8. cvičení – L'Hospitalovo pravidlo

8.1 L'Hospitalovo pravidlo (LP)

Věta 8.1

Nechť $x_0 \in \mathbb{R}^*$. Nechť je splněna jedna z podmínek:

- $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$,
- $\lim\limits_{x\to x_0} g(x) = \pm\infty$.

Existuje-li $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$, pak existuje také $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$ a platí $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$.

Poznámky:

- LP platí i pro jednostranné limity.
- LP říká, že limita $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$ se dá v případě, že se jedná o limitu typu $\left[\frac{0}{0}\right]$ nebo $\left[\frac{cokoliv}{\pm\infty}\right]$ nahradit limitou $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$, za předpokladu, že $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$ existuje.
- LP se dá využít i pro limity typu $[0 \cdot (\pm\infty)], \infty - \infty, f(x)^{g(x)}$.
  (Nejprve upravíme na $\left[\frac{0}{0}\right], \left[\frac{\pm\infty}{\pm\infty}\right], \left[\frac{\nexists}{\pm\infty}\right]$)
- **POZOR!** Pokud $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$ neexistuje, nelze LP použít!!! Rozhodně to však neznamená, že $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$ neexistuje.

Příklad 8.1

Vypočtěte následující limity:

a) $\lim\limits_{x\to 0} \frac{\sin x}{x}$ b) $\lim\limits_{x\to 0} \frac{e^x-1}{x}$ c) $\lim\limits_{x\to 0} \frac{1-\cos x}{x\cdot\sin x}$

d) $\lim\limits_{x\to 0} \frac{e^{2x}-2x-1}{\sin^2 3x}$ e) $\lim\limits_{x\to\infty} \frac{e^x-1}{x^2+1}$ f) $\lim\limits_{x\to\infty} \frac{2x^3+x-2}{3x^3+2x^2+x}$

a) $\lim\limits_{x\to 0} \frac{\sin x}{x} = \text{"}\frac{0}{0}\text{"} \overset{LP}{=} \lim\limits_{x\to 0} \frac{\cos x}{1} = \underline{\underline{1}}$

b) $\lim\limits_{x\to 0} \frac{e^x-1}{x} = \text{"}\frac{0}{0}\text{"} \overset{LP}{=} \lim\limits_{x\to 0} \frac{e^x}{1} = \underline{\underline{1}}$

c) $\lim\limits_{x\to 0} \frac{1-\cos x}{x\cdot\sin x} = \text{"}\frac{0}{0}\text{"} \overset{LP}{=} \lim\limits_{x\to 0} \frac{\sin x}{\sin x + x\cdot\cos x} = \text{"}\frac{0}{0}\text{"} \overset{LP}{=}$

$= \lim\limits_{x\to 0} \frac{\cos x}{\cos x + \cos x - x\cdot\sin x} = \frac{1}{1+1-0} = \underline{\underline{\frac{1}{2}}}$

d) $\lim\limits_{x\to 0} \dfrac{e^{2x}-2x-1}{\sin^2 3x} = \text{"}\dfrac{0}{0}\text{"} \overset{LP}{=} \lim\limits_{x\to 0} \dfrac{2\cdot e^{2x}-2}{\underbrace{2\cdot\sin 3x\cdot\cos 3x}_{\sin 6x}\cdot 3} =$

$= \lim\limits_{x\to 0} \dfrac{2e^{2x}-2}{\cancel{6}\cdot 3\cdot \sin 6x} = \text{"}\dfrac{0}{0}\text{"} \overset{LP}{=} \lim\limits_{x\to 0} \dfrac{4e^{2x}}{3\cdot\cos 6x\cdot 6} = \dfrac{4\cdot 1}{3\cdot 1\cdot 6} = \underline{\underline{\dfrac{2}{9}}}$

e) $\lim\limits_{x\to\infty} \dfrac{e^{x}-1}{x^2+1} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to\infty} \dfrac{e^{x}}{2x} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to\infty} \dfrac{e^{x}}{2} = \underline{\underline{\infty}}$

f) $\lim\limits_{x\to\infty} \dfrac{2x^3+x-2}{3x^3+2x^2+x} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to\infty} \dfrac{6x^2+1}{9x^2+4x+1} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=}$

