The Stokes Flow with Friction

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Abstract. Efficient methods for solving the Stokes problem with a friction effect simulated by the slip boundary condition are developed. The dual formulation of the problem arising from the finite element approximation leads to the minimization of the strictly quadratic function with few unknowns constrained by simple bounds. Two solution algorithms highly efficient in contact solid mechanics are tested: the active set strategy and the path-following variant of the interior point method. Numerical experiments conclude the paper.

Keywords: Stokes Problem, Slip Boundary Condition, Active Set Algorithm, Interior Point Method
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INTRODUCTION

Observing a fluid flow along a solid impermeable wall, one can notice in some applications a variable tangential velocity of the fluid that may depend on a material quality or a shape of the wall. Such behaviour of the fluid is usually simulated by the slip boundary condition. It is used for modelling the blood flow, the metal forming processes, the polymer flow, or the hydrodynamics problems; see [11, 1] and references therein. Conditions of this type are used also in contact problems of solid mechanics, where they describe friction laws between bodies [6].

Our paper deals with the slip boundary condition analogous to the Tresca friction law. To demonstrate difficulties and still to keep ideas as clear as possible, we consider the Stokes problem in a planar domain. The weak formulation leads to the variational inequality of the second kind that is equivalent to the minimum of the total potential energy [1]. The finite element approximation leads to the algebraic problem given by the minimization of the non-differentiable energy function subject to two linear equality constraints describing the impermeability of the slip part of the boundary and the incompressibility of the fluid. After eliminating the velocity components, we get the smooth dual function in terms of three Lagrange multipliers. The first Lagrange multiplier regularizes the problem. Its components are subject to simple bounds. The other two Lagrange multipliers treat the impermeability and the incompressibility conditions. The third Lagrange multiplier plays the role of the pressure in the whole fluid domain. The solution to the dual problem is computed by the active set algorithm [3, 2, 10] or by the path-following algorithm that is a variant of the interior point method [9]. These algorithms represent conceptually different strategies how to solve the dual problem. Numerical experiments illustrate computational efficiencies.

FORMULATION OF THE MODEL PROBLEM

Let Ω be a bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega$ that is split into three disjoint parts: $\partial\Omega = \overline{\gamma}_D \cup \overline{\gamma}_N \cup \overline{\gamma}_C$. We consider the model of the viscous flow of an incompressible Newtonian fluid modelled by the Stokes equations in Ω with the Dirichlet and Neumann boundary conditions on γ_D and γ_N, respectively, and with the

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impermeability and the slip boundary conditions on γ_C:

$$\left.\begin{array}{rcll}-\nu\Delta\mathbf{u}+\nabla\mathrm{p} & = & \mathbf{f} & \text{in } \Omega,\\ \nabla\cdot\mathbf{u} & = & 0 & \text{in } \Omega,\\ \mathbf{u} & = & \mathbf{u}_D & \text{on } \gamma_D,\\ \boldsymbol{\sigma} & = & \boldsymbol{\sigma}_N & \text{on } \gamma_N,\\ \mathrm{u}_n & = & 0 & \text{on } \gamma_C,\\ |\sigma_t| & \leq & \mathrm{g} & \text{on } \gamma_C,\\ |\sigma_t|<\mathrm{g}\ \Rightarrow\ \mathrm{u}_t & = & 0 & \text{on } \gamma_C,\\ |\sigma_t|=\mathrm{g}\ \Rightarrow\ \exists k\geq 0:\ \mathrm{u}_t & = & -k\sigma_t & \text{on } \gamma_C,\end{array}\right\} \tag{1}$$

where

$$\boldsymbol{\sigma}=\nu\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{n}}-\mathrm{p}\mathbf{n}.$$

