Bifurcations in contact problems with local Coulomb friction.

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Abstract This contribution illustrates the bifurcation behaviour of solutions to contact problems with local Coulomb friction. The bifurcation character of solutions is well-known for models with a low number of degrees of freedom. Our aim is to show that a similar phenomen occurs when a finite element approximation with a high number of degrees of freedom is used. We experimentally find a critical value of the coefficient of friction in which one branch of solutions splits into two ones.

1 Introduction

Contact problems with local Coulomb friction belong to challenging mathematical problems which remained unsolved for a long time. Recent results on the existence of solutions to this class of problems can be found in [1]. On the other hand, a complete description of the structure of solutions is still missing in a general case. For discrete problems the situation is slightly better. Systems with a very small number of degrees of freedom can be solved "by hand" so that all solutions are available: see for ex. [5] where the system was parametrized by applied loads P and [4] where the parametrization by a coefficient of friction $\mathscr{F}$ is used. Nevertheless it is not still clear if and how these results can be extended to finite element models with a very high number of dof. which are already close to a continuous model. In this contribution we focus on the parametrization by $\mathscr{F}$. To our knowledge there are only few results valid for any number of dof., namely (a) the existence of locally lipschitz continuous branches of solutions (see [4]) (b) the existence of a solution for any coefficient of friction and uniqueness of the solution if $\mathscr{F}$ is below a critical value which (unfortunately) depends on a discretization parametr of a finite element model

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(see [2]). In practice this means that for a given finite element partition one may have a different number of solutions depending on the value of $\mathscr{F}$. The aim of this paper is to document this phenomenon experimentally for "real" discretizations: one branch of the solutions splits into (at least) two ones for $\mathscr{F}$ passing a critical value. As a model of friction we use Coulomb's law with a coefficient which depends on a solution.

2 Setting of the problem

Let us consider an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with the Lipschitz boundary $\partial\Omega = \overline{\Gamma}_u \cap \overline{\Gamma}_p \cap \overline{\Gamma}_c$ where Γ_u, Γ_p, Γ_c are non–empty, disjoint parts of $\partial\Omega$. On each part different boundary conditions are prescribed: Ω is fixed along Γ_u, while surface tractions of density P act on Γ_p. The body is unilaterally supported by a rigid foundation S along Γ_c. For the sake of simplicity we shall suppose that S is either a half-plane $(d = 2)$ or a half-space $(d = 3)$ and there is no gap between Ω and S in the undeformed state. Finally, Ω is subject to body forces of density F. Our aim is to find an equilibrium state of Ω taking into account friction between Ω and S which obeys the classical Coulomb law with a coefficient of friction $\mathscr{F}$ *depending* on a solution. An equilibrium state is characterized by a displacement vector $u : \Omega \mapsto \mathbb{R}^d$ which satisfies the equilibrium equations of linear elasticity in Ω, the classical boundary conditions on Γ_u and Γ_p and the following unilateral and friction conditions on Γ_c:

$$T_n := T(u)\cdot n \le 0, \;\; u_n := u \cdot n \le 0, \;\; T_n u_n = 0 \;\text{ on } \Gamma_c \tag{1}$$

$$\left.\begin{array}{l} ||T_t(u)|| \le -\mathscr{F}(||u_t||)T_n(u) \;\text{ on } \Gamma_c \\ u_t(x) \neq 0 \Rightarrow T_t(u)(x) = \mathscr{F}(||u_n||)T_n(u)\frac{u_t}{||u_t||}(x), \;\; x \in \Gamma_c \end{array}\right\} \tag{2}$$

where $T_n(u), T_t(u) := T(u) - T_n(u)n$ is the normal, tangential component of a stress vector $T(u)$, respectively which corresponds to u; u_n, $u_t := u - u_n n$ is the normal, tangential component of a displacement vector u, respectively. The symbol $||\;||$ in (2) stands for the absolute value of a scalar $(d = 2)$ or the Euclidean norm of a vector $(d = 3)$. Finally, $\mathscr{F}$ is a coefficient of friction whose value depends on the magnitude of u_t on Γ_c.