$= \lim\limits_{x\to\infty} \dfrac{12x}{18x+4} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to\infty} \dfrac{12}{18} = \underline{\underline{\dfrac{2}{3}}}$

Jinak:

$\lim\limits_{x\to\infty} \dfrac{2x^3+x-2}{3x^3+2x^2+x} = \lim\limits_{x\to\infty} \dfrac{\cancel{x^3}\left(2+\frac{1}{x^2}-\frac{2}{x^3}\right)}{\cancel{x^3}\left(3+\frac{2}{x}+\frac{1}{x^2}\right)} = \dfrac{2+0-0}{3+0+0} = \underline{\underline{\dfrac{2}{3}}}$

Příklad 8.2 (Ne vždy lze LP použít!)

Vypočtěte následující limity:

a) $\lim\limits_{x\to\infty} \frac{\sin x}{x}$ b) $\lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

a) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = \text{"}\dfrac{\nexists}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to\infty} \dfrac{\cos x}{1} = \lim\limits_{x\to\infty} \cos x$ ... neexistuje, tj. takto nelze počítat

$\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = \lim\limits_{x\to\infty} \underbrace{\dfrac{1}{x}}_{\downarrow\, 0} \cdot \underbrace{\sin x}_{\text{ohraničená posl.}} = \underline{\underline{0}}$

b) $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{2}\cdot(x^2+1)^{-1/2}\cdot 2x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x}{1} = \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$ ... původní limita, toto nevede k cíli

$\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{\cancel{x}}{\cancel{x}\cdot\sqrt{1+\frac{1}{x^2}}} = \dfrac{1}{\sqrt{1+0}} = \underline{\underline{1}}$

8.2 Limity typu $[0 \cdot (\pm\infty)]$

Převedeme na typ $\left[\frac{0}{0}\right]$ nebo $\left[\frac{\pm\infty}{\pm\infty}\right]$.

Příklad 8.3

Vypočtěte následující limity:

a) $\lim\limits_{x\to 0^+}(x \cdot \ln x)$ b) $\lim\limits_{x\to\infty}(x^2 \cdot e^{-x})$ c) $\lim\limits_{x\to 0^+}\left(x \cdot e^{\frac{1}{x}}\right)$

d) $\lim\limits_{x\to 0^+} \sin x \cdot \ln\frac{1}{x}$

a) $\lim\limits_{x\to 0^+}(x\cdot \ln x) = \text{"}0\cdot(-\infty)\text{"} = \lim\limits_{x\to 0^+}\left(\dfrac{\ln x}{\left(\frac{1}{x}\right)}\right) = \text{"}\dfrac{-\infty}{\infty}\text{"} \overset{LP}{=}$

$= \lim\limits_{x\to 0^+}\dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+}(-x) = \underline{\underline{0}}$

jinak: $\lim\limits_{x\to 0^+}(x\cdot \ln x) = \lim\limits_{x\to 0^+}\dfrac{x}{\frac{1}{\ln x}} = \text{"}\dfrac{0}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to 0^+}\dfrac{1}{-\frac{1}{\ln^2 x}\cdot\frac{1}{x}} =$

$= \lim\limits_{x\to 0^+}(-x\cdot \ln^2 x) = \text{"}0\cdot(-\infty)\text{"} = \dots$

... nevhodný postup

b) $\lim\limits_{x\to\infty}(x^2\cdot e^{-x}) = \text{"}\infty\cdot 0\text{"} = \lim\limits_{x\to\infty}\dfrac{x^2}{e^x} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to\infty}\dfrac{2x}{e^x} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=}$

