Boundary Element Method for Wave Equation

Michal Merta

IT4Innovations, Dept. of Applied Mathematics

michal.merta@vsb.cz

March 4, 2014
1 Introduction

2 BEM for wave equation

3 Numerical realization

4 Conclusion
modelling of wave (acoustic/electromagnetic) propagation has numerous engineering applications

- nondestructive testing
- seismology
- radar
- ultrasonic imaging
- tomography

BEM especially suitable for modelling of wave propagation in an unbounded domain
Wave equation

Scattering problem

\[
\begin{aligned}
\frac{1}{c^2} \frac{\partial^2 u^{sc}(x, t)}{\partial t^2} - \Delta u^{sc}(x, t) &= 0 \quad \text{in } \Omega \times \mathbb{R}, \\
u^{sc}(x, 0) &= 0 \quad \text{in } \Omega, \\
\frac{\partial u^{sc}(x, 0)}{\partial t} &= 0 \quad \text{in } \Omega, \\
\mathcal{B} u^{sc}(x, t) &= -\mathcal{B} u^{inc}(x, t) \quad \text{on } \Gamma \times \mathbb{R}_+
\end{aligned}
\]

- boundary conditions
  - sound-soft scatterer: \( \mathcal{B} u = u \)
  - sound-hard scatterer: \( \mathcal{B} u = \frac{\partial u}{\partial n} \)
  - absorbing scatterer: \( \mathcal{B} u = \frac{\partial u}{\partial n} - \alpha \frac{\partial u}{\partial t} \)
Wave equation

BEM approaches to wave equation

- **Space-time integral equations**
  - use the fundamental solution of the wave equation
  - global in time
  - large system matrix
  - special integration method needed

- **Laplace transform method**
  - solve frequency domain problems and use inverse Laplace/Fourier transform for transform to time domain

- **Time-stepping methods**
  - use implicit scheme for time-discretization and BEM for the solution of resulting elliptic problems in each time step
Fundamental solutions

Lemma

The fundamental solution of the wave equation is given by

\begin{align*}
G(t, x, y) &= \frac{1}{2} H(t - |x - y|) \quad \text{in 1D}, \\
G(t, x, y) &= \frac{1}{2} \frac{H(t - |x - y|)}{\sqrt{t^2 - |x - y|^2}} \quad \text{in 2D}, \\
G(t, x, y) &= \frac{1}{4\pi} \frac{\delta(t - |x - y|)}{|x - y|} \quad \text{in 3D}.
\end{align*}
Representation theorem

Representation formula in 3D

\[ u(t, x) = \int_0^t \int_{\Gamma} \frac{\partial}{\partial n(y)} G(t - s, x - y)[u(s, y)] - G(t - s, x - y)[\frac{\partial}{\partial n} u(y)] \, d\Gamma_y \, ds \]

\[ = \int_0^t \int_{\Gamma} \frac{\partial}{\partial n(y)} \left( \frac{1}{4\pi |x - y|} \delta(t - s - |x - y|) \right) [u(s, y)] \]

\[ - \frac{1}{4\pi |x - y|} \delta(t - s - |x - y|) \frac{\partial}{\partial n} u(y) \, d\Gamma_y \, ds \]

\[ = \int_0^t \int_{\Gamma} \frac{\partial}{\partial n(y)} \frac{1}{4\pi |x - y|} \delta(t - s - |x - y|) [u(s, y)] \]

\[ - \frac{1}{4\pi |x - y|} \frac{\partial |x - y|}{\partial n(y)} \frac{\partial}{\partial t} \delta(t - s - |x - y|) [u(s, y)] \]

\[ - \frac{1}{4\pi |x - y|} \delta(t - s - |x - y|) \frac{\partial}{\partial n} u(y) \, d\Gamma_y \, ds \]

\[ = \int_{\Gamma} \frac{\partial}{\partial n(y)} \frac{1}{4\pi |x - y|} [u(t - |x - y|, y)] - \frac{1}{4\pi |x - y|} \frac{\partial |x - y|}{n(y)} \frac{\partial}{\partial t} u(t - |x - y|) \]

\[ - \frac{1}{4\pi |x - y|} \left[ \frac{\partial}{\partial n} u(t - |x - y|, y) \right] d\Gamma_y \]
Boundary layer potentials

