13. cvičení - Další typy integrálů

13.1 Integrály typu $\int R(e^x)dx$, $\int R(\ln x)dx$

$\int R(e^x)dx$ substituce: $t = e^x$, $dt = e^x dx$
nebo
substituce: $t = e^x \Rightarrow x = \ln t$, $dx = \frac{1}{t}dt$

$\int R(\ln x)dx$ substituce: $t = \ln x$, $dt = \frac{1}{x}dx$

Příklad 13.1

a) $\int \frac{e^{2x}-2e^x}{e^{2x}+1}dx = \left| \begin{matrix} t = e^x \\ dt = \underbrace{e^x}_{t} dx \Rightarrow \frac{1}{t}dt = dx \end{matrix} \right| = \int \frac{t^2-2t}{t^2+1} \cdot \frac{1}{t}dt =$

$$= \int \frac{t-2}{t^2+1}dt = \frac{1}{2}\int \frac{2t}{t^2+1}dt - 2\cdot \underbrace{\int \frac{1}{t^2+1}dt}_{\operatorname{arctg} t} =$$

$$= \frac{1}{2}\ln(t^2+1) - 2\cdot \operatorname{arctg} t =$$

$$= \underline{\underline{\frac{1}{2}\ln(e^{2x}+1) - 2\cdot \operatorname{arctg} e^x}}$$

b) $\int \frac{\ln x - 1}{(\ln^2 x+1)x}dx = \left| \begin{matrix} t = \ln x \\ dt = \frac{1}{x}dx \end{matrix} \right| = \int \frac{t-1}{t^2+1}dt =$

$$= \frac{1}{2}\int \frac{2t}{t^2+1}dt - \int \frac{1}{t^2+1}dt =$$

$$= \frac{1}{2}\ln(t^2+1) - \operatorname{arctg} t =$$

$$= \underline{\underline{\frac{1}{2}\ln(\ln^2 x+1) - \operatorname{arctg}(\ln x)}}$$

13.2 Integrály obsahující odmocniny

$\int R(x, \sqrt[s]{x})dx$ substituce: $t = \sqrt[s]{x} \Rightarrow x = t^s,\ dx = st^{s-1}dt$

$\int R(x, \sqrt[s]{ax+b})dx$ substituce: $t = \sqrt[s]{ax+b} \Rightarrow x = \frac{t^s-b}{a},\ dx = \frac{s}{a}t^{s-1}dt$

$\int R(x, \sqrt[s_1]{x}, \sqrt[s_2]{x}, \dots, \sqrt[s_k]{x})dx$ substituce: $t = \sqrt[s]{x}$, kde s je nejmenší společný násobek $s_1, s_2, \dots, s_k$

$\int R\left(x, \sqrt[s]{\frac{ax+b}{cx+d}}\right)dx$ substituce: $t = \sqrt[s]{\frac{ax+b}{cx+d}}$

Příklad 13.2

a) $\int \frac{x^2+\sqrt{x}+1}{x+\sqrt{x}}dx = \left| \begin{matrix} t=\sqrt{x} \Rightarrow t^2 = x \\ 2t\,dt = dx \end{matrix} \right| = \int \frac{t^4+t+1}{t^2+t}\cdot 2t\,dt =$

$= 2\int \frac{t^4+t+1}{t+1}dt \overset{(*)}{=} 2\int \left(t^3-t^2+t+\frac{1}{t+1}\right)dt =$

$= 2\cdot\left(\frac{t^4}{4}-\frac{t^3}{3}+\frac{t^2}{2}+\ln|t+1|\right) = \frac{1}{2}t^4-\frac{2}{3}t^3+t^2+2\ln|t+1| =$

$= \underline{\underline{\frac{1}{2}x^2-\frac{2}{3}\sqrt{x^3}+x+2\cdot\ln(\sqrt{x}+1)}}$

(*)