Assuming that Ω is made of a linear elastic material which obeys a linear Hooke law characterized by elasticity coefficients $c_{ijkl} \in L^\infty(\Omega)$, the weak form of our problem is given by the following *implicit* variational inequality:

$$\left.\begin{array}{l} \text{Find } u \in K \text{ such that} \\ a(u, v-u) + j(u,u,v) - j(u,u,u) \ge L(v-u) \quad \forall v \in K \end{array}\right\} \tag{$\mathscr{P}$}$$

The meaning of symbols is as follows (the summation convention is addopted):

$$\mathbb{V} = \{v \in (H^1(\Omega))^d \,|\, v = 0 \text{ on } \Gamma_u\}$$
$$K = \{v \in \mathbb{V} \,|\, v_n \le 0 \text{ on } \Gamma_c\}$$
$$a(u,v) := \int_\Omega c_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(v)\,dx\,, \quad \varepsilon_{kl}(u) = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big)$$
$$L(v) := \int_\Omega F_i v_i\,dx + \int_{\Gamma_p} P_i v_i\,ds\,, \quad F \in (L^2(\Omega))^d,\ P \in (L^2(\Gamma_p))^d$$
$$j(u,v,w) := -\langle \mathscr{F}(||u_t||)T_n(v), ||w_t||\rangle\,,$$

where $\langle\, , \,\rangle$ is the duality pairing between $X_t = \{\varphi \in L^2(\Gamma_c) \,|\, \exists v \in \mathbb{V}:\ \varphi = ||v_t|| \text{ on } \Gamma_c\}$ and its dual X_t'. In a similar way we define X_n as the space of $v_{n|_{\Gamma_c}}$ of $v \in \mathbb{V}$ and its dual X_n'. The cone of non–negative elements from X_t, X_n' will be denoted by X_t^+, X_{n+}', respectively.

The existence of solutions to $(\mathscr{P})$ under appropriate assumptions on data, and in particular on $\mathscr{F}$ has been established in [1]. Numerical realization of $(\mathscr{P})$ is based on an equivalent fixed–point formulation. For $(\varphi, g) \in X_t^+ \times X_{n+}'$ fixed let us consider the following contact problem with given friction and the coefficient $\mathscr{F}_\varphi := \mathscr{F}(\varphi)$:

$$\left.\begin{array}{l}\text{Find } u := u(\varphi,g) \in K \text{ such that}\\ a(u,v-u) + j(\varphi,g,v) - j(\varphi,g,u) \ge L(v-u) \quad \forall v \in K\end{array}\right\} \qquad (\mathscr{P}(\varphi,g))$$

and define the mapping $\Phi : X_t^+ \times X_{n+}' \mapsto X_t^+ \times X_{n+}'$ by

$$\Phi(\varphi,g) = (||u_{t|_{\Gamma_c}}||, -T_n(u)) \tag{3}$$

where $u \in K$ is the unique solution of $(\mathscr{P}(\varphi,g))$. Comparing the definitions of $(\mathscr{P})$ and $(\mathscr{P}(\varphi,g))$ we see that $u \in K$ solves $(\mathscr{P})$ if and only if it solves $\mathscr{P}(||u_{t|_{\Gamma_c}}||, -T_n(u))$ or equivalently, $(||u_{t|_{\Gamma_c}}||, -T_n(u))$ is a fixed point of Φ.

3 Discretization of $(\mathscr{P})$, properties of the discrete model

Let Ω be a polygonal $(d=2)$ or a polyhedral $(d=3)$ domain and $\mathscr{T}_h$ be a partition of $\overline{\Omega}$ into triangles $(d=2)$ or tetrahedra $(d=3)$ such that $diam\,T \le h\ \forall T \in \mathscr{T}_h$. With any $\mathscr{T}_h$ we associate the spaces V_h, $\mathbb{V}_h$:

$$V_h = \{v_h \in C(\overline{\Omega}) \,|\ v_{h|_T} \in P_1(T)\ \forall T \in \mathscr{T}_h,\ v_h = 0 \text{ on } \Gamma_u\}\,, \quad \mathbb{V}_h = (V_h)^d\,.$$