Representation formula in 3D

\[
\begin{align*}
    u(t, x) &= \int_{\Gamma} \frac{\partial}{\partial n(y)} \frac{1}{4\pi|x-y|} [u(t - |x-y|, y)] - \frac{1}{4\pi|x-y|} \frac{\partial|x-y|}{n(y)} \left[ \frac{\partial}{\partial t} u(t - |x-y|) \right] d\Gamma_y \\
    &\quad - \int_{\Gamma} \frac{1}{4\pi|x-y|} \left[ \frac{\partial}{\partial n} u(t - |x-y|, y) \right] d\Gamma_y = D([u]) - S([\partial_n u]), \quad x \in \Omega
\end{align*}
\]

Let \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \Gamma\). For \(p, \varphi : \mathbb{R}_+ \times \Gamma \to \mathbb{R}\) we define

- **single layer potential**
  - \((S([p]))(t, x) := \int_{\Gamma} \frac{1}{4\pi|x-y|} [p(t - |x-y|, y)] d\Gamma_y\)

- **double layer potential**
  - \((D([\varphi]))(t, x) := \int_{\Gamma} \frac{\partial}{\partial n(y)} \frac{1}{4\pi|x-y|} [\varphi(t - |x-y|, y)] - \frac{1}{4\pi|x-y|} \frac{\partial|x-y|}{n(y)} \left[ \frac{\partial}{\partial t} \varphi(t - |x-y|) \right] d\Gamma_y = \frac{1}{4\pi} \int_{\Gamma} n(y)(x-y) \left( \frac{\varphi(t-|x-y|, y)}{|x-y|^2} + \frac{\varphi(t-|x-y|, y)}{|x-y|} \right) d\Gamma_y\)
Retarded potential operators

For $x \in \Omega^-$, resp. $x \in \Omega$ going to $\Gamma$:

Traces of the potential operators

\[
\lim_{\Omega^- \ni x \to \Gamma} (S(p))(t, x) = \lim_{\Omega \ni x \to \Gamma} (S(p))(t, x) = Vp(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (I/2 + K)p(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (-I/2 + K)p(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} (D(\varphi))(t, x) = (-I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} (D(\varphi))(t, x) = (I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = \lim_{\Omega \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = W\varphi(t, x)
\]
Retarded potential operators

For $x \in \Omega^-$, resp. $x \in \Omega$ going to $\Gamma$:

Traces of the potential operators

\[
\lim_{\Omega^- \ni x \to \Gamma} (S(p))(t, x) = \lim_{\Omega \ni x \to \Gamma} (S(p))(t, x) = Vp(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (I/2 + K)p(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (-I/2 + K)p(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} (D(\varphi))(t, x) = (-I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} (D(\varphi))(t, x) = (I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = \lim_{\Omega \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = W\varphi(t, x)
\]
Retarded potential operators

For \( x \in \Omega^- \), resp. \( x \in \Omega \) going to \( \Gamma \):

**Traces of the potential operators**

\[
\lim_{\Omega^- \ni x \to \Gamma} (S(p))(t, x) = \lim_{\Omega \ni x \to \Gamma} (S(p))(t, x) = Vp(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (I/2 + K)p(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (-I/2 + K)p(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} (D(\varphi))(t, x) = (-I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} (D(\varphi))(t, x) = (I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = \lim_{\Omega \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = W\varphi(t, x)
\]
Retarded potential operators

For \( x \in \Omega^- \), resp. \( x \in \Omega \) going to \( \Gamma \):

**Traces of the potential operators**

\[
\lim_{\Omega^- \ni x \to \Gamma} (S(p))(t, x) = \lim_{\Omega \ni x \to \Gamma} (S(p))(t, x) = Vp(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (I/2 + K)p(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (-I/2 + K)p(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} (D(\varphi))(t, x) = (-I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} (D(\varphi))(t, x) = (I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = \lim_{\Omega \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = W\varphi(t, x)
\]
Retarded potential operators

For \( x \in \Omega^- \), resp. \( x \in \Omega \) going to \( \Gamma \):

