v}_j = \delta_{ij} \Rightarrow (\hat{A}^{1/2}\bar{\varphi}_i)\cdot(\hat{A}^{1/2}\bar{\varphi}_j) = \boxed{\bar{\varphi}_i \hat{A} \bar{\varphi}_j = \delta_{ij}}$

$\Rightarrow \boxed{\bar{\varphi}_i A \bar{\varphi}_j = \lambda_i \delta_{ij}}$

$(\bar{\varphi}_i)$ je tzv. bi-ortogonální báze vzhledem k A a $\hat{A}$

$\|\bar{u}\|^2_A = (\sum_i u_i \bar{\varphi}_i) \cdot A \cdot (\sum_j u_j \bar{\varphi}_j) = \sum_i \lambda_i (u_i)^2$

$\|\bar{u}\|^2_{\hat{A}} = (\sum_i u_i \bar{\varphi}_i) \hat{A} (\sum_j u_j \bar{\varphi}_j) = \sum_i (u_i)^2$

$\Rightarrow \boxed{\hat{A}^{-1} A \bar{\varphi}_i = \lambda_i \bar{\varphi}_i}$

- Smoothing step

$\bar{b} - A\hat{u}_{1+i-1} = A(\bar{u} - \hat{u}_{1+i-1})$

$\hat{u}_{1+i} = \hat{u}_{1+i-1} + (\hat{A})^{-1} \cancel{A}_{1+i-1} \Rightarrow$

$\hat{e}_{1+i} := \bar{u} - \hat{u}_{1+i} = \underbrace{\bar{u} - \hat{u}_{1+i-1}}_{\hat{e}_{1+i-1}} - (\hat{A})^{-1} A \hat{e}_{1+i-1} = (I - \hat{A}^{-1}A)\hat{e}_{1+i-1}$

$$\boxed{\hat{e}_{1+m} = (I - \hat{A}^{-1}A)^m \hat{e}_1}$$

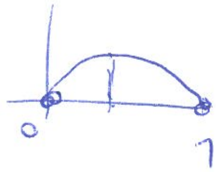
$\|\hat{e}_{1+m}\|^2_A = \|\sum_i (1-\lambda_i)^m b_i \bar{\varphi}_i\|^2_A = \sum_i \lambda_i (1-\lambda_i)^{2m} (b_i)^2$

Smooth. prop. $\forall \bar{v}: \|\bar{v}\|_A \le \|\bar{v}\|_{\hat{A}} \iff \lambda_{max}(\hat{A}^{-1}A) \le 1$

$$\cancel{\bar{v}A\bar{v} \le \bar{v}\hat{A}\bar{v}}$$

tj. $\lambda_i \in (0,1\rangle$

tj. $\max_i \lambda_i(1-\lambda_i)^{2m} \le \max_{\lambda\in(0,1)} \underbrace{\lambda(1-\lambda)^{2m}}_{=:f(\lambda)}$



$$f'(\lambda) = (1-\lambda)^{2m} - 2m\lambda(1-\lambda)^{2m-1} \overset{!}{=} 0$$

$$1-\lambda = 2m\lambda$$

$$\lambda = \frac{1}{1+2m} : f(\lambda) = \frac{1}{1+2m}\left(1-\frac{1}{1+2m}\right)^{2m} =: \gamma(m)$$

$$\|\hat{e}_{1+m}\|_A^2 \le \gamma(m)\cdot\sum(b_i)^2 = \gamma(m)\cdot\|\hat{e}_1\|_{\hat{A}}^2 = \gamma(m)\|(I-P^0)\bar{e}_0\|_{\hat{A}}^2$$

$$\le K\gamma(m)\|(I-P^0)\bar{e}_0\|_A^2 \quad \text{approx. prop. } \|(I-P^0)v\|_{\hat{A}} \le K\|(I-P^0)v\|_A$$

(Projekce)

$$\le K\gamma(m)\|\bar{e}_0\|_A^2$$

$\square$

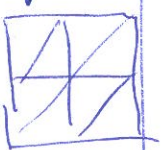
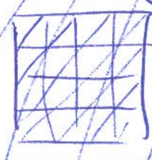
<u>Smoothing property</u> $\|\bar{v}\|_A \le \|\bar{v}\|_{\hat{A}}$

nejjednodušší např. $\hat{A} := \frac{1}{\Lambda} I$, kde $\lambda_{max}(A) \le \Lambda$