Here, $\mathbf{u}=(\mathrm{u}_1,\mathrm{u}_2)$ is the flow velocity function, p is the pressure function, $\mathbf{f}=(\mathrm{f}_1,\mathrm{f}_2)$ describes forces acting on the fluid, $\nu>0$ is the kinematic viscosity, $\mathbf{u}_D$ and $\boldsymbol{\sigma}_N$ are the Dirichlet and Neumann boundary data, respectively, $\mathbf{n}$ and $\mathbf{t}$ are the unit outer normal and tangential vectors to $\partial\Omega$, respectively, $\mathrm{u}_n=\mathbf{u}\cdot\mathbf{n}$ and $\mathrm{u}_t=\mathbf{u}\cdot\mathbf{t}$, $\sigma_t=\boldsymbol{\sigma}\cdot\mathbf{t}$ are the normal and tangential components of $\mathbf{u}$, $\boldsymbol{\sigma}$ along γ_C, respectively, and $\mathrm{g}\geq 0$ is the slip bound function on γ_C. We will always assume that $\gamma_D\neq\emptyset$ and $\gamma_C\neq\emptyset$. The existence of an unique (weak) solution component $\mathbf{u}$ is proved in [1]. The existence of an unique p is guaranteed, e.g., when $\gamma_N\neq\emptyset$ [4]

ALGEBRAIC FORMULATIONS

The finite element approximation of (1) leads to the following algebraic problem:

$$\textit{Find } \bar{u}\in\mathbb{V} \textit{ such that } \mathscr{J}(\bar{u})\leq\mathscr{J}(u)\quad\forall u\in\mathbb{V}, \tag{2}$$

where $\mathscr{J}(u)=\frac{1}{2}u^\top Au-u^\top b+g^\top|Tu|$, $\mathbb{V}=\{u\in\mathbb{R}^{n_u}:\ Nu=0,\ Bu=0\}$, $A\in\mathbb{R}^{n_u\times n_u}$ is symmetric, positive definite, $T,N\in\mathbb{R}^{n_c\times n_u}$, $B\in\mathbb{R}^{n_p\times n_u}$ are full row-rank, $b\in\mathbb{R}^{n_u}$, and $g\in\mathbb{R}^{n_c}_+$; n_p is the number of nodes, n_c is the number of nodes lying on $\overline{\gamma}_C\backslash\overline{\gamma}_D$, and n_u is the dimension of the algebraic solution representing the velocity. The constraints in $\mathbb{V}$ describe the impermeability and incompressibility conditions. Nota that (2) is generated from (1) using finite elements satisfying the *inf-sup* stability condition [4].

Let us introduce the Lagrangian $\mathscr{L}:\mathbb{R}^{n_u}\times\Lambda\mapsto\mathbb{R}$ to (2) defined by

$$\mathscr{L}(u,\lambda)=\frac{1}{2}u^\top Au-u^\top b+\lambda^\top Cu,$$

where $\Lambda=\{\lambda_t\in\mathbb{R}^{n_c}:\ |\lambda_t|\leq g\}\times\mathbb{R}^{n_c+n_p}$, $\lambda=(\lambda_t^\top,\lambda_n^\top,p^\top)^\top\in\Lambda$ is the Lagrange multiplier, and $C=(T^\top,N^\top,B^\top)^\top$. The minimization problem (2) is equivalent to the following saddle-point formulation:

$$\textit{Find } (\bar{u},\bar{\lambda})\in\mathbb{R}^{n_u}\times\Lambda \textit{ such that } \mathscr{L}(\bar{u},\lambda)\leq\mathscr{L}(\bar{u},\bar{\lambda})\leq\mathscr{L}(u,\bar{\lambda})\quad\forall(u,\lambda)\in\mathbb{R}^{n_u}\times\Lambda. \tag{3}$$

Eliminating the velocity component by $\bar{u}=A^{-1}(b-C^\top\bar{\lambda})$, we get the dual problem:

$$\textit{Find } \bar{\lambda}\in\Lambda \textit{ such that } q(\bar{\lambda})\leq q(\lambda)\quad\forall\lambda\in\Lambda, \tag{4}$$

where $q(\lambda)=\frac{1}{2}\lambda^\top F\lambda-\lambda^\top d$ with the symmetric, positive definite dual Hessian $F=CA^{-1}C^\top$ and $d=CA^{-1}b$.

