Gaussova eliminace, maticový počet

Př.: $2x_1 - x_2 = 1$
$1x_1 + x_2 = 2$ ($\bar b$)

1. rov.: / 2. rov.:
$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 2 \end{array}\right) \quad \bar r_2 := 2\bar r_2 - \bar r_1 \quad \left(\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 3 & 3 \end{array}\right)$$
(x_1, x_2, $\bar b$; $2 \neq 0$)

$2x_1 - x_2 = 1$
$3x_2 = 3$

$x_2 = 1$

Zpětně: 1. rov.: $2x_1 - 1 = 1$
$2x_1 = 2$
$x_1 = 1$

Př.: $x_1 + x_2 - x_3 = 1$
$x_1 - x_2 + 2x_3 = 2$
$x_1 - 3x_2 + 5x_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & -3 & 5 & 0 \end{array}\right) \begin{array}{c} \\ -\bar r_1 \\ -\bar r_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & -4 & 6 & -1 \end{array}\right) \quad \bar r_3 := \bar r_3 - 2\bar r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

3. rovnice: $0 = -3$ **Nemá řešení!**

Př.: $x_1 + x_2 - x_3 = 1$
$x_1 - x_2 + 2x_3 = 2$
$x_1 - 3x_2 + 5x_3 = 3$

$(\bar r_3 := -\bar r_3 + 2\bar r_2)$

$\bar r_3 := \bar r_3 - 2\bar r_2$ ($0-2\cdot 0$; $-4-2\cdot(-2)$)

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & -3 & 5 & 3 \end{array}\right) \begin{array}{c} \\ -\bar r_1 \\ -\bar r_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & -4 & 6 & 2 \end{array}\right) -2\bar r_2 \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

3. rov.: $0 = 0$ ✓

2 rovnice o 3 neznámých: $x_1 + x_2 - x_3 = 1$

$-2x_2 + 3x_3 = 1$

$6 - 2 \cdot 3$

Průnik 2 rovin

Parametrizace

Zvolím parametr

$x_3 = t \in \mathbb{R}$

Zpětné dosazování:

2. rov.: $-2x_2 + 3t = 1$

$$x_2 = \frac{1-3t}{-2} = -\frac{1}{2} + \frac{3}{2}t, \quad t \in \mathbb{R}$$

1. rov.: $x_1 + \left(-\frac{1}{2} + \frac{3}{2}t\right) - t = 1$

$$x_1 = \frac{3}{2} - \frac{1}{2}t, \quad t \in \mathbb{R}$$

$$\bar{x} = \begin{pmatrix} 3/2 - 1/2\,t \\ -1/2 + 3/2\,t \\ 0 + t \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

parametrická rovnice přímky ve 3d

$t = 0$: $\bar{x} = \begin{pmatrix} 3/2 \\ -1/2 \\ 0 \end{pmatrix}$ je řešení

$t = 1$: $\bar{x} = \begin{pmatrix} 3/2 - 1/2 \\ -1/2 + 3/2 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Př.: $1x_1 + 1x_2 + x_3 = 1$

$(1, 1, 1 \mid 1)$

G.e. ✓

Parametrizovat

$x_3 = t \in \mathbb{R}$

$x_2 = s \in \mathbb{R}$

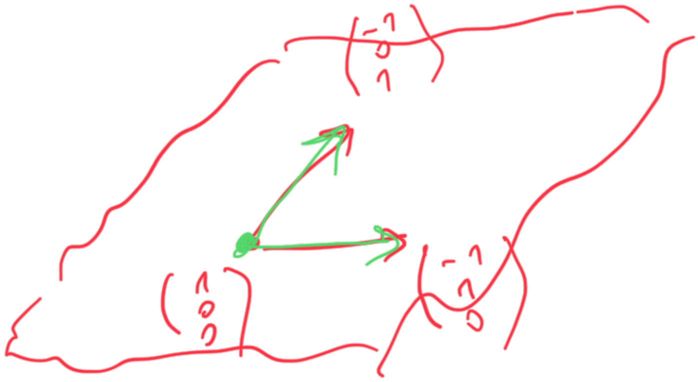
Zp. dos.: $x_1 + s + t = 1$

$x_1 = 1 - s - t$

$$\bar{x} = \begin{pmatrix} 1 & -1s & -1t \\ 0 & +1s & +0t \\ 0 & +0s & +1t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$s, t \in \mathbb{R}$

parametrický popis roviny



Př. $0 + 2x_2 - x_3 = 1$

$x_1 \quad + x_3 = 0$

$0 + x_2 - x_3 = 1$

Násobení matice krát vektor

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$a x = b$

$\mathbb{A} \cdot \bar{x} = \bar{b}$

$\bar{u} = \bar{v}$

$\Updownarrow$

$\bar{u}, \bar{v} \in \mathbb{R}^n$

$\forall i \in \{1..n\}: u_i = v_i$

$(a_{11}\ a_{12} \dots a_{1n})$ $(x_1$ def.