By $\mathscr{V}_h = V_{h|_{\Gamma_c}}$ we denote the space of restrictions on Γ_c of functions from V_h while $\mathscr{V}_h^+$ stands for the set of non–negative elements of $\mathscr{V}_h$. Further, let $\mathscr{T}_H$ be a partition of $\overline{\Gamma}_c$ into segments S_H, $diam\,S_h \le H\ \forall S_H \in \mathscr{T}_H$. On any $\mathscr{T}_H$ we construct the space L_H of piecewise constant functions:

$$L_H = \{\mu_H \in L^2(\Gamma_c) \mid \quad \mu_{H|_{S_H}} \in P_0(S_H)\ \forall S_H \in \mathscr{T}_H\}$$

and its subset Λ_H of all non–negative functions. For any $(\varphi_h, g_H) \in \mathscr{V}_h^+ \times \Lambda_H$ given we define the following auxiliary problem:

$$\left.\begin{array}{l}\text{Find } (u_h, \lambda_H) \in \mathbb{V}_h \times \Lambda_H \text{ such that} \\ a(u_h, v_h - u_h) + j(\varphi_h, g_H, v_h) - j(\varphi_h, g_H, u_h) \geq \\ \qquad L(v_h - u_h) - (\lambda_H, v_{hn} - u_{hn})_{0,\Gamma_c} \quad \forall v_h \in \mathscr{V}_h \\ (\mu_H - \lambda_H, u_{hn})_{0,\Gamma_c} \leq 0 \quad \forall \mu_H \in \Lambda_H \end{array}\right\} \qquad (\mathscr{P}(\varphi_h, g_H))_h^H$$

$(\mathscr{P}(\varphi_h, g_H))_h^H$ is a mixed formulation of the contact problem with given friction and the coefficient $\mathscr{F}_{\varphi_h} := \mathscr{F} \circ \varphi_h$ which uses the dualization of the unilateral constraint $u_{hn} \leq 0$ on Γ_c. Next we shall suppose that $\mathbb{V}_h$ and Λ_H are such that the following condition guaranteeing the uniqueness of a solution to $(\mathscr{P}(\varphi_h, g_H))_h^H$ is satisfied:

$$(\mu_H, v_{hn})_{0,\Gamma_c} = 0 \ \ \forall v_h \in \mathscr{V}_h \quad \Rightarrow \quad \mu_H = 0\,. \tag{4}$$

This enables us to define the mapping $\Phi_{hH} : \mathscr{V}_h^+ \times \Lambda_H \mapsto \mathscr{V}_h^+ \times \Lambda_H$ by

$$\Phi_{hH}(\varphi_h, g_H) = (r_h ||u_{ht|_{\Gamma_c}}||, \lambda_H)\,,$$

where (u_h, λ_H) is the solution of $(\mathscr{P}(\varphi_h, g_H))_h^H$ and $r_h : C(\overline{\Gamma}_c) \mapsto \mathscr{V}_h$ is a linear approximation operator preserving the monotonicity property: $v \geq 0$ on $\overline{\Gamma}_c \Rightarrow r_h v \in \mathscr{V}_h^+$ (the Lagrange interpolation operator, e.g.). Since $-\lambda_H$ can be interpreted as the discrete normal stress on Γ_c the mapping Φ_{hH} can be viewed to be a discretization of Φ defined by (3).

Definition 1. By a discrete solution of the contact problem with Coulomb friction and the coefficient depending on a solution we call any function $u_h \in \mathbb{V}_h$ such that (u_h, λ_H) is a solution of $(\mathscr{P}(r_h||u_{ht|_{\Gamma_c}}||, \lambda_H))_h^H$, i.e. $(r_h||u_{ht|_{\Gamma_c}}||, \lambda_H)$ is a fixed point of Φ_{hH}.

Let us recall main results concerning the existence and uniqueness of the fixed point of Φ_{hH}. Proofs for 2D problems can be found in [3] but their adaptation to the 3D case is easy.

Theorem 1. *It holds:*

(a) if $\mathscr{F} \in C(\mathbb{R}^1_+)$, $0 \leq \mathscr{F}(t) \leq \mathscr{F}_{max}\ \forall t \in \mathbb{R}^1_+$, where $\mathscr{F}_{max}$ is given then there exists at least one fixed point of Φ_{hH};

(b) if, in addition to (a), $\mathscr{F}$ is Lipschitz continuous in $\mathbb{R}^1_+$:

$$|\mathscr{F}(t_1) - \mathscr{F}(t_2)| \leq l|t_1 - t_2| \quad \forall t_1, t_2 \in \mathbb{R}^1_+$$

so Φ_{hH} is in $\mathscr{V}_h^+ \times \Lambda_H$: $\exists q > 0$ such that

$$||\Phi_{hH}(\varphi_h, g_H) - \Phi_{hH}(\overline{\varphi}_h, \overline{g}_H)|| \leq q||(\varphi_h, g_H) - (\overline{\varphi}_h, \overline{g}_H)|| \tag{5}$$

holds for every $(\varphi_h, g_H), (\overline{\varphi}_h, \overline{g}_H) \in \mathscr{V}_h^+ \times \Lambda_H$, where

$$||(\varphi_h, g_H)|| := ||\varphi_h||_{0,\Gamma_c} + ||g_H||_h\,, \quad ||g_H||_h := \sup_{\mathbb{V}_h} \frac{(g_h, v_{hn})_{0,\Gamma_c}}{||v_h||_{1,\Omega}}\,.$$