Although $\mathscr{J}$ is non-differentiable due to the last term, the dual problem consists in minimizing the strictly quadratic function that is smooth. The minimum to the dual problem will be computed by algorithms highly efficient in contact problems of solid mechanics [3, 9]. Note that the dual problem (4) contains n_c constrained components (of λ_t), while remaining n_c+n_p components (of λ_n and p) are unconstrained. As n_p corresponds to the whole fluid domain and n_c only to the slip part of the boundary, it is typically $n_p\gg n_c$ for finer meshes. Therefore, only few unknowns are subject to constraints for large algebraic problems. This fact influences considerably the efficiency of computations.

NUMERICAL EXPERIMENTS

The problem (1) is approximated by the P1-bubble/P1 [7] and P2/P1 [4] finite elements on triangular meshes. The *inf-sup* stability of the Lagrange multipliers is proved in [1] for the P1-bubble/P1 finite elements. In the case of the P2/P1 finite elements, the stability is observed experimentally, if the friction effect is considered at vertices of triangles lying on γ_C [8].

The geometry of Ω is seen in Figure 1. The decomposition of the boundary $\partial\Omega$ is as follows: $\gamma_D = (0,1) \times \{1\}$, $\gamma_{N_{left}} = \{0\} \times (0,1)$, $\gamma_{N_{right}} = \{1\} \times (0,1)$, $\gamma_N = \gamma_{N_{left}} \cup \gamma_{N_{right}}$, and $\gamma_C = \{(x, -0.1\sin(2\pi x)) : x \in (0,1)\}$. The forces $\mathbf{f}$, and the boundary data $\mathbf{u}_D$, $\boldsymbol{\sigma}_N$ are defined by the analytic solution of [7] that is known for the pure Dirichlet-Neumann problem (with the Dirichlet boundary condition considered also on γ_C). The analytic solution is unknown in our case that is the situation typical for problems with friction. We prescribe different g on γ_C in order to demonstrate all friction effects; see Figure 2 .

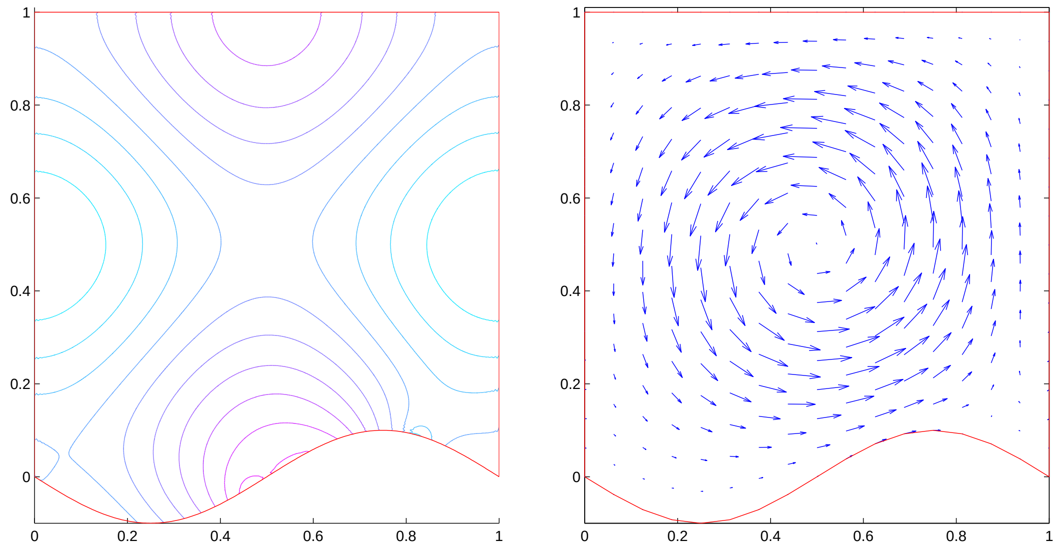

FIGURE 1. Solution for the P1-bubble/P1 finite elements, $g = 10$: isobar lines (left) and velocity field (right).

