Gaussova eliminační metoda

Př. $2x_1 + x_2 = 1$

$x_1 - 2x_2 = 0 \quad / \cdot 2$

$2x_1 + x_2 = 1$

$2x_1 - 4x_2 = 0 \quad / -r_1$

$$\boxed{\begin{array}{l} 2x_1 + x_2 = 1 \\ \cancel{0x_1} \; -5x_2 = -1 \end{array}}$$

G.E. ✓

z 2. rovnice: $x_2 = 1/5$

Zpětné dosazování:

z 1. rovnice: $2x_1 + 1/5 = 1$

$2x_1 = 4/5$

$x_1 = 2/5$

2. Násobení rovnice $\boxed{\alpha \neq 0}$

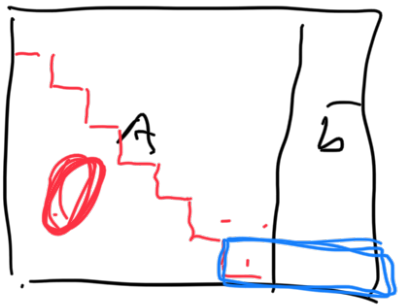
$x = 2 \quad / \cdot 2$

$2x = 4 \quad \cancel{\cdot 0}$

1. Přičtení jiné rovnice

$r_i := \alpha_1 r_1 + \alpha_2 r_2 + \dots + \alpha_n r_n$

$\boxed{\alpha_i \neq 0}$



Zkouška: 1.r.: $2x_1 + x_2 = 1$: $2 \cdot 2/5 + 1/5 = 4/5 + 1/5 = 1$ ✓

2.r.: $x_1 - 2x_2 = 0$: $2/5 - 2 \cdot 1/5 = 2/5 - 2/5 = 0$ ✓

Řešení soustav lin. rovnic Gauss. elim. metodou

Př.

$$\begin{array}{l} 1x_1 - 2x_2 + 1x_3 = 1 \\ 2x_1 - 1x_2 + 1x_3 = -1 \\ 1x_1 - 1x_2 - 1x_3 = 2 \end{array} \iff \begin{array}{l} x_1 - 2x_2 + x_3 = 1 \\ \quad \square x_2 + \square x_3 = \square \\ \qquad \square x_3 = \square \end{array}$$

tzv. (schodový tvar)

Ekvivalentní úpravy

1. Přičtení jiné rovnice k dané rovnici:

$$r_i := r_i + r_j \ , \ i \neq j$$

2. Vynásobení rovnice nenulovým číslem

$$r_2 := r_2 - 2r_1$$

$$x_1 - 2x_2 + x_3 = 1$$

$$\underbrace{(2x_1 - 2x_1)}_{0\ ✓} \underbrace{- x_2 + 4x_2}_{3x_2} + \underbrace{x_3 - 2x_3}_{-x_3} = \underbrace{-1-2}_{-3}$$

$$\begin{aligned} \boxed{x_1} - 2x_2 + x_3 &= 1 \\ 3x_2 - x_3 &= -3 \\ x_1 - x_2 - x_3 &= 2 \end{aligned}$$

$$r_3 := r_3 - r_1$$

$$x_1 - 2x_2 + x_3 = 1$$

$$3x_2 - x_3 = -3$$

$$\underbrace{x_1 - x_1}_{=0\ ✓} \underbrace{- x_2 + 2x_2}_{x_2} \underbrace{- x_3 - x_3}_{-2x_3} = \underbrace{2-1}_{1}$$

$$\Longleftrightarrow$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ \boxed{3x_2} - x_3 &= -3 \\ x_2 - 2x_3 &= 1 \end{aligned}$$

$$r_3 := 3r_3 - r_2$$

$$x_1 - 2x_2 + x_3 = 1$$

$$3x_2 - x_3 = -3$$

$$\underbrace{3x_2 - 3x_2}_{0\ ✓} \underbrace{-6x_3 + x_3}_{-5x_3} = \underbrace{3-(-3)}_{6}$$

$$\Longleftrightarrow \quad \begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ 3x_2 - x_3 &= -3 \\ -5x_3 &= 6 \end{aligned}$$

g.e. ✓

Zpětné dosazování

3. rov.: $x_3 = -\frac{6}{5}$

Dosadím do 2. rov.: $3x_2 - \left(-\frac{6}{5}\right) = -3$

$$3x_2 = -\frac{15}{5} - \frac{6}{5} = -\frac{21}{5}$$

$$x_2 = -\frac{7}{5}$$

Dosadím do 1. rov.: $x_1 - 2\cdot\left(-\frac{7}{5}\right) + \left(-\frac{6}{5}\right) = 1$

$$x_1 = 1 - \frac{14}{5} + \frac{6}{5}$$

$$\underline{\underline{x_1}} = \frac{5-14+6}{5} = \underline{\underline{-\frac{3}{5}}}$$

Zkouška. Dosadím do soustavy ze zadání:

1. rov.: $x_1 - 2x_2 + x_3 = 1$: $-\frac{3}{5} - 2\cdot\left(-\frac{7}{5}\right) + \left(-\frac{6}{5}\right) = \frac{-3+14-6}{5} = \frac{5}{5} = 1$ ✓

2. rov.: $2x_1 - x_2 + x_3 = -1$: $2\cdot\left(-\frac{3}{5}\right) - \left(-\frac{7}{5}\right) + \left(-\frac{6}{5}\right) = \frac{-6+7-6}{5} = -\frac{5}{5} = -1$ ✓