The constant q in (5) *depends on $\Omega, h, H, \mathscr{F}_{max}$ and l in such a way that for Ω, h, H fixed, $q \to 0+$ if $\mathscr{F}_{max}, l \to 0+$.*

Remark 1. There exist $\overline{\mathscr{F}} > 0, \bar{l} > 0$ both depending on Ω, h and H such that if $\mathscr{F}_{max} \le \overline{\mathscr{F}}$ and $l \le \bar{l}$ the mapping Φ_{hH} is contractive in $\mathscr{V}_h^+ \times \Lambda_H$ so that Φ_{hH} has a unique fixed point and the method of successive approximations converges.

Remark 2. If the following Babuška–Brezzi condition and the inverse inequality are satisfied, i.e.

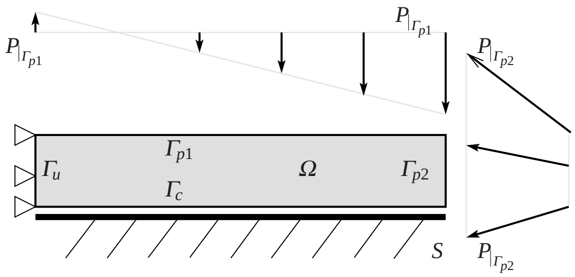
$$||\mu_H||_h \ge \beta ||\mu_H||_{X'_n}\,, \quad ||\mu_H||_{0,\Gamma_c} \le \overline{\beta} H^{-1/2} ||\mu_H||_{X'_n}\,, \quad \forall \mu_H \in L_H$$

where $\beta, \overline{\beta} > 0$ do not depend on $h, H > 0$ then the bounds $\overline{\mathscr{F}}, \bar{l}$ guaranteeing the uniqueness of the solution are bounded from above by $\sqrt{hH}$, i.e. are mesh–dependent ([3]).

Let us comment on the previous results. Unlike to the continuous setting in which the existence of a solution has been shown for $\mathscr{F}$ small enough, a solution to the discrete model exists for any $\mathscr{F}$ satisfying (a) of Theorem 1 regardless of the shape of $\Omega, \mathscr{F}_{max}, l$ and the applied forces F and P. Moreover, if $\mathscr{F}_{max}$ and l are small enough, the solution to the discrete model is unique. Unfortunately, this uniqueness result depends on the mesh norms h, H as follows from Remark 2. One of ways how a possible non–uniqueness comes to light is that the method of successive approximations used for finding fixed points of Φ_{hH} depends on the choice of initial approximations. In the next section we illustrate this phenomenon on model examples in 2D and 3D: starting from two different initial approximations we find two different fixed points for a particular coefficient of friction $\mathscr{F}$. Then taking the same examples (with the same $\mathscr{T}_h$ and $\mathscr{T}_H$) but replacing $\mathscr{F}$ by $\xi\mathscr{F}$, where $\xi \to 0+$ we find (accordingly to our theoretical results) a critical value $\overline{\xi} > 0$ for which originally two different fixed points will coincide for $\xi < \overline{\xi}$ using the same initial approximations as before.

4 Examples with branching solutions

We start with a 2D problem. The body represented by $\Omega = (0,10) \times (0,1)\,[m]$ is made of an elastic material characterized by the Young modulus $E = 21.19e10\,[Pa]$ and Poisson's ratio $\sigma = 0.277$. The partition of $\partial\Omega$ into Γ_u, $\Gamma_p = \Gamma_{p1} \cup \Gamma_{p2}$ and Γ_c is seen from Fig. 1. The surface tractions P are linearly distributed along Γ_{p1} and Γ_{p2} starting from the following values: $P_{|\Gamma_{p1}}(0,1) = (0, 1.e6)\,[N]$, $P_{|\Gamma_{p1}}(10,1) = (0, -8.e6)\,[N]$, $P_{|\Gamma_{p2}}(10,1) = (-10.e6, 10.e6)\,[N]$ and $P_{|\Gamma_{p2}}(10,0) = (-10.e6, -3.e6)\,[N]$. The body forces are neglected. The graph of the coefficient of friction $\mathscr{F}$ is shown in Fig. 2.