$= \lim\limits_{x\to\infty}\dfrac{2}{e^x} = \dfrac{2}{\infty} = \underline{\underline{0}}$

c) $\lim\limits_{x\to 0^+}(x\cdot e^{1/x}) = \text{"}0\cdot\infty\text{"} = \lim\limits_{x\to 0^+}\dfrac{e^{1/x}}{1/x} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=} \lim\limits_{x\to 0^+}\dfrac{e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} =$

$= \lim\limits_{x\to 0^+} e^{\frac{1}{x}} = e^{\infty} = \underline{\underline{\infty}}$

jinak: $\lim\limits_{x\to 0^+}(x\cdot e^{1/x}) = \lim\limits_{x\to 0^+}\dfrac{x}{e^{-1/x}} = \text{"}\dfrac{0}{0}\text{"} \overset{LP}{=} \lim\limits_{x\to 0^+}\dfrac{1}{e^{-1/x}\cdot\frac{1}{x^2}} =$

$= \lim\limits_{x\to 0^+}\dfrac{x^2}{e^{-1/x}}$ ... nevhodné (těžší než původní f-ce)

d) $\lim\limits_{x\to 0^+}\left(\sin x\cdot \ln\frac{1}{x}\right) = \text{"}0\cdot\infty\text{"} = \lim\limits_{x\to 0^+}\dfrac{\ln\frac{1}{x}}{\sin^{-1}x} = \text{"}\dfrac{\infty}{\infty}\text{"} \overset{LP}{=}$

$= \lim\limits_{x\to 0^+}\dfrac{x\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{\sin^2 x}\cdot\cos x} = \lim\limits_{x\to 0^+}\dfrac{\sin^2 x}{x\cdot\cos x} = \lim\limits_{x\to 0^+}\dfrac{\overbrace{\frac{\sin x}{x}}^{1}\cdot\sin x}{\cos x} =$

$= \dfrac{1\cdot 0}{1} = \underline{0}$

8.3 Limity typu $[\infty - \infty]$

Převedeme na společného jmenovatele.

Příklad 8.4

Vypočtěte následující limity:

a) $\lim\limits_{x\to 0^+}\left(\frac{1}{x} - \operatorname{cotg} x\right)$ b) $\lim\limits_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{e^x-1}\right)$ c) $\lim\limits_{x\to 0}\left(\frac{1}{x\cdot\sin x} - \frac{1}{x^2}\right)$

a)
$$\lim_{x\to 0^+}\left(\frac{1}{x} - \operatorname{cotg} x\right) = \text{"}\infty - \infty\text{"} = \lim_{x\to 0^+}\frac{1 - x\cdot\operatorname{cotg} x}{x} = \text{"}\frac{1-\infty\cdot 0}{0}\text{"} =$$

$$= \lim_{x\to 0^+}\frac{1 - x\cdot\frac{\cos x}{\sin x}}{x} = \lim_{x\to 0^+}\frac{\sin x - x\cdot\cos x}{x\cdot\sin x} = \text{"}\frac{0}{0}\text{"}\ LP =$$

$$= \lim_{x\to 0^+}\frac{\cos x - (\cos x - x\cdot\sin x)}{\sin x + x\cdot\cos x} = \lim_{x\to 0^+}\frac{x\cdot\sin x}{\sin x + x\cdot\cos x} = \text{"}\frac{0}{0}\text{"}\ LP =$$

$$= \lim_{x\to 0^+}\frac{\sin x + x\cdot\cos x}{\cos x + (\cos x - x\cdot\sin x)} = \frac{0}{1+(1-0)} = \frac{0}{2} = \underline{\underline{0}}$$

b)
$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{e^x-1}\right) = \lim_{x\to 0}\frac{e^x-1-\sin x}{\sin x\,(e^x-1)} = \text{"}\frac{0}{0}\text{"}\ LP =$$