$(t^4+t+1):(t+1) = t^3-t^2+t+\frac{1}{t+1}$
$-(t^4+t^3)$
$\overline{\quad -t^3+t+1}$
$-(-t^3-t^2)$
$\overline{\quad t^2+t+1}$
$-(t^2+t)$
$\overline{\quad 1}$

b) $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx = \left| \begin{matrix} \sqrt{x+1} = t \Rightarrow x+1 = t^2 \\ dx = 2t\,dt \end{matrix} \right| = \int \frac{t+1}{t-1}\cdot 2t\,dt =$

$= 2\int \frac{t^2+t}{t-1}dt \overset{(*)}{=} 2\int\left(t+2+\frac{2}{t-1}\right)dt =$

$= 2\cdot\left(\frac{t^2}{2}+2t+2\cdot\ln|t-1|\right) = t^2+4t+4\ln|t-1| =$

$= \underline{\underline{x+1+4\sqrt{x+1}+4\cdot\ln|\sqrt{x-1}-1|}}$

není nutno psát - obecně všude je "+ konst."

(*)

$(t^2+t):(t-1) = t+2+\frac{2}{t-1}$
$-(t^2-t)$
$\overline{\quad 2t}$
$-(2t-2)$
$\overline{\quad 2}$

c) $\int x\sqrt{3-x}dx = \left| \begin{array}{l} t=\sqrt{3-x} \Rightarrow t^2 = 3-x \Rightarrow x = 3-t^2 \\ \qquad\qquad\qquad\qquad\qquad\quad dx = -2t\,dt \end{array} \right| =$

$$= \int (3-t^2)\cdot t\cdot(-2t)\,dt = 2\int(t^4-3t^2)\,dt =$$

$$= \frac{2}{5}t^5 - 2t^3 = \frac{2}{5}t^3(t^2-5) = \frac{2}{5}\sqrt{(3-x)^3}\,(\underbrace{3-x-5}_{-x-2}) =$$

$$= -\frac{2}{5}\sqrt{(3-x)^3}\cdot(x+2)$$

d) $\int \frac{1}{(1+\sqrt[3]{x})\sqrt{x}}dx = \left| \begin{array}{l} t=\sqrt[6]{x} \Rightarrow t^6 = x \\ \qquad\qquad\quad 6t^5dt = dx \end{array} \right| = \int \frac{1}{(1+t^2)t^3}\cdot 6t^5\,dt =$

$$= 6\int \frac{t^2+1-1}{t^2+1}\,dt = 6\int\left(1-\frac{1}{t^2+1}\right)dt =$$

$$= 6t - 6\,\mathrm{arctg}\,t = 6\cdot\sqrt[6]{x} - 6\cdot\mathrm{arctg}\,\sqrt[6]{x}$$

e) $\int \frac{1+\sqrt{x}-\sqrt[3]{x}}{x+\sqrt[6]{x^5}}dx = \left| \begin{array}{l} t=\sqrt[6]{x} \Rightarrow t^6 = x \\ 6t^5 dt = dx \end{array} \right| =$

$= \int \frac{1+t^3-t^2}{t^6+t^5}\cdot 6t^5\,dt = 6\int \frac{t^3-t^2+1}{t+1}\,dt \overset{(*)}{=}$

$= 6\cdot\int\left(t^2-2t+2-\frac{1}{t+1}\right)dt = 6\left(\frac{t^3}{3}-t^2+2t-\ln|t+1|\right) =$

$= 2\cdot\sqrt{x}-6\sqrt[3]{x}+12\cdot\sqrt[6]{x}-6\cdot\ln\left(\sqrt[6]{x}+1\right)$

$$
\begin{array}{l}
(t^3-t^2+1):(t+1) = t^2-2t+2-\frac{1}{t+1} \\
-(t^3+t^2) \\
\quad -2t^2+1 \\
\quad -(-2t^2-2t) \\
\qquad 2t+1 \\
\qquad -(2t+2) \\
\qquad\quad -1
\end{array}
$$

13.3 Integrály obsahující goniometrické funkce

Speciálním případem je $\int sin^n x \cdot cos^m x\, dx$, který řešíme takto:

