Bilineární formy

$B : V \times V \longrightarrow \mathbb{R}$

- lineární v 1. argumentu
- lineární ve 2. argumentu

Př. $B(\bar{u},\bar{v}) := u_1 v_1 + u_2 v_1 - 2 u_2 v_2$ na $\mathbb{R}^2$

$$= u_1 (1 v_1 + 0 v_2)$$
$$+ u_2 (1 v_1 - 2 v_2)$$
$$= (u_1, u_2) \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}}_{\mathbb{B}} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow B \text{ je bilineární forma}$$

Obecně

$B : V \times V \to \mathbb{R}$ .. bilineární forma

$E = (\bar{e}_1, \dots, \bar{e}_n)$ .. báze V

$\bar{u}, \bar{v} \in V \Rightarrow \exists! \bar{\alpha}, \bar{\beta} \in \mathbb{R}^n :$ $\bar{u} = \alpha_1 \bar{e}_1 + \dots + \alpha_n \bar{e}_n$, $\bar{v} = \beta_1 \bar{e}_1 + \dots + \beta_n \bar{e}_n$

$$\boxed{B(\bar{u},\bar{v}) = [\bar{u}]_E^T \cdot \mathbb{B}_E \cdot [\bar{v}]_E}$$

$$\| $$

$$B(\alpha_1 \bar{e}_1 + \dots + \alpha_n \bar{e}_n, \beta_1 \bar{e}_1 + \dots + \beta_n \bar{e}_n)$$

$$\| \text{ lin. v 2. argumentu}$$

$$\sum^n \beta_j \cdot B(\alpha_1 \bar{e}_1 + \dots + \alpha_n \bar{e}_n, \bar{e}_j)$$

$j=1$

$\Vert$ lin. v 1. argumentu

$$\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \mathbb{B}(\bar{e}_i, \bar{e}_j)\, \beta_j$$

$\Vert$

$$[\bar{u}]_E^T = \bar{\alpha}^T \cdot \underbrace{\begin{bmatrix} \mathbb{B}(\bar{e}_1,\bar{e}_1) & \mathbb{B}(\bar{e}_1,\bar{e}_2) & \dots & \mathbb{B}(\bar{e}_1,\bar{e}_n) \\ \mathbb{B}(\bar{e}_2,\bar{e}_1) & \mathbb{B}(\bar{e}_2,\bar{e}_2) & \dots & \mathbb{B}(\bar{e}_2,\bar{e}_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{B}(\bar{e}_n,\bar{e}_1) & \mathbb{B}(\bar{e}_n,\bar{e}_2) & \dots & \mathbb{B}(\bar{e}_n,\bar{e}_n) \end{bmatrix}} \cdot \bar{\beta} = [\bar{v}]_E$$

$\mathbb{B}_E$ .. matice bilineární formy vzhledem k bázi E

Změna matice $\mathbb{B}$ při změně báze

Změna souřadnic (při změně báze) je lineární transformace $\tau_{E,F}(\bar{\alpha}) = \underbrace{T_{E,F} \cdot [\bar{u}]_E}_{[\bar{u}]_F}$

$\bar{\alpha} = [\bar{u}]_E$

$E = (\bar{e}_1, \dots, \bar{e}_n) \to F = (\bar{f}_1, \dots, \bar{f}_n)$

$\bar{u} \in V$

$\underbrace{\bar{u} = \alpha_1 \bar{e}_1 + \dots + \alpha_n \bar{e}_n \qquad \bar{u} = \beta_1 \bar{f}_1 + \dots + \beta_n \bar{f}_n}$

$$\boxed{T_{E,F} \cdot \bar{\alpha} = \bar{\beta}}$$

$\Rightarrow \mathbb{B}(\bar{u}, \bar{v}) = [\bar{u}]_E^T \cdot \mathbb{B}_E \cdot [\bar{v}]_E$

$\mathbb{B}(\bar{u}, \bar{v}) = [\bar{u}]_F^T \cdot \mathbb{B}_F \cdot [\bar{v}]_F$