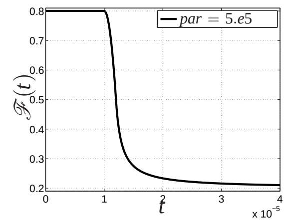
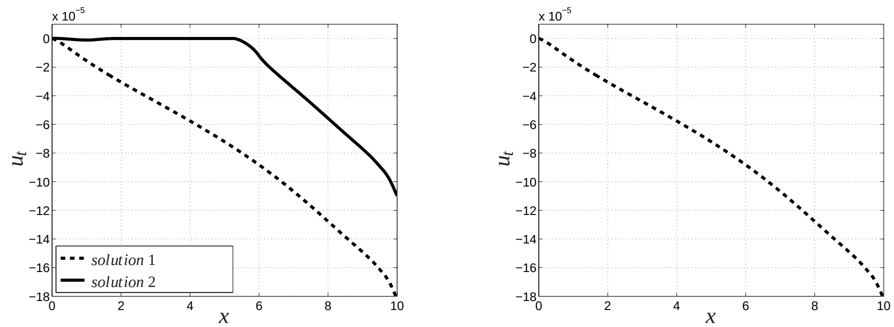


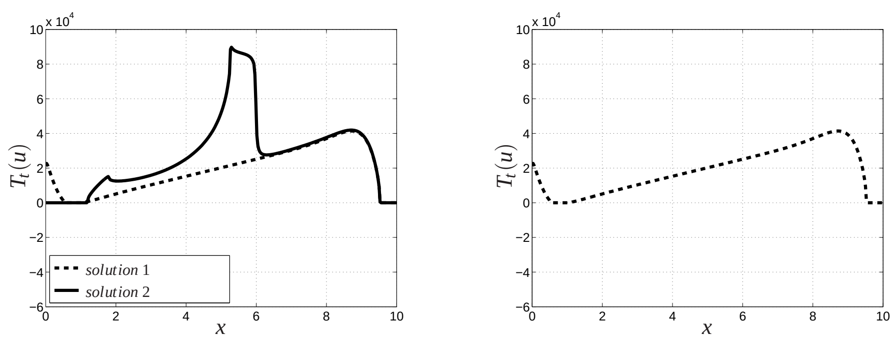
Fig. 1 Geometry of the problem

Fig. 2 The graph of $\mathscr{F}$



$a)$ $\xi = 0.98581$ $b)$ $\xi = 0.9858$

Fig. 3 Tangential displacements on Γ_c



$a)$ $\xi = 0.98581$ $b)$ $\xi = 0.9858$

Fig. 4 Friction force $T_t(u)$ on Γ_c

Numerical realization of each iterative step of the method of successive approximations is based on its dual formulation (for details see [3]). The used partitions of $\overline{\Omega}$ and $\overline{\Gamma}_c$ give 26 640 primal variables and 720 dual variables (discrete contact stresses). Two different initial approximations were used, namely $(\varphi_h^{(0)}, g_H^{(0)}) = (0,0)$ corresponding to a contact problem without friction and $(\overline{\varphi}_h^{(0)}, \overline{g}_H^{(0)}) = (0, 1.e8)$

(a contact problem with a high slip bound). Starting from them two different fixed points of Φ_h^H were obtained. Since the most significant differences are in the tangential direction we focus on it. One of these solutions is such that a slip occurs along the whole Γ_c $(solution1)$ while both stick and slip zone are present for the second one $(solution2)$. Now instead of $\mathscr{F}$ we take $\xi\mathscr{F}$. If $\xi = 0.98581$ than again two solutions with the same character as for $\xi = 1$ appear. On the other hand if $\xi = 0.9858$ both solutions meet together and only one solution with a slip along the whole Γ_c is obtained (see Figs. 3 and 4).

