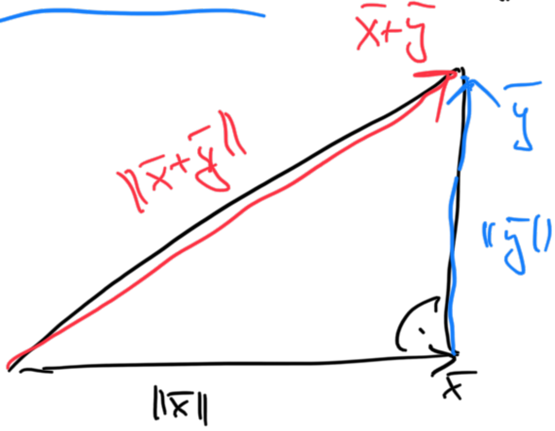
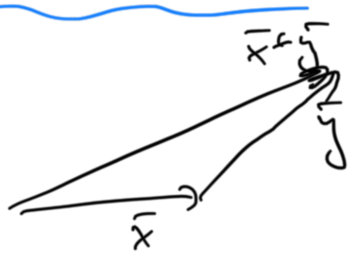
Ortogonalita

Pythagorova věta

$$\|\bar{x}\|^2 + \|\bar{y}\|^2 = \|\bar{x}+\bar{y}\|^2$$



Cosinová věta



$$\|\bar{x}+\bar{y}\|^2 = (\bar{x}+\bar{y})\cdot(\bar{x}+\bar{y})$$

$$= \bar{x}\cdot\bar{x} + \underbrace{\bar{y}\cdot\bar{x} + \bar{x}\cdot\bar{y}}_{=2\bar{x}\cdot\bar{y}} + \bar{y}\cdot\bar{y}$$

$$= \|\bar{x}\|^2 + 2\bar{x}\cdot\bar{y} + \|\bar{y}\|^2$$

$$\boxed{\|\bar{x}\| := \sqrt{\bar{x}\cdot\bar{x}}}$$

Vektory $\bar{x} \perp \bar{y}$ jsou ortogonální

$\Leftrightarrow$ def. $\quad \bar{x},\bar{y} \neq \bar{0}$

$$\underbrace{\boxed{\bar{x}\cdot\bar{y} = 0}}_{= x_1y_1 + x_2y_2 + \dots}$$

euklidovský skalární souči…

Př. Najděte vektor ortogonální k $\bar{y} = \begin{pmatrix}1\\2\end{pmatrix}$.

Hledám $\bar{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$: $\bar{x}\cdot\bar{y} = 0$ 1 rovnice

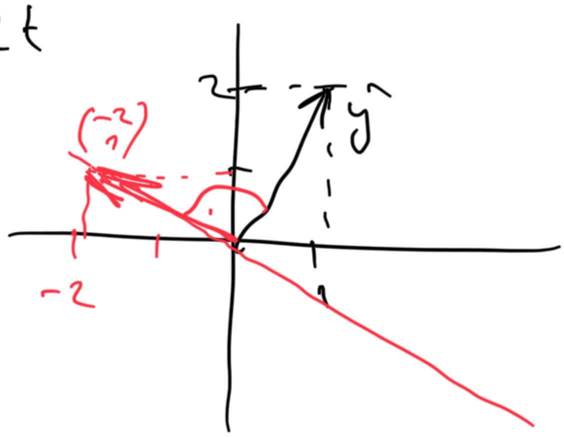
$\cdot\ -1 \qquad \| \ 0$ ∘ 2 nesvedu…

$1x_1 + 2x_2$

$x_2 = t \in \mathbb{R} \neq 0$

$x_1 = -2t$

$\bar{x} = t \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ t \neq 0$



Př. Najděte $\bar{x} \in \mathbb{R}^3 : \ \bar{x} \perp \bar{y} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

