Semi–Smooth Newton Method for Solving 2D Contact Problems with Tresca and Coulomb Friction

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Abstract. *The contribution deals with contact problems for two elastic bodies with friction. After the description of the problem we present its discretization based on linear or bilinear finite elements. The semi–smooth Newton method is used to find the solution, from which we derive active sets algorithms. Finally, we arrive at the globally convergent dual implementation of the algorithms in terms of the Langrange multipliers for the Tresca problem. Numerical experiments conclude the paper.*

Keywords

contact problems, friction, semi-smooth Newton method, quadratic programming.

1. Introduction

Nowadays, solving contact problems with friction counts among very challenging tasks in mechanics and is of crucial importance in various practical applications. As an example we can state biological processes, the design of machines and transportation systems, and metal forming. This is a strong motivation for the development of methods allowing a reliable and fast simulation, i.e., to implement robust numerical solvers.

As for the history, in 1933, Signorini was one of the first savants, who considered the frictionless contact of an elastic body with a rigid foundation. These problems are often named after him.

Contact problems with friction are inherently nonlinear. This fact complicates the teoretical analysis and the proposal of an efficient numerical solver. General information about contact problems one can find,

e.g., in [15]. The main difficulties of contact problems are two nonlinear conditions: the non–penetration of bodies and friction law. Predominant friction laws are called Coulomb or Tresca.

Newton–type methods for solving contact problems with friction were used already in 1991, see [1]. In more recent contributions is shown that the performance of generalized Newton–like methods is superior to interior point methods (see works of P. Christensen, F. Facchinei and J. Pang). Semi–smooth Newton method in finite dimensions appears already in 1993, see [20], and also more recently in infinite dimensions. For that, the concept of the slant differentiability was introduced; see [2] and the references given therein. The proof of the superlinear convergence of the semi–smooth Newton method is stated in [21].

Names as B. Wohlmuth, K. Kunish, S. Hüeber and G. Stadler or J. Haslinger are quite closely connected to the topic of contact problems with friction and the semi–smooth Newton method.

2. Formulation

Let us consider two homogeneous isotropic elastic bodies represented by bounded domains $\Omega^k \subset \mathbb{R}^2$ with sufficiently smooth boundaries $\partial \Omega^k$, $k = 1, 2$. Each boundary consists of three disjoint parts Γ_u^k, Γ_p^k, and Γ_c^k open in $\partial \Omega^k$ so that $\partial \Omega^k = \overline{\Gamma}_u^k \cup \overline{\Gamma}_p^k \cup \overline{\Gamma}_c^k$ and $\overline{\Gamma}_u^k \neq \emptyset$; see Figure 1. The zero displacements are prescribed on Γ_u^k while surface tractions $p^k \in (L^2(\Gamma_p^k))^2$ act on Γ_p^k. The bodies may get into contact on the *contact interface* given by Γ_c^1 and Γ_c^2, where we consider three contact conditions: the non–penetration of bodies (the unilateral contact law), the transmission of contact stresses, and the Coulomb friction law. Elastic properties of

Ω^k are described by the Lamè constants $\lambda^k, \mu^k > 0$. Finally, assume the volume forces $\boldsymbol{f}^k \in (L^2(\Omega^k))^2$.

Fig. 1: Geometry of the bodies.
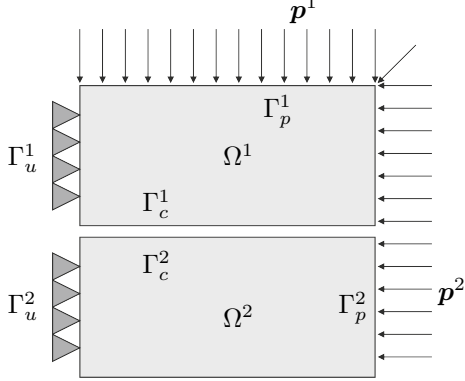
Our aim is to find an equilibrium state of Ω^1 and Ω^2. By the solution of this problem we mean displacement vector fields $\boldsymbol{u}^k = (u_1^k, u_2^k)^\top$, $k = 1, 2$, satisfying the equilibrium equations and the Dirichlet and Neumann conditions:

$$\left.\begin{array}{rcll} \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}^k) + \boldsymbol{f}^k &=& \boldsymbol{0} & \text{in} \quad \Omega^k \\ \boldsymbol{u}^k &=& \boldsymbol{0} & \text{on} \quad \Gamma_u^k \\ \boldsymbol{\sigma}^k \boldsymbol{n}^k - \boldsymbol{p}^k &=& \boldsymbol{0} & \text{on} \quad \Gamma_p^k \end{array}\right\} \quad k = 1, 2, \quad (1)$$