$$m\left\{\begin{pmatrix} \vdots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\right. \cdot \begin{pmatrix} \vdots \\ x_n \end{pmatrix} = x_1 \bar{a}_1^s + x_2 \bar{a}_2^s + \cdots + x_n \bar{a}_n^s$$

(lin. kombinace sloupců)

$$A \cdot x$$

$$= (\bar{a}_1^s, \bar{a}_2^s, \ldots, \bar{a}_n^s) = \begin{pmatrix} \bar{a}_1^r \\ \vdots \\ \bar{a}_m^r \end{pmatrix}$$

$$= \begin{pmatrix} \bar{a}_1^r \cdot \bar{x} \\ \bar{a}_2^r \cdot \bar{x} \\ \vdots \\ \bar{a}_m^r \cdot \bar{x} \end{pmatrix} \quad \text{.. praktičtější}$$

Př.
$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

1) $= (-3) \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \\ 1 \end{pmatrix}$

2) $= \begin{pmatrix} (2,1) \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ (1,-1) \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ (1,2) \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 \cdot (-3) + 1 \cdot 2 \\ 1 \cdot (-3) - 1 \cdot 2 \\ 1 \cdot (-3) + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \\ 1 \end{pmatrix}$ ✓

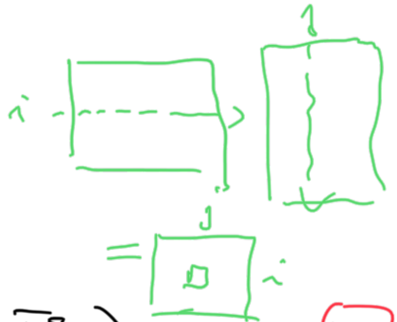
Násobení matice krát matice

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \end{pmatrix}$$

$$\begin{pmatrix} \vdots & & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \quad \begin{pmatrix} \vdots & & \vdots \\ b_{n1} & b_{n2} \cdots & b_{np} \end{pmatrix}$$

$A \in \mathbb{R}^{m \times n}$ $\qquad B \in \mathbb{R}^{n \times p}$

1) $$= \begin{pmatrix} \overline{a}_1^r \cdot B \\ \overline{a}_2^r \cdot B \\ \vdots \\ \overline{a}_m^r \cdot B \end{pmatrix} \in \mathbb{R}^{m \times p}$$



2) $$= \left(A \cdot \overline{b}_1^s, A \cdot \overline{b}_2^s, \dots, A \cdot \overline{b}_p^s \right) \in \mathbb{R}^{m \times p}$$

3) Praktický:

$$\begin{pmatrix} \overline{a}_1^r \cdot \overline{b}_1^s, & \overline{a}_1^r \cdot \overline{b}_2^s, & \dots, & \overline{a}_1^r \cdot \overline{b}_p^s \\ \overline{a}_2^r \cdot \overline{b}_1^s, & \overline{a}_2^r \cdot \overline{b}_2^s, & \dots, & \overline{a}_2^r \cdot \overline{b}_p^s \\ \vdots & \vdots & & \vdots \\ \overline{a}_m^r \cdot \overline{b}_1^s, & \overline{a}_m^r \cdot \overline{b}_2^s, & \dots, & \overline{a}_m^r \cdot \overline{b}_p^s \end{pmatrix}$$

4) fajnový:

$$\overline{a}_1^s \cdot \overline{b}_1^r + \dots + \overline{a}_n^s \cdot \overline{b}_n^r$$

Kroneckerův součin: $\overline{u} \cdot \overline{v} = \begin{pmatrix} u_1 v_1, & u_1 v_2, & \dots, & u_1 v_p \\ u_2 v_1, & u_2 v_2, & \dots, & u_2 v_p \\ \vdots & \vdots & & \vdots \\ u_m v_1, & u_m v_2, & \dots, & u_m v_p \end{pmatrix}$

Př. $$\underset{A}{\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 1 \end{pmatrix}} \cdot \underset{B}{\begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}}$$

1) $$= \begin{pmatrix} (1,2) \cdot B \\ (-1,1) \cdot B \\ (2,1) \cdot B \end{pmatrix} = \begin{pmatrix} (1,2) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}, & (1,2) \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ (-1,1) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}, & (-1,1) \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ (2,1) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}, & (2,1) \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -2, & 6 \\ -1 & 0 \end{pmatrix}$$

$$\begin{matrix} \dots \\ -1 , 6 \end{matrix}$$

$$\overset{2)}{=} \left(A \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \,\middle|\, A \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right)$$

$$= \left(\begin{array}{c|c} (1,2)\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix} & (1,2)\cdot\begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ (-1,1)\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix} & (-1,1)\cdot\begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ (2,1)\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix} & (2,1)\cdot\begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{array} \right)$$

$$\overset{3)}{=} \begin{pmatrix} (1,2)\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix}, & (1,2)\cdot\begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ (-1,1)\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix}, & (-1,1)\cdot\begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ (2,1)\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix}, & (2,1)\cdot\begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{pmatrix}$$