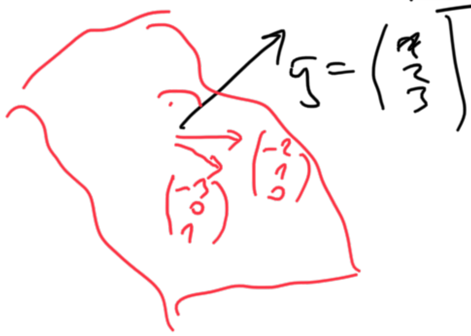
$\bar{x} \cdot \bar{y} = 0$

$$\boxed{1x_1 + 2x_2 + 3x_3 = 0}$$

$$\left.\begin{aligned} x_3 &= t \neq 0 \\ x_2 &= s \neq 0 \\ x_1 &= -2s - 3t \end{aligned}\right\} \quad \bar{x} = \begin{pmatrix} -2s - 3t \\ 1s + 0t \\ 0s + 1t \end{pmatrix} =$$

$$\bar{x} = s \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$s, t \in \mathbb{R}$



Př. Najděte $\bar{x} \in \mathbb{R}^3 : \ \bar{x} \perp \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \ \bar{x} \perp \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$$\boxed{\begin{aligned} -1x_1 + 1x_2 + 2x_3 &= 0 \\ 2x_1 + 1x_2 \qquad &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} \boxed{-1} & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right) +2r_1 \sim \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 3 & 4 & 0 \end{array}\right)$$

$$x_3 = t \neq 0$$

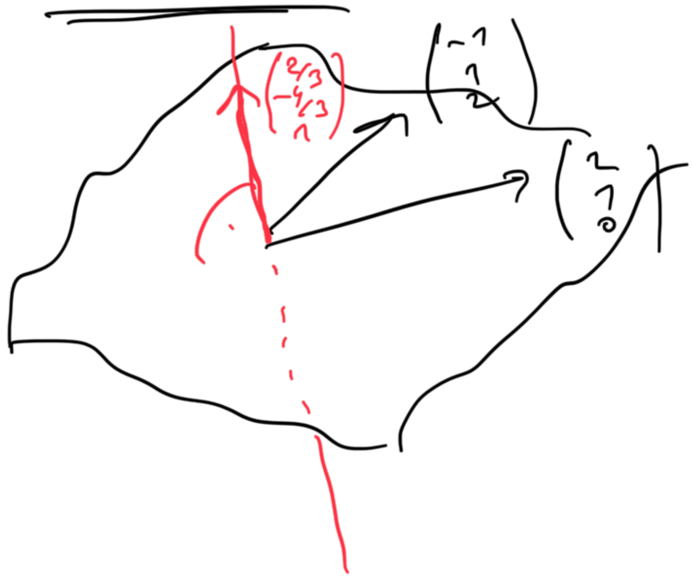
2. rovnice: $3x_2 = -4t$

$$x_2 = -\frac{4}{3}t$$

1. rovnice: $x_1 = -\frac{4}{3}t + 2t$

$$x_1 = \frac{2}{3}t$$

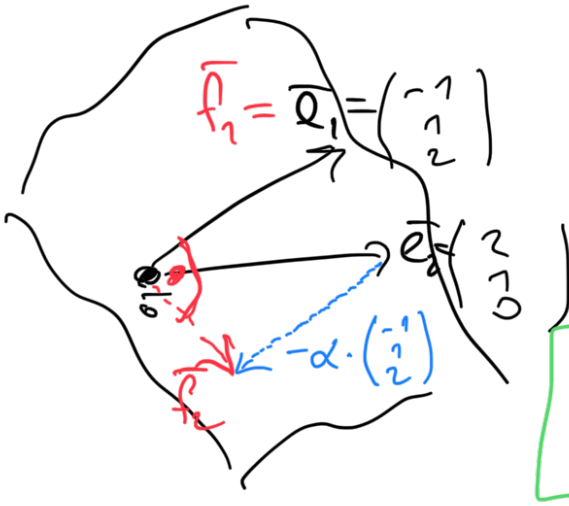
$$\bar{x} = t \cdot \begin{pmatrix} 2/3 \\ -4/3 \\ 1 \end{pmatrix}, \quad t \neq 0$$