**Traces of the potential operators**

\[
\lim_{\Omega^- \ni x \to \Gamma} (S(p))(t, x) = \lim_{\Omega \ni x \to \Gamma} (S(p))(t, x) = Vp(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (I/2 + K)p(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (-I/2 + K)p(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} (D(\varphi))(t, x) = (-I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} (D(\varphi))(t, x) = (I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = \lim_{\Omega \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = W\varphi(t, x)
\]
Retarded potential operators

For \( x \in \Omega^- \), resp. \( x \in \Omega \) going to \( \Gamma \):

Traces of the potential operators

\[
\lim_{\Omega^- \ni x \to \Gamma} (S(p))(t, x) = \lim_{\Omega \ni x \to \Gamma} (S(p))(t, x) = V p(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (I/2 + K)p(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} \frac{\partial (S(p))}{\partial n}(t, x) = (-I/2 + K)p(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} (D(\varphi))(t, x) = (-I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega \ni x \to \Gamma} (D(\varphi))(t, x) = (I/2 + K')\varphi(t, x)
\]

\[
\lim_{\Omega^- \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = \lim_{\Omega \ni x \to \Gamma} \frac{\partial (D(\varphi))}{\partial n}(t, x) = W \varphi(t, x)
\]
Retarded potential operators

\[ V_p(t, x) := \frac{1}{4\pi} \int_{\Gamma} \frac{p(\tau, y)}{|x - y|} d\Gamma_y \]

\[ K_p(t, x) := \frac{1}{4\pi} \int_{\Gamma} \frac{n(x)(x - y)}{|x - y|} \left( \frac{p(\tau, y)}{|x - y|^2} + \frac{\dot{p}(\tau, y)}{|x - y|} \right) d\Gamma_y \]

\[ K' \varphi(t, x) := \frac{1}{4\pi} \int_{\Gamma} \frac{n(y)(x - y)}{|x - y|} \left( \frac{\varphi(\tau, y)}{|x - y|^2} + \frac{\dot{\varphi}(\tau, y)}{|x - y|} \right) d\Gamma_y \]

\[ W \varphi(t, x) := \lim_{\Omega \ni x' \to x} n(x) \nabla_{x'} \left( -\frac{1}{4\pi} \int_{\Gamma} n(y) \nabla_{x'} \frac{\varphi(t - |x' - y|, y)}{|x' - y|} d\Gamma_y \right) \]

\[ \tau := t - |x - y| \]

- time domain single layer operator
- time domain double layer operator
- time domain adjoint double layer operator
- time domain hypersingular boundary integral operator
### Direct formulation

Let $u(t, x) = 0$ in $\Omega^-$. Then

$$u(t, x) = D(u|_\Gamma) - S(\partial_n u|_\Gamma) \quad \text{in } \mathbb{R}_+ \times \Omega.$$ 

#### $\gamma^e_0$

$$\gamma^e_0 u(t, x) = \gamma^e_0 (D(u|_\Gamma) - S(\partial_n u|_\Gamma))$$

$$u|_\Gamma = (I/2 + K')(u|_\Gamma) - V(\partial_n u|_\Gamma)$$

$$(K' - I/2)(u|_\Gamma) = V(\partial_n u|_\Gamma)$$

#### $\gamma^e_1$

$$\gamma^e_1 u(t, x) = \gamma^e_1 (D(u|_\Gamma) - S(\partial_n u|_\Gamma))$$

$$\partial_n u|_\Gamma = W(u|_\Gamma) - (-I/2 + K)(\partial_n u|_\Gamma)$$

$$(K + I/2)(\partial_n u|_\Gamma) = W(u|_\Gamma)$$
## Retarded potential boundary integral equations

### Indirect formulation

\[ u(t, x) = (S(p))(t, x) \quad \text{in} \quad \mathbb{R}_+ \times \Omega. \]

#### \( \gamma_0^{ex} \)

\[ \gamma_0^{ex} u(t, x) = \gamma_0^{ex} (S(p))(t, x) \]
\[ u|_\Gamma = V(p) \]

