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Scalable algorithm for solving 3D contact problems with friction

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A scalable algorithm for solving contact problems with Coulomb friction is presented. It combines fixed point iterations with an augmented Lagrangian loop and an active set strategy. Numerical experiments indicate scalability.

1 Introduction

The contribution deals with solving contact problems with *Coulomb* friction for 3D elastic bodies. First we introduce an auxiliary problem with *Tresca* friction defining a mapping Φ, which associates with a given slip bound a corresponding normal contact stress in the equilibrium state. A solution to the contact problem with Coulomb friction is a fixed point of Φ and the method of successive approximations can be used for its computation. In a discrete case, it converges for a sufficiently small coefficient of friction [4].

As the problem with Tresca friction is described by a variational inequality of the 2^{nd} kind, its finite element discretization leads to a constrained minimization of a non-smooth function so that a regularization is needed. To this end we use the duality theory that enables us to rewrite the problem in terms of normal contact stresses as a dual minimization of a (smooth) strictly convex quadratic function with separable convex constraints. The solution may be computed by an active set algorithm based on a suitable optimality criterion, e.g. on the KKT-condition [6, 8] or on the projected gradient [7]. While the function representing the KKT condition is discontinuous, the projected gradient is continuous that leads to more robust computations. Moreover the algorithm of [7] has a convergence rate in terms of the spectral condition number of the Hessian matrix.

If the FETI domain decomposition method [3] is used, then the corresponding stiffness matrix is positive semidefinite that imposes in addition linear equality constraints in the dual formulation of the Tresca problem. The solution can be computed by the iterative augmented Lagrangian algorithm, in which the outer loop updates Lagrange multipliers for the equality constraints, while the inner loop is represented by the algorithm of [7]. Moreover an appropriate penalty update enables us to find the solution at $O(1)$ matrix-vector multiplications. This generalizes scalability results proved originally for frictionless contact problems [1].

Combining the augmented Lagrangian algorithm with the method of successive approximation, we arrive at the iterative scheme for solving problems with Coulomb friction. Computations are considerably more efficient, if an inexact implementation is used so that the one successive approximation performs the only step of augmented Lagrangian algorithm. Numerical experiments indicate that also this algorithm is scalable.

2 Model contact problem with Coulomb friction

Let us consider a brick lying on a rigid foundation. The brick occupies in the reference configuration the domain $\omega \subset R^3$, whose boundary $\partial\omega$ split into three non-empty disjoint parts γ_u, γ_p and γ_c with different boundary conditions. The zero displacements are prescribed on γ_u, whereas the surface traction act on γ_p. On γ_c, we consider the contact conditions, i.e., the non-penetration and the effect of friction. The elastic behavior of the brick is described by Lamé equations that, after FETI discretization, lead to a symmetric positive semidefinite stiffness matrix $K \in R^{3n_c\times 3n_c}$ and to a load vector $f \in R^{3n_c}$. As the FETI procedure divides the brick into subbricks, we interconnect corresponding parts of the solution by a "gluing" matrix $B_g \in R^{m_g\times 3n_c}$ and, moreover, we enforce the Dirichlet boundary condition by $B_d \in R^{m_d\times 3n_c}$ [2]. Finally we introduce full rank matrices N and $T_1, T_2 \in R^{m_c\times 3n_c}$ projecting displacements at contact nodes to normal and tangential directions, respectively, and we denote $B = \left(B_d^\top, B_g^\top, N^\top, T_1^\top, T_2^\top\right)^\top \in R^{m_d+m_g+3m_c\times 3n_c}$. For more details about this model problem, we refer to [5].

The discrete contact problem with *Tresca friction* reads as

$$\left.\begin{array}{ll} \text{minimize} & \frac{1}{2}\lambda^\top F\lambda - \lambda^\top h, \\ \text{subject to} & G\lambda = e,\ \lambda_{\nu,i} \geq 0,\ \lambda_{t_1,i}^2 + \lambda_{t_2,i}^2 \leq r_i^2,\ i = 1,\dots,m_c, \\ & \lambda = (\lambda_d^\top, \lambda_g^\top, \lambda_\nu^\top, \lambda_{t_1}^\top, \lambda_{t_2}^\top)^\top,\ \lambda_d^\top \in R^{m_d}, \lambda_g^\top \in R^{m_g}, \lambda_\nu, \lambda_{t_1}, \lambda_{t_2} \in R^{m_c}, \end{array}\right\} \quad (1)$$

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where $F = BK^{\dagger}B^{\top}$, $h = BK^{\dagger}f$, $G = R^{\top}B^{\top}$, $e = R^{\top}f$ and $r_i \geq 0$ are given slip bound values at contact nodes. Here $K^{\dagger}$ denotes a generalized inverse to K and R is a matrix whose columns span the null-space of K. Point out that λ_{ν} and $\lambda_{t_1}, \lambda_{t_2}$ represent normal and tangential contact stresses, respectively.

We shall denote $R_{+}^{m_c} = \{s \in R^{m_c} : s_i \geq 0\}$. The contact problem with *Coulomb friction* uses the friction law, in which the slip bound $r \in R_{+}^{m_c}$ depends on the normal contact stress $\lambda_{\nu} \in R_{+}^{m_c}$ by

$$r \equiv \mathcal{F}\lambda_{\nu},$$

where $\mathcal{F} > 0$ is a coefficient of friction. As r is the input for (1) and λ_{ν} is the output, the Tresca problem defines the mapping

$$\Phi : R_{+}^{m_c} \longmapsto R_{+}^{m_c} : r \longmapsto \mathcal{F}\lambda_{\nu}$$

and a fixed point of Φ solves the problem with Coulomb friction. It can be computed by successive approximations:

$$r^0 \in R_{+}^{m_c} \text{ given; } \quad \text{for } k = 1, 2, \dots \text{ set } r^k = \Psi(r^{k-1}).$$

3 Numerical experiments

In our numerical experiments, we consider the steel brick $\omega = (0,3) \times (0,1) \times (0,1)$ partitioned into $3N \times N \times N$ cubes with trilinear finite elements for $N = 4, 6, 8, 10, 12, 14$. In Table 1 we report CPU time in seconds (*time*), the number of successive approximations (*iter*), the total complexity by matrix-vector multiplications (n_Q) and the relative complexity (n_Q/n). The computations are carried out in Matlab 7 on Pentium(R)4, 3GHz, 512MB with the terminating criterion

$$\|r^k - r^{k-1}\|/\|r^k\| \leq 10^{-4}.$$

The obtained results are promising, especially, n_Q is only mildly dependent on the finite element discretization so that the relative complexity considerably decreases for finer grids.

Table 1 Contact problem with Coulomb friction.

dof		$\mathcal{F} = 0.3$				$\mathcal{F} = 0.6$			
$3n_c$	$3m_c$	*time*	*iter*	n_Q	n_Q/n	*time*	*iter*	n_Q	n_Q/n
900	180	4	5	**535**	2.97	6	7	**801**	4.45
2646	378	24	5	**638**	1.68	35	6	**906**	2.40
5832	648	104	5	**758**	1.17	136	6	**1001**	1.54
10890	990	317	5	**814**	0.82	443	6	**1145**	1.16
18252	1404	789	5	**854**	0.61	1122	6	**1232**	0.88
28350	1890	1833	5	**947**	0.50	2222	6	**1169**	0.62

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