<u>Approximation property</u> $\|(I-P^0)\bar{v}\|_{\hat{A}} \le K\|(I-P^0)\bar{v}\|_A$

plyne pro H^2-regulární úlohy z Aubin-Nitscheho lemma.
(příště)

$\boxed{\text{Metoda více sítí (multigrid)}}$

V^0  c V^1  V^L : $A^L \bar{u}^L = \bar{b}^L$

$\bar{u}^\ell := M^\ell(\bar{r}_0^\ell)$:

if $\ell = 0$

$\bar{u}^\ell := (A^0)^{-1}\bar{r}_0^0$

else

~~$\bar{u}_1^\ell := I^{\ell-1} M^{\ell-1}((I^{\ell-1})^T \bar{r}_0^\ell$~~

for $i = 1:p$ (coarse grid c.)

$\bar{u}_i^\ell := \bar{u}_{i-1}^\ell + I^{\ell-1} M^{\ell-1}((I^{\ell-1})^T \bar{r}_{i-1}^\ell)$

$\bar{r}_i^\ell := \bar{r}_{i-1}^\ell - A^\ell \bar{u}_i^\ell$

end for

for $i = 1:m$ (smoothing)

$\bar{u}_{p+i}^\ell := \bar{u}_{p+i-1}^\ell + (\hat{A}^\ell)^{-1}\bar{r}_{p+i-1}^\ell$

$\bar{r}_{p+i}^\ell := \bar{r}_{p+i-1}^\ell - A^\ell \bar{v}_{p+i-1}^\ell$

end for

Důsledek Aubin–Nitscheho lemma: approx. property

$\hat{A} := \Delta \cdot I$, kde $\Delta \geq \lambda_{max}(A)$, ~~2d úloha~~ dimenze $d=2,3$

$\Downarrow$

viz čísla podmíněnosti M_Ω A

$\|\bar{w}\|^2_{\hat{A}} = \bar{w}\hat{A}\bar{w} = \Delta \cdot \|\bar{w}\|^2 \leq \frac{C_A h^{d-2}}{c_M h^d} \int_\Omega [w(x)]^2$

$\bar{w} = \bar{u} - P\bar{u}$

$\bar{u}$ .. MKP-souřadnice $u^1(x)$ řešící úlohu na $V^1 \equiv V$

$P\bar{u}$ .. MKP-souřadnice $u^0(x)$ = Galerkinova projekce $u^1(x)$ na $I^0(V^0)$

Tedy $\|(I-P)\bar{u}\|^2_{\hat{A}} \leq \frac{C_A}{c_M h^2} \int_\Omega [u^1(x) - u^0(x)]^2\, dx$ $\quad \bar{u}A\bar{u} = |a(u,u)| \geq m\|u\|_1^2$

Důsledek A-N lemma

$\leq \frac{C_A}{c_M h^2} M^2 C^2 \underbrace{(2h)^2}_{H^2} \|u^1 - u^0\|_1 \leq \underbrace{\frac{4 C_A M^2 C^2}{c_M}}_{=:K} \|(I-P)\bar{u}\|^2_A$

$\square$

Zaměníme-li v 2-grid algoritmu pořadí korekce na hrubé síti a shlazování na jemné síti, pak ~~platí~~ lze ukázat, že platí: $\bar{e}_{k+1} = (I-P)(I-\hat{A}^{-1}A)^m \bar{e}_k$