Soustavy lin. rovnic s více pravými stranami

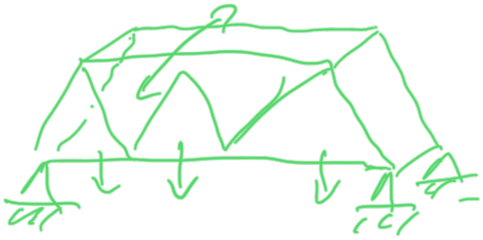
Př.

$$\begin{array}{l} 1u_1 - 1u_2 = 1 \\ 2u_1 + 1u_2 = 0 \end{array} \qquad \begin{array}{l} 1v_1 - 1v_2 = 2 \\ 2v_1 + 1v_2 = 1 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 2 \\ 2 & 1 & 0 & 1 \end{array} \right) \underset{r_2 := r_2 - 2r_1}{\sim} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 2 \\ 0 & 3 & -2 & -3 \end{array} \right) \; r_1 := 3r_1 + r_2$$

$$\sim \left(\begin{array}{cc|cc} 3 & 0 & 1 & 3 \\ 0 & 3 & -2 & -3 \end{array} \right) \begin{array}{l} \cdot 1/3 \\ \cdot 1/3 \end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 1 \\ 0 & 1 & -2/3 & -1 \end{array} \right)$$

I (the left block); $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix}$, $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Inverzní matice

$2x = 1 \qquad ax = b \qquad\qquad A \cdot \bar{x} = \bar{b}$

$\Downarrow$ $a \neq 0$

$$x = a^{-1} \cdot b$$

$\Updownarrow$ A regulární

$$\bar{x} = A^{-1} \cdot \bar{b}$$

Def. Mějme $A \in \mathbb{R}^{n \times n}$.

Pokud existuje $X \in \mathbb{R}^{n \times n}$: $X \cdot A = I = \begin{pmatrix} 1 & & 0 \\ 0 & 1 & \\ \vdots & & \ddots \\ 0 & 0 & & 1 \end{pmatrix}$,

pak říkáme, že A je regulární ($A^T \cdot X^T = I$)

a $A^{-1} := X$ je inverzní matice k A.

Platí:

$$X \cdot A = I \iff A \cdot X = I$$

(ekvivalentní definice)

Výpočet X $\equiv$ řešení soustav s více pravými str.

$$A \cdot [\bar{x}_1^s, \bar{x}_2^s, \dots, \bar{x}_n^s] = \left[\begin{pmatrix}1\\0\\ \vdots \\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\ \vdots \\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\ \vdots \\0\\1\end{pmatrix}\right]$$

Př. Vypočítejte $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1}$.

$$\left(\begin{array}{cc|c} 1 & 2 & \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \\ 2 & 1 & \end{array}\right) \xrightarrow[r_2 := 2r_1 - r_2]{\text{Gauss-elim.}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{array}\right)$$

Jordan $\sim$ $r_1 := 3r_1 - 2r_2$

$$\left(\begin{array}{cc|cc} 3 & 0 & -1 & 2 \\ 0 & 3 & 2 & -1 \end{array}\right) \begin{matrix} / \cdot 1/3 \\ / \cdot 1/3 \end{matrix} \sim \left(\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right)$$

$\Downarrow$ A^{-1}

Zk.: $A^{-1} \cdot A = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & \dots \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ \dots \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad ✓$$

2) Řešte pomocí A^{-1} úlohu $\quad x_1 + 2x_2 = 1$, $\quad 2x_1 + x_2 = 2$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\boxed{\bar{x} = A^{-1} \cdot \bar{b}} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/3 + 4/3 \\ 2/3 - 2/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Zk. $A \cdot \bar{x} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 \\ 2 \cdot 1 + 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \bar{b}$ ✓

Souvislost maticového počtu s Gaussovou eliminací (LU rozklad)

Př.:

$$\underbrace{\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & 1 & 1 & -2 \\ 0 & 1 & 2 & 2 \end{array}\right)}_{(A|\bar{b})} \quad r_2' := -1 r_2 - 2 r_1 \quad \sim \quad \underbrace{\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & -3 & -1 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right)}_{(A_1|\bar{b}_1)} \sim \quad r_3' := 3 r_3' + r_2'$$

lin. kombinace řádků

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$r_1' := 1 r_1 + 0 r_2 + 0 r_3$

$r_3' := 0 r_1 + 0 r_2 + 1 r_3$

$$L_1 \cdot (A|\bar{b}) = (A_1|\bar{b}_1)$$

(· realizuje lineár. kombinace řádků)

$$r_3' := 3 r_3' + r_2' \qquad \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 5 & 6 \end{array}\right)$$

$\sim \quad \begin{pmatrix} 1 & 0 & 0 \\ \ldots \end{pmatrix}$

$L_2 = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 3 \end{array}\right)$ $\quad (U \mid \bar{c})$

lower triangular $\quad$ upper triangular

$\underbrace{\tilde{L}_2 \cdot \tilde{L}_1}_{\tilde{L}} \cdot (A \mid \bar{b}) = (U \mid \bar{c})$

$\tilde{L} \cdot A = U \Leftrightarrow A = L \cdot U$

$\tilde{L}^{-1} = L$