#### \( \gamma_1^{ex} \)

\[ \gamma_1^{ex} u(t, x) = \gamma_1^{ex} (S(p))(t, x) \]
\[ \partial_n u|_\Gamma = (-I/2 + K)(p) \]
Indirect formulation

\[ u(t, x) = (\mathcal{D}(\varphi))(t, x) \quad \text{in} \quad \mathbb{R}_+ \times \Omega. \]

\[ \gamma_0^{ex} \]

\[ \gamma_0^{ex} u(t, x) = \gamma_0^{ex} (\mathcal{D}(\varphi))(t, x) \]

\[ u|_{\Gamma} = (I/2 + K')(\varphi) \]

\[ \gamma_1^{ex} \]

\[ \gamma_1^{ex} u(t, x) = \gamma_1^{ex} (\mathcal{D}(\varphi))(t, x) \]

\[ \partial_n u|_{\Gamma} = W(\varphi) \]
usually done via Laplace transform to frequency domain

\[(\mathcal{L} f)(\omega) = \hat{f} = \int_{-\infty}^{\infty} e^{i\omega t} f(t) \, dt\]

e.g.

\[(\mathcal{L}(Vp))(\omega) = \frac{1}{4\pi} \int_{\Gamma} \frac{e^{i\omega|x-y|}}{|x-y|} \hat{p}(y, \omega) \, d\Gamma_y = \hat{V}_\omega \hat{p}(\omega, x)\]

RPBIE $\xrightarrow{\mathcal{L}}$ BIE (Helmholtz equation) $\xrightarrow{\mathcal{L}^{-1}}$ RPBIE
Variational formulation

Space-time variational formulation for soft scattering

- indirect formulation using single layer potential for Dirichlet problem

\[
V(\phi) = u|_{\Gamma} \\
\int_{\Gamma} \frac{\phi(t - |x - y|, y)}{4\pi|x - y|} \, d\Gamma_y = g(t, x)
\]

Weak formulation

Find \( \phi \in H^{-1/2,-1/2}([0, T] \times \Gamma) := L^2(0, T, H^{-1/2}(\Gamma)) + H^{-1/2}(0, T, L^2(\Gamma)) \) such that

\[
\int_0^T \int_{\Gamma} \int_{\Gamma} \frac{\phi(t - |x - y|, y) \xi(t, x)}{4\pi|x - y|} \, d\Gamma_y \, d\Gamma_x \, dt = \int_0^T \int_{\Gamma} \dot{g}(x, t) \xi(x, t) \, d\Gamma_x \, dt
\]

holds for all \( \xi \).
Space-time Galerkin discretization

\[ \phi_{\text{Galerkin}} = \sum_{i=1}^{N} \sum_{j=1}^{M} \alpha_i^j \varphi_j(x) b_i(t), \quad (x, t) \in \Gamma \]

- \( \{b_i\}_{i=1}^{N} \) . . . basis functions in time (with compact supports)
- \( \{\varphi_j\}_{j=1}^{M} \) . . . basis functions in space (with compact supports)
- \( \alpha_i^j \) . . . unknown coefficients
Galerkin discretization

Find $\alpha^j_i, i = 1, \ldots, N, j = 1, \ldots, M$ such that

$$\int_0^T \int_G \int_G \sum_{i=1}^N \sum_{j=1}^M \frac{\alpha^j_i \phi_j(y) \dot{b}_i(t - |x - y|) \phi_l(x) b_k(t)}{4\pi|x - y|} \ d\Gamma_y \ d\Gamma_x \ dt$$

$$= \int_0^T \int_G \dot{g}(x, t) \phi_l(x) b_k(t) d\Gamma_x \ dt$$

for $k = 1, \ldots, N, l = 1, \ldots, M$.

$$\psi_{i,k}(r) := \int_0^T \frac{\dot{b}_i(t - r) b_k(t)}{4\pi r} \ dt$$

$$A^i_{j,l} := \int_G \int_G \phi_j(y) \phi_l(x) \psi_{i,k}(|x - y|) \ d\Gamma_y \ d\Gamma_x$$

$$= \int_{\text{supp}(\phi_l)} \int_{\text{supp}(\phi_j)} \phi_j(y) \phi_l(x) \psi_{i,k}(|x - y|) \ d\Gamma_y \ d\Gamma_x$$
Galerkin discretization

Find $\alpha^j_i, i = 1, \ldots, N, j = 1, \ldots, M$ such that

$$\sum_{i=1}^{N} \sum_{j=1}^{M} A^{i,k}_{j,l} \alpha^j_i = \int_0^T \int_{\Gamma} \dot{g}(x,t) \varphi_l(x) b_k(t) \, d\Gamma_x \, dt$$

for $k = 1, \ldots, N, l = 1, \ldots, M$. 
Temporal basis functions

How to efficiently evaluate $A_{j,l}^{i,k}$?