Now we switch to 3D problems. Let the body be represented by $\Omega = (0,10) \times (0,1) \times (0,1)\,[m]$. The decomposition of the boundary $\partial\Omega$ into Γ_u, Γ_p and Γ_c, as well as the applied surface tractions P are seen from Fig. 5. The Young modulus E, Poisson's ratio σ and the coefficient of friction $\mathscr{F}$ are the same as in the 2D case. The body forces are neglected again. Discretizations of $\overline{\Omega}$ and $\overline{\Gamma}_c$ are such that the total number of the primal, dual variables is 30 000 and 12 700, respectively. The initial approximations for the method of successive approximations are the same as before. Denote $\mathscr{F}_\xi := \xi\mathscr{F}$. For $\xi = 1.37689$ we get two different solutions: the one sliding along the whole Γ_c, the other one with a stick and slip zone as shown in Fig. 6. The norm of $T_t(u)$ on Γ_c is depicted in Fig. 7 and the distribution of $T_n(u)$ and u_n on Γ_c are shown in Figs. 8 and 9 for $(solution\ 2)$. Setting $\xi = 1.37688$ both solutions joint together. The obtained solution slides along the whole Γ_c, i.e. has the character of $(solution\ 1)$.

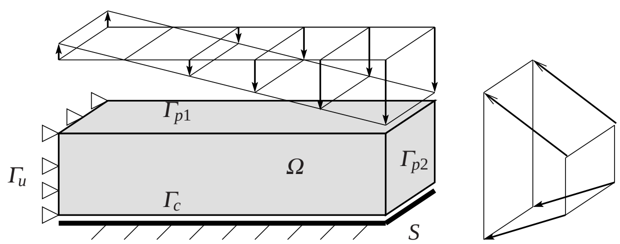


Fig. 5 Geometry of the problem

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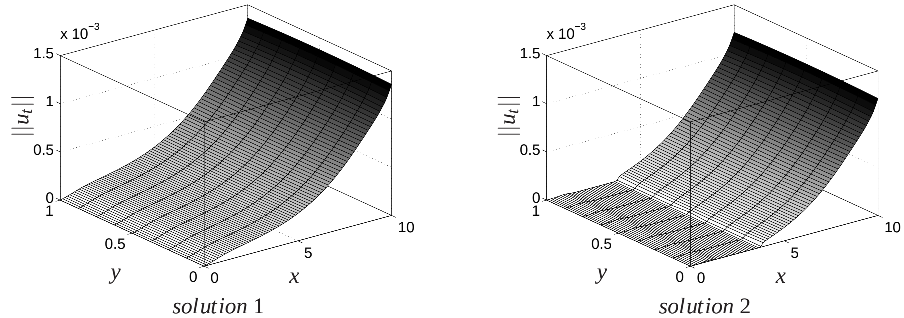

Fig. 6 The norm of the tangential displacements for $\xi = 1.37689$

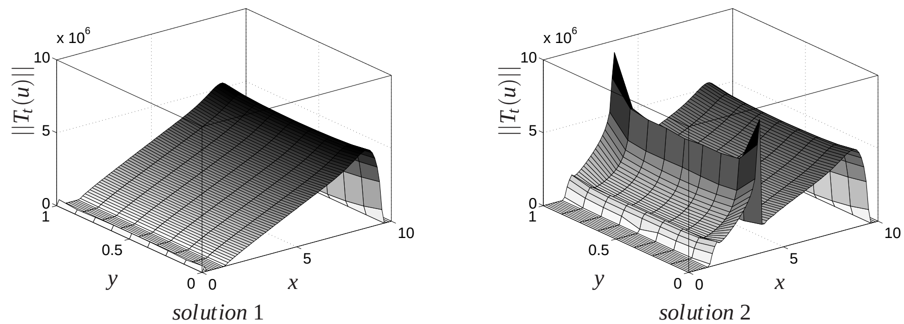

Fig. 7 The norm of the tangential stresses for $\xi = 1.37689$

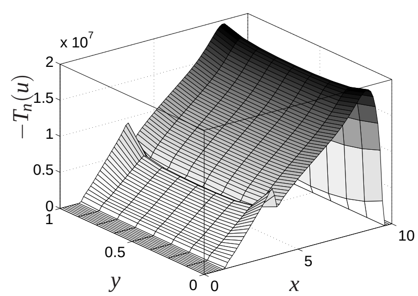

Fig. 8 The normal stresses for $\xi = 1.37689$

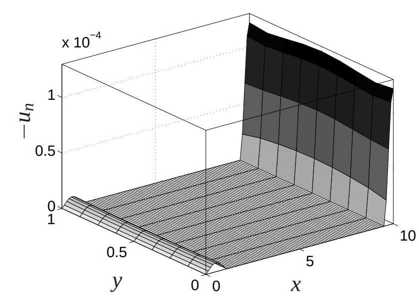

Fig. 9 The normal displacement for $\xi = 1.37689$

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