$\Vert$

$$(T\cdot[\vec u]_E)^T \cdot B_F \cdot T\cdot[\vec v]_E$$

$$[\vec u]_E^T\cdot T^T\cdot B_F\cdot T\cdot[\vec v]_E$$

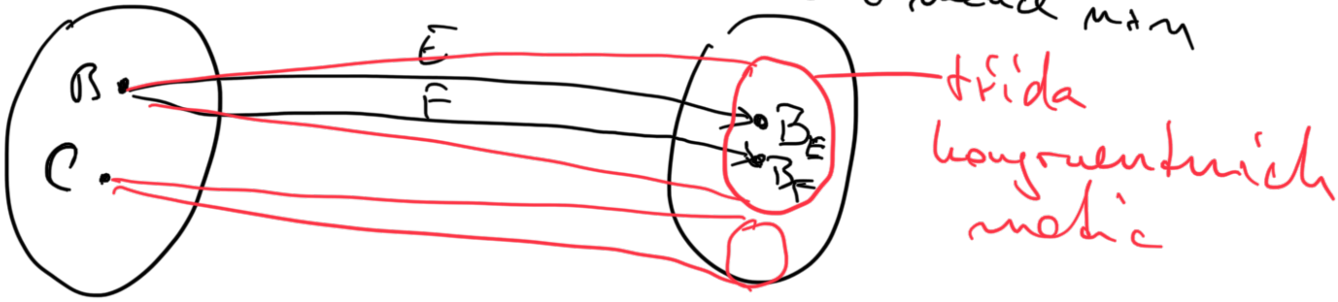
regulární matice

$$B_E = T^T\cdot B_F\cdot T$$

kongruentní matice

Bilineární f.ce V

čtvercové matice $n\times n$



Symetrické matice (bilineární formy)

$$\forall i,j:\ (B)_{ij} = (B)_{ji} \iff \boxed{B = B^T}$$

Př. a) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ je symetrická

b) $\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$

Antisymetrické matice

$$\forall i,j:\ (B)_{ij} = -(B)_{ji} \iff \boxed{B = -B^T}$$

Př. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ +2 & -3 & 0 \end{pmatrix}$

Věta: Každou čtvercovou lze jednoznačně

rozložit na součet symetrické a antisym.

$$\boxed{\mathbb{B} = \underbrace{\tfrac{1}{2}(\mathbb{B} + \mathbb{B}^T)}_{=\mathbb{B}^S} + \underbrace{\tfrac{1}{2}(\mathbb{B} - \mathbb{B}^T)}_{=\mathbb{B}^A}}$$

$\mathbb{B}^S$: $\frac{1}{2}(\mathbb{B}+\mathbb{B}^T)^T \overset{\shortparallel}{} \frac{1}{2}(\mathbb{B}+\mathbb{B}^T)$

$\mathbb{B}^A$: $\frac{1}{2}(\mathbb{B}-\mathbb{B}^T)^T = -\frac{1}{2}(\mathbb{B}-\mathbb{B}^T)$ $\left((\mathbb{B}^T)^T = \mathbb{B}\right)$

Př. Rozložte $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$ na $\mathbb{B}^S + \mathbb{B}^A$

$$\mathbb{B}^S = \frac{1}{2}\left(\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}^T\right) = \dots$$

$$= \begin{pmatrix} 1 & 1/2 & 5/2 \\ 1/2 & 2 & 1/2 \\ 5/2 & 1/2 & 0 \end{pmatrix}$$

$$\mathbb{B}^A = \mathbb{B} - \mathbb{B}^S = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1/2 & 5/2 \\ 1/2 & 2 & 1/2 \\ 5/2 & 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3/2 & 1/2 \\ -3/2 & 0 & 1/2 \\ -1/2 & -1/2 & 0 \end{pmatrix}$$

Determinanty

1. $\begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} = 1$

2. $\left|\bar a_1^s,\dots,\bar a_{i-1}^s,\ \alpha\bar u+\beta\bar v,\ \bar a_{i+1}^s,\dots,\bar a_n^s\right|$ (multi-lin.)

$$=\alpha\cdot\left|\bar a_1^s,\dots,\bar u,\dots\bar a_n^s\right|+\beta\cdot\left|\bar a_1^s,\dots,\bar v,\dots\bar a_n^s\right|$$

3. $\left|\bar a_1^s,\dots,\bar a_i^s,\dots,\bar a_j^s,\dots,\bar a_n^s\right| = -\left|\bar a_1^s,\dots,\bar a_j^s,\dots,\bar a_i^s,\dots,\bar a_n^s\right|$ (antisimetrija)