a $\boxed{\|\bar{e}_{k+1}\|_A \leq \frac{K}{\sqrt{m}} \|\bar{e}_k\|_A}$

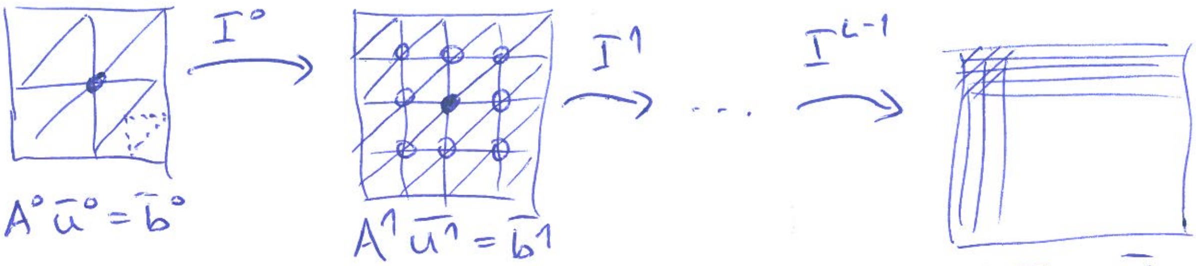
$\boxed{\hat{q} := T(\bar{r}_0)}$

```
ū_0 := 0̄
for i=1:m
    ū_i := ū_{i-1} + Â^{-1} r̄_{i-1}      (=: v̄_i = Â^{-1} r̄_{i-1})
    r̄_i := r̄_{i-1} - A v̄_i
end for
û_{m+1} := ū_m + I^0 (A^0)^{-1} (I^0)^T r̄_m
q̂ := ū_{m+1}
```

Metoda více sítí (multigrid)

$V^0 \subset V^1 \subset \dots \subset V^L$

$V^0 : A^0$ $\quad$ $V^1 : A^1, \hat{A}^1$ $\quad$ $V^L : \underset{\equiv A}{A^L}, \underset{\equiv \hat{A}}{\hat{A}^L}$



$A^0 \bar{u}^0 = \bar{b}^0$ $\quad$ $A^1 \bar{u}^1 = \bar{b}^1$ $\quad$ $A^L \bar{u}^L = \bar{b}^L$, $\quad \bar{u}^L \equiv \bar{u}$, $\bar{b}^L \equiv \bar{b}$

Multigrid: $\bar{v} := M_p^\ell(\bar{r}_0)$

if $\ell = 0$

$\quad \bar{v} := (A_0)^{-1} \bar{r}_0$

else

$\quad \bar{u}_0 := \bar{o}$

$\quad$ for $i = 1:m$

$\quad\quad \bar{u}_i := \bar{u}_{i-1} + \underbrace{(\hat{A}^\ell)^{-1} \bar{r}_{i-1}}_{=: \bar{v}_i}$

$\quad\quad \bar{r}_i := \bar{r}_{i-1} - A^\ell \bar{v}_i$

$\quad$ end for

$\quad$ for $i = 1:p$

$\quad\quad \bar{u}_{m+i} := \bar{u}_{m+i-1} + \underbrace{I^{\ell-1} M_p^{\ell-1}((I^{\ell-1})^T \bar{r}_{m+i-1})}_{=: \bar{v}_i}$