$x\to 0^+$: $\left(\frac{1}{0^+} - \frac{1}{0^+}\right)$, $\infty - \infty$

$x\to 0^-$: $\left(\frac{1}{0^-} - \frac{1}{0^-}\right)$, $-\infty - (-\infty)$

$$= \lim_{x\to 0}\frac{e^x - \cos x}{\cos x\,(e^x-1) + \sin x\cdot e^x} = \text{"}\frac{0}{0}\text{"}\ LP =$$

$$= \lim_{x\to 0}\frac{e^x + \sin x}{-\sin x\,(e^x-1) + \cos x\cdot e^x + \cos x\cdot e^x + \sin x\cdot e^x} =$$

$$= \frac{1+0}{0\cdot 0 + 1\cdot 1 + 1\cdot 1 + 0\cdot 1} = \underline{\underline{\frac{1}{2}}}$$

c)
$$\lim_{x\to 0}\left(\underset{\frac{1}{0}}{\frac{1}{x\cdot\sin x}} - \underset{\frac{1}{0}}{\frac{1}{x^2}}\right) = \lim_{x\to 0}\frac{x - \sin x}{x^2\cdot\sin x} = \text{"}\frac{0}{0}\text{"}\ LP =$$

$$= \lim_{x\to 0}\frac{1-\cos x}{2x\cdot\sin x + x^2\cdot\cos x} = \text{"}\frac{0}{0}\text{"}\ LP = \lim_{x\to 0}\frac{\sin x}{2\cdot\sin x + \underbrace{2x\cos x + 2x\cos x}_{4x\cos x} - x^2\sin x} =$$

$$= \text{"}\frac{0}{0}\text{"}\ LP = \lim_{x\to 0}\frac{\cos x}{2\cos x + 4\cos x - 4x\sin x - (2x\sin x + x^2\cos x)} = \frac{1}{2+4-0-(0+0)} = \underline{\underline{\frac{1}{6}}}$$

8.4 Limity typu $\left[f(x)^{g(x)}\right]$

$\lim\limits_{x\to x_0} f(x)^{g(x)}$ převedeme na $\lim\limits_{x\to x_0} e^{g(x)\cdot\ln f(x)} = e^{\lim\limits_{x\to x_0} g(x)\cdot\ln f(x)}$.

(Plyne z věty o limitě složené funkce (věta 6.10).)

Poznámka:

- Typ $[0^0] = 1$,
- Typ $[\infty^\infty]$ vede na $[\infty\cdot\infty] = \infty$.

Příklad 8.5

Vypočtěte následující limity:

a) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ b) $\lim\limits_{x\to\infty}(1+x^2)^{\frac{1}{\ln x}}$ c) $\lim\limits_{x\to\infty}\left(\frac{1+x}{2+x}\right)^x$

a) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \text{"}1^\infty\text{"} = \lim\limits_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = \lim\limits_{x\to\infty} e^{x\cdot\ln\left(1+\frac{1}{x}\right)} =$

$= e^{\lim\limits_{x\to\infty}\left(x\cdot\ln\left(1+\frac{1}{x}\right)\right)^{*)}} = e^1 = \underline{\underline{e}}$

$^{*)}\ \lim\limits_{x\to\infty}\left(x\cdot\ln\left(1+\frac{1}{x}\right)\right) = \text{"}\infty\cdot 0\text{"} = \lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \text{"}\frac{0}{0}\text{"}\ LP =$

$= \lim\limits_{x\to\infty}\dfrac{\frac{1}{1+1/x}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = \dfrac{1}{1+0} = 1$

b) $\lim\limits_{x\to\infty}(1+x^2)^{\frac{1}{\ln x}} = \text{"}\infty^0\text{"} = \lim\limits_{x\to\infty} e^{\ln(1+x^2)^{\frac{1}{\ln x}}} =$