3. rov.: $x_1 - x_2 - x_3 = 2$: $-\frac{3}{5} - \left(-\frac{7}{5}\right) - \left(-\frac{6}{5}\right) = \frac{-3+7+6}{5} = \frac{10}{5} = 2$ ✓

Vše jsme mohli provádět pouze s koeficienty soustavy tzv. rozšířená matice soustavy

Pivot

$$\begin{array}{r|c}
 & \begin{array}{ccc|c} x_1 & x_2 & x_3 & \vec{b} \end{array} \\
\begin{array}{r} \text{1. rov.:} \\ \text{2. rov.:} \\ \text{3. rov.:} \end{array} & \left(\begin{array}{ccc|c} \boxed{1} & -2 & 1 & 1 \\ 2 & -1 & 1 & -1 \\ 1 & -1 & -1 & 2 \end{array}\right)
\end{array}
\qquad
\begin{array}{l} \check{r}_2 := \check{r}_2 - 2\check{r}_1 \\ \check{r}_3 := \check{r}_3 - \check{r}_1 \end{array}$$

$n = 3$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & \boxed{3} & -1 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right)
\qquad
\check{r}_3 := 3\check{r}_3 - \bar{r}_2$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 3 & -1 & -3 \\ 0 & 0 & -5 & 6 \end{array}\right)$$

Zpětné dos.: 3. rov.: $-5x_3 = 6 \Leftrightarrow \underline{\underline{x_3 = -\frac{6}{5}}}$

2. rov.: $3x_2 - \left(-\frac{6}{5}\right) = -3 \Leftrightarrow \underline{\underline{x_2 = -\frac{7}{5}}}$

1. rov.: $x_1 - 2\cdot\left(-\frac{7}{5}\right) + \left(-\frac{6}{5}\right) = 1$

Výpočetní složitost G.E. $\approx n^3$ vers.

Kramerovo pravidlo $\approx (n+1)!$ vers.

Kramer. pr. + Gauss. elim. $\approx n^4$

Př.
$$\begin{aligned} \boxed{0} \quad x_2 + x_3 &= 1 \\ x_1 - x_2 + x_3 &= 0 \\ 2x_1 - x_2 - x_3 &= -1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & -1 & -1 & -1 \end{array}\right) \underset{r_1 \leftrightarrow r_2}{\sim} \left(\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & -1 & -1 \end{array}\right) \underset{r_3 := r_3 - 2r_1}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 1 & -3 & -1 \end{array}\right) \underset{r_3 := r_3 - r_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -4 & -2 \end{array}\right)$$

Zp. dos.: 3. rov.: $-4x_3 = -2$

$x_3 = 1/2$

2. rov.: $x_2 + 1/2 = 1$

$x_2 = 1/2$

1. rov.: $x_1 - 1/2 + 1/2 = 0$

$x_1 = 0$

$x_1 = 1 - \frac{14}{5} + \frac{6}{5}$

$x_1 = -\frac{3}{5}$

Př.
$$\begin{aligned} x_2 &= 1 \\ x_1 + x_2 &= 1 \end{aligned}$$

3. Záměna řádků

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 &= 1 \end{aligned}$$

Zp. dos.

1. rov.: $x_1 + 1 =$

$x_1 = 0$

2x2:

a_2, b, a_1

$x_r + i x_{im}$ $y_r + i y_{im}$

Př.
$$\begin{aligned} (1+i)\,x + i\,y &= -i \\ i\,x + y &= -1 \end{aligned}$$

= reálná soustava 4x4

$\left(\boxed{1+i} \quad i \quad \middle| \; -i \right)$

$$\left(\begin{array}{cc|c} i & 1 & -1 \end{array}\right) \qquad \vec{r}_2 := (1+i)\vec{r}_2 - i\vec{r}_1$$

$$\left(\begin{array}{cc|c} 1+i & i & -i \\ 0 & 2+i & -2-i \end{array}\right)$$

$(1+i)i - i(1+i) = 0$ ✓

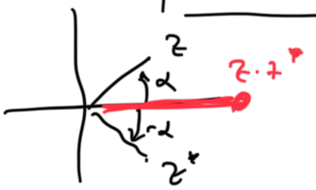
$(1+i)\cdot(-1) - i(-i) = -1 - i + \underbrace{i^2}_{=-1}$
$= -2 - i$

$(1+i)\cdot 1 - \underbrace{i \cdot i}_{=-1} = 1 + i - (-1) = 2 + i$

Zpětné dosazování

2\. rov.: $(2+i)\,y = -2-i$

$z = a+bi$: $\boxed{z \cdot z^* = a^2 + b^2}$



$y = \dfrac{-2-i}{2+i} \cdot \dfrac{2-i}{2-i} = \dfrac{-4+2i-2i-1}{2^2 + \underbrace{2i-2i}_{=0} + \underbrace{1^2}_{-i^2}} = \dfrac{-5}{5} = \underline{\underline{-1}}$

$(2+i)^* = 2-i$

1\. rov.: $(1+i)x + \cancel{i(-1)} = \cancel{-i}$

$(1+i)x = 0$

$\underline{\underline{x = 0}}$

Zk.: 1. r.: $(1+i)\cdot 0 + i(-1) = -i$ ✓

2\. r.: $i \cdot 0 + (-1) = -1$ ✓

$L = i \cdot x + y\,, \quad -1 = P$
$x = 0, \quad y = -1$

$\underbrace{i \cdot 0 + (-1)}\ = -1$

$= -1$ ✓

$L = P$ ✓