- je $-$ li n liché: substituce: $t = \cos x$, $dt = -\sin x\, dx$
- je-li m liché: substituce: $t = \sin x$, $dt = \cos x\, dx$
- je-li n, m liché: substituce: $t = \cos x$, $dt = -\sin x\, dx$
  nebo
  substituce: $t = \sin x$, $dt = \cos x\, dx$
- je-li n, m sudé: upravíme pomocí vztahů:
  $sin^2 x = \frac{1-\cos 2x}{2}$, $cos^2 x = \frac{1+\cos 2x}{2}$

Příklad 13.3

a) $\int \sin x \cdot \cos^3 x \, dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x \, dx \end{array} \right| = -\int t^3 dt = -\frac{t^4}{4} =$

$$= -\frac{1}{4} \cos^4 x$$

b) $\int \sin^2 x \cdot \cos^3 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx = \int \sin^2 x \cdot (1 - \sin^2 x) \cdot \cos x \, dx$

$$= \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right| = \int t^2 (1 - t^2) dt =$$

$$= \int (t^2 - t^4) dt = \frac{t^3}{3} - \frac{t^5}{5} = \frac{1}{15} t^3 (5 - 3t^2) =$$

$$= \frac{1}{15} \cdot \sin^3 x \cdot (5 - 3 \cdot \sin^2 x)$$

c) $\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx = \int \left(\frac{1}{2} - \frac{1}{2} \cdot \cos 2x \right) dx =$

$$= \frac{1}{2} x - \frac{1}{2} \underbrace{\int \cos 2x \, dx} = \frac{1}{2} x - \frac{1}{4} \sin 2x$$

$$t = 2x$$
$$dt = 2 \, dx$$
$$\Downarrow$$
$$\int \frac{1}{2} \cos t \, dt$$
$$\Downarrow$$
$$\frac{1}{2} \sin t$$
$$\Downarrow$$
$$\frac{1}{2} \sin 2x$$

d) $\int \sin^4 x \cdot \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \cdot \left(\frac{1+\cos 2x}{2}\right) dx =$

$$= \frac{1}{8} \int (1 - 2\cdot\cos 2x + \cos^2 2x)\cdot(1+\cos 2x)\, dx =$$

$$= \frac{1}{8} \int (1 - 2\cdot\cos 2x + \cos^2 2x + \cos 2x - 2\cdot\cos^2 2x + \cos^3 2x)\, dx$$

$$= \frac{1}{8} \int (\cos^3 2x - \cos^2 2x - \cos 2x + 1)\, dx =$$

$$= \frac{1}{8} \underbrace{\int \cos^3 2x \, dx}_{(***)} - \frac{1}{8} \underbrace{\int \cos^2 2x \, dx}_{(**)} - \frac{1}{8} \underbrace{\int \cos 2x \, dx}_{(*)} + \frac{1}{8} x$$

$$= \frac{1}{16} \sin 2x - \frac{1}{48} \sin^3 2x - \frac{1}{16} x - \frac{1}{32} \sin 2x - \frac{1}{16} \sin 2x + \frac{1}{8} x =$$

$$= -\frac{1}{48} \cdot \sin^3 2x - \frac{1}{32} \sin 2x + \frac{1}{16} x$$

(*) $\int \cos 2x \, dx = \left| \begin{matrix} t = 2x \\ dt = 2\,dx \end{matrix} \right| = \frac{1}{2} \int \cos t \, dt = \frac{1}{2} \sin t = \frac{1}{2} \sin 2x$

(**) $\int \cos^2 2x \, dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2} \int (1 + \underbrace{\cos 2x}_{\text{viz } *})\, dx = \frac{1}{2}\left(x + \frac{1}{2} \sin 2x\right)$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x$$

(***) $\int \cos^3 2x \, dx = \int \cos 2x \cdot \underbrace{(1 - \sin^2 2x)}_{\cos^2 2x} dx = \left| \begin{matrix} t = \sin 2x \\ dt = 2\cos x \, dx \end{matrix} \right| =$

$$= \frac{1}{2} \int (1 - t^2)\, dt = \frac{1}{2}\left(t - \frac{t^3}{3}\right) = \cancel{\sin x}$$

$$= \frac{1}{2} \sin 2x - \frac{1}{6} \sin^3 2x$$