$= \lim\limits_{x\to\infty} e^{\frac{1}{\ln x}\cdot\ln(1+x^2)} = e^{\lim\limits_{x\to\infty}\left(\frac{1}{\ln x}\cdot\ln(1+x^2)\right)^{*)}} = \underline{\underline{e^2}}$

$^{*)}\ \lim\limits_{x\to\infty}\dfrac{\ln(1+x^2)}{\ln x} = \text{"}\frac{\infty}{\infty}\text{"}\ LP = \lim\limits_{x\to\infty}\dfrac{\frac{1}{1+x^2}\cdot 2x}{\frac{1}{x}} =$

$= \lim\limits_{x\to\infty}\dfrac{2x^2}{1+x^2} = \lim\limits_{x\to\infty}\dfrac{2x^2}{x^2\left(1+\frac{1}{x^2}\right)} = \dfrac{2}{1+0} = 2$

c) $\lim\limits_{x\to\infty}\left(\frac{1+x}{2+x}\right)^x = \lim\limits_{x\to\infty} e^{\ln\left(\frac{1+x}{2+x}\right)^x} = \lim\limits_{x\to\infty} e^{x\cdot\ln\left(\frac{1+x}{2+x}\right)} =$

$= e^{\lim\limits_{x\to\infty}\left(x\cdot\ln\left(\frac{1+x}{2+x}\right)\right)}\ {*)} = \underline{\underline{e^{-1}}}$

*) $\lim\limits_{x\to\infty}\frac{\ln\left(\frac{1+x}{2+x}\right)}{\frac{1}{x}} = \text{"}\frac{0}{0}\text{" LP} = \lim\limits_{x\to\infty}\frac{\frac{2+x}{1+x}\cdot\frac{1\cdot(2+x)-(1+x)\cdot 1}{(2+x)^{2}}}{-\frac{1}{x^2}} =$

$= \lim\limits_{x\to\infty}\frac{-x^2}{(1+x)(2+x)} = \lim\limits_{x\to\infty}\frac{-x^2}{x^2+3x+2} = -1$

Jinak:

$\lim\limits_{x\to\infty}\left(\frac{1+x}{2+x}\right)^x = \lim\limits_{x\to\infty}\left(\frac{1+x+2-2}{x+2}\right)^x = \lim\limits_{x\to\infty}\left(1-\frac{1}{x+2}\right)^x$ ← není vyhrané z $(1+\frac{1}{x})^x$

$\lim\limits_{x\to\infty}\left(\frac{1+x}{2+x}\right)^x = \lim\limits_{x\to\infty}\left(\frac{2+x}{1+x}\right)^{-x} = \lim\limits_{x\to\infty}\left[\left(1+\frac{1}{1+x}\right)^x\right]^{-1} =$

$= \lim\limits_{x\to\infty}\left[\left(1+\frac{1}{1+x}\right)^{x+1}\cdot\left(1+\frac{1}{1+x}\right)^{-1}\right]^{-1} = (e\cdot 1)^{-1} = \underline{\underline{e^{-1}}}$

8.5 Spojitost funkce

Příklad 8.6

Určete, zda je funkce f spojitá v bodě x_0.

a) $f(x) = \begin{cases}\cos x + \frac{\sin\left(x-\frac{\pi}{2}\right)}{2x-\pi} & \text{pro } x \neq \frac{\pi}{2}\\ 1 & \text{pro } x = \frac{\pi}{2}\end{cases}\ ;\ x_0 = \frac{\pi}{2}$

- $f(\frac{\pi}{2}) = 1$
- $\lim\limits_{x\to\frac{\pi}{2}}\left(\cos x + \frac{\sin(x-\frac{\pi}{2})}{2x-\pi}\right) = \underbrace{\lim\limits_{x\to\pi/2}\cos x}_{0} + \lim\limits_{x\to\frac{\pi}{2}}\underbrace{\frac{\sin(x-\frac{\pi}{2})}{2x-\pi}}_{\text{"}\frac{0}{0}\text{" LP}} =$