In Tables 1 and 2 we summarize results of our numerical tests. In columns labeled by AS and PF we introduce the number of matrix-vector multiplications by the dual Hessian F and the CPU time (in seconds) for the active set algorithm [3] and the path-following algorithm [9], respectively. We use the different terminating tolerance (10^{-5} for AS and 10^{-3} for PF) leading to comparable residua of computed solutions. Note that the size of the dual problem treated by these algorithms is $n_p + 2n_c$. The symbol ">5000" stands for situations, when the terminating criterion is not achieved for the default number of iterations. All codes are implemented in Matlab 2013b and the computations are performed by ANSELM supercomputer at VŠB-TU Ostrava.

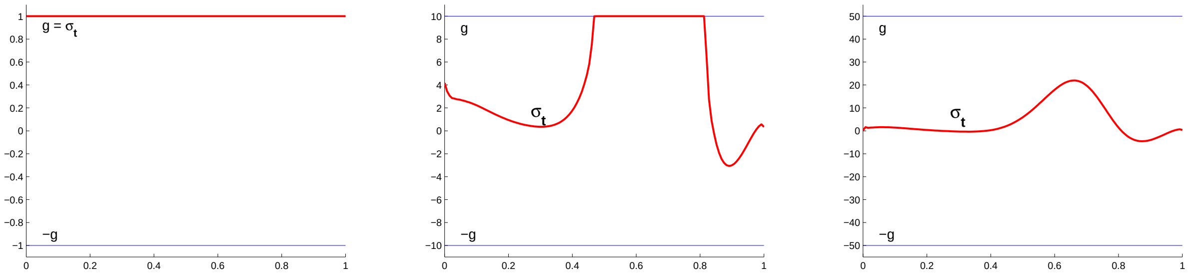

FIGURE 2. $g = 1$ (left): the slip bound is achieved all on γ_C; $g = 10$ (middle): the slip bound is achieved on a part of γ_C; $g = 50$ (right): the slip bound is not achieved on γ_C

TABLE 1. P1-bubble/P1 finite elements

slip bound	g = 1		g = 10		g = 50	
$n_u/n_p/n_c$	AS	PF	AS	PF	AS	PF
544 / 289 / 17	**91** (0.03)	**115** (0.04)	**250** (0.60)	**127** (0.10)	**125** (0.12)	**157** (0.04)
2112 / 1089 / 33	**111** (0.15)	**117** (0.15)	**1094** (1.98)	**178** (0.28)	**151** (0.20)	**153** (0.18)
8320 / 4225 / 65	**151** (1.88)	**113** (1.63)	**4993** (54.03)	**194** (2.24)	**205** (2.32)	**216** (2.43)
33024 / 16641/ 129	**123** (8.67)	**170** (13.71)	> 5000	**259** (18.98)	**676** (46.00)	**180** (13.27)
131584 / 66049 / 257	**184** (99.71)	**173** (108.09)	> 5000	**252** (148.51)	**2090** (1121.5)	**319** (181.49)

TABLE 2. P2/P1 finite elements

slip bound	g = 1		g = 10		g = 50	
$n_u/n_p/n_c$	AS	PF	AS	PF	AS	PF
544 / 81 / 9	**58** (0.03)	**80** (0.04)	**128** (0.28)	**98** (0.06)	**67** (0.03)	**93** (0.04)
2112 / 289 / 17	**78** (0.37)	**97** (0.35)	**271** (0.87)	**110** (0.34)	**85** (0.43)	**99** (0.39)
8320 / 1089 / 33	**82** (1.43)	**96** (1.76)	**870** (12.90)	**119** (1.96)	**100** (1.49)	**87** (1.45)
33024 / 4225 / 65	**102** (10.56)	**105** (12.83)	**3153** (304.15)	**107** (12.44)	**150** (15.02)	**123** (13.66)
131584 / 16641 / 129	**111** (97.45)	**116** (122.33)	> 5000	**187** (184.90)	**273** (236.42)	**114** (114.73)

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