Př. Ortogonalizujte $\left\langle \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ .. tzv. lin. obal

$$\Vert$$

$$\left\{ \bar{x} = \alpha_1 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$



$$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \not\perp \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} : -1\cdot 2 + 1\cdot 1 + 2\cdot 0 = -1 \neq 0$$

$$\boxed{\bar{f}_2 = \bar{e}_2 - \alpha \cdot \bar{e}_1}$$

kde $\alpha \in \mathbb{R}$: $\bar{f}_2 \perp \bar{e}_1$

tj. $\boxed{0 = \bar{f}_2 \cdot \bar{e}_1} = (\bar{e}_2 - \alpha \bar{e}_1) \cdot \bar{e}_1$

$$= \bar{e}_2 \cdot \bar{e}_1 - \alpha\, \bar{e}_1 \cdot \bar{e}_1$$

$$\boxed{\alpha = \frac{\bar{e}_2 \cdot \bar{e}_1}{\bar{e}_1 \cdot \bar{e}_1}}$$

$$\alpha = \frac{\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\2\end{pmatrix}}{\begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\2\end{pmatrix}} = \frac{2\cdot(-1)+1\cdot 1+0\cdot 2}{(-1)^2+1^2+2^2} = \frac{-1}{6}$$

$$\bar{f}_2 = \bar{e}_2 - \alpha\,\bar{e}_1 = \begin{pmatrix}2\\1\\0\end{pmatrix} + \frac{1}{6}\cdot\begin{pmatrix}-1\\1\\2\end{pmatrix} = \frac{1}{6}\begin{pmatrix}11\\7\\2\end{pmatrix}$$

$$\langle \bar{e}_1, \bar{e}_2\rangle = \left\langle \bar{f}_1 := \bar{e}_1 = \begin{pmatrix}-1\\1\\2\end{pmatrix},\ \bar{f}_2 = \frac{1}{6}\cdot\begin{pmatrix}11\\7\\2\end{pmatrix}\right\rangle$$

Zk. $\bar{f}_1\cdot\bar{f}_2 = \begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\frac{1}{6}\begin{pmatrix}11\\7\\2\end{pmatrix} = \frac{1}{6}\cdot(\underbrace{-11+7+4}_{=0})$ ✓

Př. Ortogonalizujte $\left\langle \bar{e}_1 := \begin{pmatrix}1\\1\\1\end{pmatrix}, \bar{e}_2 := \begin{pmatrix}-1\\1\\1\end{pmatrix}, \bar{e}_3 := \begin{pmatrix}2\\1\\0\end{pmatrix}\right\rangle$

Ř: $\bar{f}_1 := \bar{e}_1$, $\bar{f}_2 \in \langle \bar{e}_1, \bar{e}_2\rangle$, $\bar{f}_3 \in \langle \bar{e}_1, \bar{e}_2, \bar{e}_3\rangle$

$\bar{f}_1 \perp \bar{f}_2$ $\quad$ $\bar{f}_2 \perp \bar{f}_3$, $\bar{f}_1 \perp \bar{f}_3$

Gram-Schmidtův ortogonalizační proces

1. $\bar{f}_1 := \bar{e}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$

2. $\bar{f}_2 = \bar{e}_2 - \alpha\,\bar{f}_1 = \begin{pmatrix}-1\\1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\cdot\begin{pmatrix}-4\\2\\2\end{pmatrix}$

$$\alpha = \frac{\bar{e}_2\cdot\bar{f}_1}{\bar{f}_1\cdot\bar{f}_1} = \frac{\begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{1}{3}$$