$\quad\quad \bar{r}_{m+i} := \bar{r}_{m+i-1} - A^\ell \bar{v}_i$

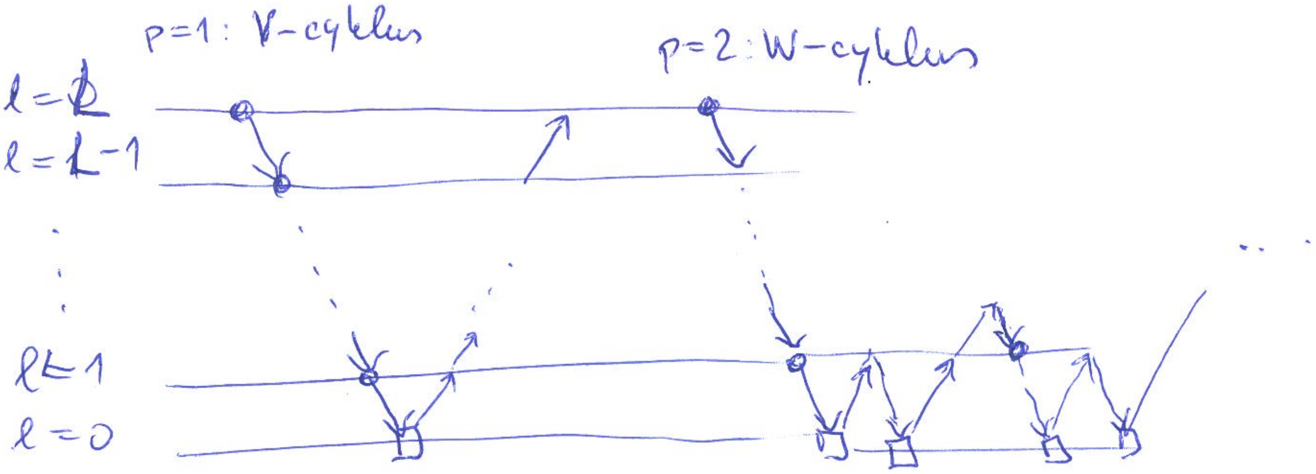
$\quad$ end for

$\quad \bar{v} := \bar{u}_{m+p}$

$\bar{u}_{m+p} := \bar{u}_m + \bar{q}_p$

Richardson

$\bar{u}_{k+1} := \bar{u}_k + M_p^\ell(\bar{r}_k)$



Platí:

$$\|\hat{q} - \bar{q}_1^\ell\|_A = \| I^{\ell-1} \{ M_p^{\ell-1} - (A^{\ell-1})^{-1} \} (I^{\ell-1})^T \bar{r}_m^\ell \|_A$$

$$= \|(M_p^{\ell-1} A^{\ell-1} - I)\, \bar{e}_m^{\ell-1}\|_A$$

$$\|\hat{q} - \bar{q}_p^\ell\|_A = \|(M_p^{\ell-1} A^{\ell-1} - I)^p\, \bar{e}_m^{\ell-1}\|_A$$

Věta. (konvergence multigridu)

$$\left.\begin{array}{l} \circ\ p \geqq 2 \\ \circ\ \gamma \in (0,1) \end{array}\right\} \Rightarrow \forall m \geqq \left\lceil \frac{K}{\gamma - \gamma^p} \right\rceil^2 : \|\bar{e}_{k+1}^\ell\| \leqq \gamma \|\bar{e}_k^\ell\|_A$$

nezávisle na ℓ.

Důkaz. (indukcí)

$\ell = 0$: $\bar{e}_{k+1}^\ell = \bar{o}$

$\ell > 0$: $\bar{e}_{k+1}^\ell = \bar{u}^\ell - \bar{u}_{m+p}^\ell = \bar{u}^\ell - \bar{u}_m^\ell - \bar{q}_p^\ell \pm \hat{q}^\ell$

$$\|\bar{e}_{k+1}^\ell\|_A \leqq \underbrace{\|\bar{u}^\ell - \bar{u}_m^\ell - \hat{q}^\ell\|_A}_{\leqq \frac{K}{\sqrt{m}}\|\bar{e}_k^\ell\|_A} + \underbrace{\|\hat{q}^\ell - \bar{q}_p^\ell\|_A}_{\leqq \gamma^p \cdot \|\bar{e}_k^\ell\|_A} \leqq \gamma \|\bar{e}_k^\ell\|$$

konvergence 2-gridu

indukční předpoklad + předchozí pozorování

□

Výpočetní náročnost 1 iterace multigridu

$$W^k \leqq C \cdot m \cdot n^k + p \cdot W^{k-1} \leqq C m n^k + p\left[C m n^{k-1} + p W^{\ell-2}\right] \leqq \cdots$$

$$\leqq C m \left[n^k + p n^{k-1} + \cdots + p^{\ell-1} n^1 + K\right]$$

CPU$((A^0)^{-1})$

Předp. $\cancel{n^\ell}$ $n^\ell \geqq \underset{\geqq 2}{D} \cdot n^{\ell-1}$

$$\leqq C m n^L \underbrace{\left[1 + \left(\frac{p}{D}\right) + \left(\frac{p}{D}\right)^2 + \cdots \left(\frac{p}{D}\right)^{L-1}\right]}_{\leqq \sum_{i=0}^{\infty} q^i \in \mathbb{R} \text{ pro } q := \boxed{\frac{p}{D} < 1}} + C m K$$

A tedy $\boxed{\mathrm{CPU}(M_p^k(..)) = O(n^L)}$