$\Downarrow$

4. $$\begin{array}{c} \\ i\dots \\ \\ j\dots \\ \\ \end{array}\begin{vmatrix} \bar a_1^r \\ \vdots \\ \bar a \\ \vdots \\ \bar a \\ \vdots \\ \bar a_n^r \end{vmatrix} = -\begin{vmatrix} \bar a_1 \\ \vdots \\ \bar a \\ \vdots \\ \bar a \\ \vdots \\ \bar a_n^r \end{vmatrix} = 0$$

$$c = -c \iff c = 0$$

Gaussovski upreasa

$$\begin{vmatrix} \bar a_1^r \\ \vdots \\ \bar a_{i-1}^r \\ \sum_{j=1}^n \alpha_j \bar a_j^r \\ \bar a_{i+1}^r \\ \vdots \\ \bar a_n^r \end{vmatrix} \underset{\text{linearite}}{=} \alpha_1 \underbrace{\begin{vmatrix} \bar a_1^r \\ \vdots \\ \bar a_{i-1}^r \\ \bar a_1^r \\ \vdots \end{vmatrix}}_{=0} + 0 + \dots + 0 + \alpha_i \begin{vmatrix} \bar a_1^r \\ \vdots \\ \bar a_{i-1}^r \\ \bar a_i^r \\ \vdots \end{vmatrix} + 0 + \dots$$

$\Downarrow$

$$A \xrightarrow[\alpha_i \neq 0]{\bar{a}_i^r := \sum_{j=1}^{m} \alpha_j \bar{a}_j^r} \tilde{A} \quad : \quad |A| = \frac{1}{\alpha_i} \cdot |\tilde{A}|$$

Determinant trojúhelníkové matice

$$\begin{vmatrix} u_{11} & \cdots & \cdots \\ & \ddots & \cdots \\ 0 & u_{n-1,n-1} & u_{n-1,n} \\ & & u_{n,n} \end{vmatrix} \quad \underline{\underline{r^r_{n-1} := 1 \cdot r^r_{n-1} - \frac{u_{n-1,n}}{u_{nn}} \cdot r^r_n}}$$

$$= \begin{vmatrix} u_{11} & \cdots & \cdots & \cdots \\ & \ddots & \cdots & \cdots \\ 0 & & u_{n-1,n-1} & 0 \\ & & & u_{n,n} \end{vmatrix} = \cdots$$

$$= \begin{vmatrix} u_{11} & & 0 \\ & \ddots & \\ 0 & & u_{nn} \end{vmatrix} \overset{\text{lé...}}{=} \underline{\underline{u_{11} u_{22} \cdots u_{nn}}} \cdot \underbrace{\begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix}}_{=1}$$

Př.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \underset{-3r_1}{=\!=} \frac{1}{1} \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 1 \cdot (-2) = \underline{\underline{-2}}$$

$$r_2 = 1 \, r_2 - 3 r_1$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & 0 & -1 \end{vmatrix} \quad \begin{matrix} r_2 := 1\cdot r_2 - 1 r_1 \\ r_3 := 1\cdot r_3 - 2 r_1 \end{matrix} \quad = \frac{1}{1\cdot 1}\begin{vmatrix} 1 & -1 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 2 & -5 \end{vmatrix} \quad r_3 := 3 r_3 - 2 r_2$$

$$= \frac{1}{3}\begin{vmatrix} 1 & \cdot & \cdot \\ 0 & 3 & \cdot \\ 0 & 0 & -17 \end{vmatrix} = \frac{1\cdot \cancel{3}\cdot(-17)}{\cancel{3}} = \underline{\underline{-17}}$$

$$\begin{vmatrix} \boxed{1} & -1 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ 2 & 2 & 0 & 1 \\ 1 & -1 & 3 & 2 \end{vmatrix} \begin{matrix} \\ -2r_1 \\ -2r_1 \\ -r_1 \end{matrix} = \begin{vmatrix} 1 & & & \\ 0 & \boxed{3} & -5 & -2 \\ 0 & 4 & -4 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$r_3 := 3 r_3 - 4 r_2$$

$$= \frac{1}{3}\cdot \begin{vmatrix} 1 & & & \\ & 3 & & \\ & 0 & 8 & 5 \\ & 0 & 1 & 1 \end{vmatrix} \quad r_4 := 8 r_4 - r_3$$

$$= \frac{1}{3}\cdot\frac{1}{8}\begin{vmatrix} 1 & \cdot & \cdot & \cdot \\ & 3 & \cdot & \cdot \\ & & 8 & \cdot \\ & & 0 & \boxed{3} \end{vmatrix} = \frac{1}{\cancel{3}\cdot\cancel{8}}\cdot 1\cdot\cancel{3}\cdot\cancel{8}\cdot 3 = \underline{\underline{3}}$$

antisimetrie

$$\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = - \underbrace{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}}_{\substack{\| \\ 2\cdot 1 - 1\cdot 0 \\ \| \\ +2}} = -2\cdot 1 = \underline{\underline{-2}}$$

$$\| \\ 0\cdot 1 - 1\cdot 2 \\ = -2$$