$\bar{f}_2 \perp \bar{f}_1$ ✓

3. $\bar{f}_3 = \bar{e}_3 - \beta\,\bar{f}_1 - \gamma\,\bar{f}_2$

$$\bar f_3 \perp \bar f_1 \quad \boxed{: 0 = \bar f_3 \cdot \bar f_1 = \bar e_3 \cdot \bar f_1 - \beta\, \bar f_1 \cdot \bar f_1} - \cancel{\gamma\, \underbrace{\bar f_2 \cdot \bar f_1}_{=0}}$$

$$\boxed{\beta = \frac{\bar e_3 \cdot \bar f_1}{\bar f_1 \cdot \bar f_1}} = \frac{\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{3}{3} = 1$$

$$\bar f_3 \perp \bar f_2 : \boxed{0 = \bar f_3 \cdot \bar f_2 = \bar e_3 \cdot \bar f_2 - \cancel{\beta\, \underbrace{\bar f_1 \cdot \bar f_2}} - \gamma\, \bar f_2 \cdot \bar f_2}$$

$$\boxed{\gamma = \frac{\bar e_3 \cdot \bar f_2}{\bar f_2 \cdot \bar f_2}} = \frac{\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot \cancel{\frac{1}{3}}\begin{pmatrix}-4\\2\\2\end{pmatrix}}{\cancel{\frac{1}{3}}\begin{pmatrix}-4\\2\\2\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}-4\\2\\2\end{pmatrix}} = 3\cdot\frac{-8+2+0}{16+4+4}$$

$$= \cancel{3}\cdot\frac{-6}{24}$$

$$\bar f_3 = \begin{pmatrix}2\\1\\0\end{pmatrix} - 1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{3}{4}\frac{1}{\cancel{3}}\begin{pmatrix}-4\\2\\2\end{pmatrix} \qquad = -\frac{6}{8} = -\frac{3}{4}$$

$$\underline{\underline{\bar f_3}} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}-4\\2\\2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}-1\\1/2\\1/2\end{pmatrix} = \underline{\underline{\begin{pmatrix}0\\1/2\\-1/2\end{pmatrix}}}$$

$$\langle \bar e_1, \bar e_2, \bar e_3 \rangle = \underline{\underline{\left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}-4\\2\\2\end{pmatrix}, \begin{pmatrix}0\\1/2\\-1/2\end{pmatrix} \right\rangle}}$$

Př. Ortogonalizujte $\left\langle \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\rangle$

$$\bar f_1 := \bar e_1 = \begin{pmatrix}2\\1\end{pmatrix}, \quad \bar q_1 = \frac{\bar f_1}{\|\bar f_1\|} = \frac{1}{\sqrt{2^2+1^2}}\begin{pmatrix}2\\1\end{pmatrix}$$

$$\bar q_1 = \frac{1}{\sqrt5}\begin{pmatrix}2\\1\end{pmatrix}$$

$$\bar f_2 = \bar e_2 - \alpha\, \bar f_1 = \begin{pmatrix}1\\2\end{pmatrix} - \frac{4}{5}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-3/5\\6/5\end{pmatrix}$$

$\bar e_2 \cdot \bar f_1 \quad 2+2 \quad 4$

$$\alpha = \frac{\ldots}{\bar{f}_1 \cdot \bar{f}_1} = \frac{\ldots}{4+1} = \frac{1}{5}$$

$$\underline{\bar{q}_2} = \frac{\bar{f}_2}{\|\bar{f}_2\|} = \frac{1}{\sqrt{\frac{9}{25} + \frac{36}{25}}} \begin{pmatrix} -3/5 \\ 6/5 \end{pmatrix}$$

$$= \frac{\cancel{5}}{\sqrt{45}} \cdot \frac{1}{\cancel{5}} \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

$$\underline{\underline{Q = \left[\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{45}} \begin{pmatrix} -3 \\ 6 \end{pmatrix} \right]}} \quad : Q^T \cdot Q = \underline{I}$$