$= 0 + \underbrace{\lim\limits_{x\to\pi/2}\frac{\cos(x-\frac{\pi}{2})}{2}}_{\frac{1}{2}} = \frac{1}{2} \neq f(\frac{\pi}{2}) \Rightarrow$

$\Rightarrow$ $\underline{\underline{f \text{ není spojitá v } x=\frac{\pi}{2}}}$

b) $f(x) = \begin{cases} 2x + \frac{tg\, 2x}{x} & pro\ x \neq 0 \\ 2 & pro\ x = 0 \end{cases}$; $x_0 = 0$

- $f(0) = 2$
- $$\lim_{x\to 0}\left(2x + \frac{tg\, 2x}{x}\right) = \underbrace{\lim_{x\to 0} 2x}_{0} + \underbrace{\lim_{x\to 0}\frac{tg\, 2x}{x}}_{\text{"}\frac{0}{0}\text{" LP}} = 0 + \lim_{x\to 0}\frac{\frac{1}{\cos^2 2x}\cdot 2}{1} =$$

$$= 2 = f(0) \Rightarrow$$

$\Rightarrow$ $\underline{f \text{ je v } x = 0 \text{ spojitá}}$

8.6 Další příklady na LP

Příklad 8.7

Vypočtěte následující limity:

a) $\lim\limits_{x\to 1^-}\frac{\ln(1-x^2)}{\ln(\sin \pi x)}$ b) $\lim\limits_{x\to\infty}\left(\frac{\pi}{2} - arctg\, x\right)\ln x$ c) $\lim\limits_{x\to 0^+}\frac{\ln(\sin x)}{\ln x}$

d) $\lim\limits_{x\to 1}\frac{x^n - x}{x^n - 1}$ $(n \in \mathbb{R}\setminus\{0\})$ e) $\lim\limits_{x\to 0}\left(x^2 \cdot e^{\frac{1}{x^2}}\right)$

a) $$\lim_{x\to 1^-}\frac{\ln(1-x^2)}{\ln(\sin \pi x)} = \text{"}\frac{-\infty}{-\infty}\text{" LP} = \lim_{x\to 1^-}\frac{\frac{1}{1-x^2}\cdot(-2x)}{\frac{1}{\sin \pi x}\cdot \cos \pi x \cdot \pi} =$$

$$= \lim_{x\to 1^-}\frac{-2x \cdot \sin \pi x}{\pi\cdot(1-x^2)\cdot\cos \pi x} = \text{"}\frac{0}{0}\text{" LP} = \lim_{x\to 1^-}\frac{-2\cdot\sin \pi x - 2x\cdot\cos \pi x\cdot \pi}{-2x\pi\cdot\cos \pi x - \pi(1-x^2)\cdot\sin \pi x\cdot \pi}$$

$$= \cancel{\lim_{x\to 1^-}}\frac{-2\cdot 0 - 2\cdot 1\cdot \pi}{-2\pi\cdot 1 - \pi\cdot 0\cdot 0\cdot \pi} = \frac{-2\pi}{-2\pi} = \underline{\underline{1}}$$

b) $$\lim_{x\to\infty}\left(\frac{\pi}{2} - arctg\, x\right)\cdot\ln x = \text{"}0\cdot\infty\text{"} = \lim_{x\to\infty}\frac{\ln x}{\left(\frac{\pi}{2} - arctg\, x\right)^{-1}} \overset{LP}{=}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{-\left(\frac{\pi}{2} - arctg\, x\right)^{-2}\cdot\left(-\frac{1}{1+x^2}\right)} = \lim_{x\to\infty}\frac{1+x^2}{x\cdot\left(\frac{\pi}{2} - arctg\, x\right)^{-2}} \ldots$$ nevede k cíli