\[ \psi_{i,k}(r) := \int_0^T \frac{\dot{b}_i(t-r)b_k(t)}{4\pi r} \, dt \]

is non-zero only for $r = |x - y|$ such that

\[ \text{supp}(\dot{b}_i(t-r)) \cap \text{supp}(b_j(t)) \neq \emptyset \]
construction of infinitely smooth temporal basis functions using partition of unity method (PUM), [Sauter, Veit]

Let us start with the $C^\infty$ function

$$f(t) := \begin{cases} 
  \text{erf}(2\text{arctanh}(t)), & \text{for } |t| < 1, \\
  -1, & \text{for } t \leq -1, \\
  1, & \text{for } t \geq 1.
\end{cases}$$

Then

$$h_{a,b}(t) := \frac{1}{2} f \left( 2 \frac{t - a}{b - a} - 1 \right) + \frac{1}{2},$$

and

$$\rho_{a,b,c}(t) := \begin{cases} 
  h_{a,b}(t), & \text{for } t \leq b, \\
  1 - h_{b,c}(t), & \text{for } t \geq b.
\end{cases}$$
Partition of unity functions

Let $\Theta = \langle 0, T \rangle$ and $0 = t_0 < t_1 < t_2 < \ldots < t_{N-2} < t_{N-1} = T$, $\tau_i := \langle t_{i-1}, t_i \rangle$. Let $\Theta_1 := \tau_1, \Theta_1 := \tau_1, \Theta_i := \tau_{i-1} \cup \tau_i, i = 2, \ldots N - 2, \Theta_N := \tau_{N-1}$. Then a smooth partition of unity subordinate to the cover $\{\Theta_i\}$ is defined as

$$\varphi_1(t) := 1 - h_{t_0,t_1}(t),$$
$$\varphi_i(t) := \rho_{t_{i-2},t_{i-1},t_i}(t), \quad \text{for } i = 2, \ldots, N - 1,$$
$$\varphi_N(t) := h_{t_{N-2},t_{N-1}}(t).$$

Temporal basis functions

The temporal basis functions are defined as

$$b_1(t) := \varphi_1(t)t^2,$$
$$b_i(t) := \varphi_i(t), \quad \text{for } i = 2, \ldots, N - 1,$$
$$b_N(t) := \varphi_N(t).$$
Algorithm 1 System matrix assembly

Require: A triangulation \( \{\tau_i : 1 \leq i \leq M\} \) of \( \Gamma \), number of time-steps \( N \), time derivative \( g \) of RHS

1: for \( k = 1 \) to \( N \) do
2: \( g_k \leftarrow \left( \int_0^T \int_{\Gamma} \dot{g}(x,t) \varphi_l(x)b_k(t) \, d\Gamma_x \, dt \right)_{l=1}^M \in \mathbb{R}^M \)
3: for \( i = 1 \) to \( N \) do
4: if \( \min \text{ supp } b_i \geq \max \text{ supp } b_k \) then
5: \( A_{k,i} \leftarrow 0 \in \mathbb{R}^{M \times M} \)
6: else
7: for \( j, l = 1 \) to \( M \) do
8: \( A_{j,i} \leftarrow \int_0^T \int_{\Gamma} \int_{\Gamma} \varphi_j(y) \varphi_l(x) \psi(r) \, d\Gamma_y \, d\Gamma_x \)
9: end for
10: end if
11: end for
12: end for
Matrix structure

nz = 50020

Michal Merta (IT4I)
Matrix structure
What kind of solver should we use?

- iterative (GMRES, BiCGStab)
  - would be ideal because of low memory requirements
  - missing suitable preconditioners
- direct (PARDISO, SuperLU, MUMPS)
  - high memory requirements
Current work

- optimizing and parallelizing system matrix assembly
- tests of direct solvers
  - MUMPS - 5120 elements, 25 time steps - approx. 15 min. on ANSELM
- MPI parallelization necessary
- matrix approximation?
- preconditioners?

Figure: Assembly of hypersingular operator matrix


Thank you for your attention!