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$$\lim_{x\to\infty}\left(\frac{\pi}{2} - arctg\, x\right)\cdot\ln x = \lim_{x\to\infty}\frac{\frac{\pi}{2} - arctg\, x}{\ln^{-1} x} = \text{"}\frac{0}{0}\text{" LP} =$$

$$= \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\ln^{-2}x\cdot\frac{1}{x}} = \lim_{x\to\infty}\frac{x\cdot\ln^2 x}{1+x^2} = \text{"}\frac{\infty}{\infty}\text{"} \overset{LP}{=}$$

$$= \lim_{x\to\infty} \frac{\ln^2 x + x\cdot 2\cdot\ln x\cdot\frac{1}{x}}{2x} = \lim_{x\to\infty}\frac{\ln^2 x+2\ln x}{2x} = \text{"}\frac{\infty}{\infty}\text{"} \overset{LP}{=}$$

$$= \lim_{x\to\infty} \frac{2\cdot\ln x\cdot\frac{1}{x} + 2\cdot\frac{1}{x}}{2} = \underbrace{\lim_{x\to\infty}\frac{\ln x}{x}}_{\text{"}\frac{\infty}{\infty}\text{"}\ LP} + \underbrace{\lim_{x\to\infty}\frac{1}{x}}_{0} =$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \underline{\underline{0}}$$

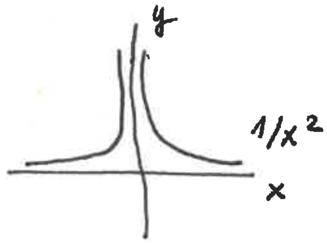
e) $$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\ln x} = \text{"}\frac{-\infty}{-\infty}\text{"} \overset{LP}{=} \lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cdot\cos x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{x\cdot\cos x}{\sin x} =$$

$$= \text{"}\frac{0}{0}\text{"} \overset{LP}{=} \lim_{x\to 0^+}\frac{\cos x - x\cdot\sin x}{\cos x} = \frac{1-0\cdot 0}{1} = \underline{\underline{1}}$$

d) $$\lim_{x\to 1}\frac{x^n - x}{x^n - 1} = \text{"}\frac{0}{0}\text{"} = \lim_{x\to 1}\frac{n\cdot x^{n-1} - 1}{n\cdot x^{n-1}} = \frac{n-1}{n} = \underline{\underline{1-\frac{1}{n}}}$$

f) $$\lim_{x\to 0}\left(x^2\cdot e^{\frac{1}{x^2}}\right) = \text{"}0\cdot\infty\text{"} = \lim_{x\to 0}\frac{e^{1/x^2}}{1/x^2} = \text{"}\frac{\infty}{\infty}\text{"} \overset{LP}{=}$$

$$= \lim_{x\to 0}\frac{e^{\frac{1}{x^2}}\cdot\cancel{\left(\frac{-2}{x^3}\right)}}{\cancel{\left(\frac{-2}{x^3}\right)}} = \underline{\underline{\infty}}$$



Bonusový příklad:

$$\lim_{x\to 0}\frac{(1-\cos x)\cdot e^x\cdot(x^2+5)}{x^2\cdot\cos(7x)\cdot(x^3+10)} = \text{"}\frac{0}{0}\text{"}$$ ~ není vhodné přímo použít LP (složité)

$$= \underbrace{\lim_{x\to 0}\frac{1-\cos x}{x^2}}_{\text{"}\frac{0}{0}\text{"} \Rightarrow LP} \cdot \underbrace{\lim_{x\to 0}\frac{e^x\cdot(x^2+5)}{\cos(7x)\cdot(x^3+10)}}_{\frac{1\cdot(0+5)}{1\cdot(0+10)} = \frac{1}{2}} = \underline{\underline{\frac{1}{4}}}$$

$$\underbrace{\lim_{x\to 0}\frac{\sin x}{2\cdot x}}_{\frac{1}{2}\cdot 1}$$