where $\boldsymbol{\sigma}^k := \boldsymbol{\sigma}(\boldsymbol{u}^k)$ is the stress tensor in Ω^k and $\boldsymbol{n}^k$ stands for the unit outward normal vector to $\partial \Omega^k$, $k = 1, 2$. The tress tensor is related to the linearized strain tensor $\boldsymbol{\varepsilon}^k := \boldsymbol{\varepsilon}(\boldsymbol{u}^k) = 1/2(\nabla \boldsymbol{u}^k + \nabla^\top \boldsymbol{u}^k)$ by the Hooke law for linear isotropic materials:

$$\boldsymbol{\sigma}^k := \lambda^k \operatorname{tr}(\boldsymbol{\varepsilon}^k) \boldsymbol{I} + 2\mu^k \boldsymbol{\varepsilon}^k \quad \text{in} \quad \Omega^k,$$

where "tr" denotes the trace of matrices and $\boldsymbol{I} \in \mathbb{R}^{2 \times 2}$ is the identity matrix. Let us note that λ^k, μ^k are given by the Young modulus $E^k > 0$ and the Poisson ratio $\nu^k \in (0, 0.5)$ as $\lambda^k := E^k \nu^k / (1 - (\nu^k)^2)$, $\mu^k := E^k / (2(1 + \nu^k))$, respectively.

To formulate the contact conditions we introduce a predefined one–to–one transfer mapping $\boldsymbol{\chi} : \Gamma_c^1 \mapsto \Gamma_c^2$ by means of which we define the initial distance between the bodies at $\boldsymbol{x} \in \Gamma_c^1$ as $d(\boldsymbol{x}) := \|\boldsymbol{\chi}(\boldsymbol{x}) - \boldsymbol{x}\|$ and the critical direction $\boldsymbol{\nu}(\boldsymbol{x}) := (\boldsymbol{\chi}(\boldsymbol{x}) - \boldsymbol{x})/d(\boldsymbol{x})$, if $d(\boldsymbol{x}) \neq 0$, or $\boldsymbol{\nu}(\boldsymbol{x}) := \boldsymbol{n}^1(\boldsymbol{x})$, if $d(\boldsymbol{x}) = 0$. The symbol $\|.\|$ stands for the Euclidean norm in $\mathbb{R}^2$. The unilateral contact law reads as follows:

$$u_\nu - d \leq 0, \ \sigma_\nu \leq 0, \ \sigma_\nu(u_\nu - d) = 0 \quad \text{on} \quad \Gamma_c^1, \quad (2)$$

where $u_\nu(\boldsymbol{x}) := (\boldsymbol{u}^1(\boldsymbol{x}) - \boldsymbol{u}^2(\boldsymbol{\chi}(\boldsymbol{x})))^\top \boldsymbol{\nu}(\boldsymbol{x})$ stands for the relative normal contact displacement and $\sigma_\nu(\boldsymbol{x}) := \boldsymbol{\nu}(\boldsymbol{x})^\top \boldsymbol{\sigma}^1(\boldsymbol{x}) \boldsymbol{\nu}(\boldsymbol{x})$ is the normal contact stress at $\boldsymbol{x} \in \Gamma_c^1$.

Let $\boldsymbol{t} := \boldsymbol{t}(\boldsymbol{x})$ be an unit vector orthogonal to $\boldsymbol{\nu} := \boldsymbol{\nu}(\boldsymbol{x})$ so that the pair $\{\boldsymbol{\nu}, \boldsymbol{t}\}$ is a local orthonormal basis

in $\mathbb{R}^2$ with the origin at $\boldsymbol{x} \in \Gamma_c^1$. By $\mathcal{F} := \mathcal{F}(\boldsymbol{x})$ we denote given positive coefficient of friction. The Coulomb friction law for $\boldsymbol{x} \in \Gamma_c^1$ reads as follows:

$$\begin{array}{l} |\sigma_t(\boldsymbol{x})| \leq -\mathcal{F}\sigma_\nu(\boldsymbol{x}) \\ |\sigma_t(\boldsymbol{x})| < -\mathcal{F}\sigma_\nu(\boldsymbol{x}) \quad \Rightarrow \quad u_t(\boldsymbol{x}) = \boldsymbol{0} \\ |\sigma_t(\boldsymbol{x})| = -\mathcal{F}\sigma_\nu(\boldsymbol{x}) \quad \Rightarrow \quad \exists c_t \geq 0 : u_t(\boldsymbol{x}) = -c_t \sigma_t(\boldsymbol{x}), \end{array} \quad (3)$$

where $u_t(\boldsymbol{x}) := (\boldsymbol{u}^1(\boldsymbol{x}) - \boldsymbol{u}^2(\boldsymbol{\chi}(\boldsymbol{x})))^\top \boldsymbol{t}(\boldsymbol{x})$ and $\sigma_t(\boldsymbol{x}) := \boldsymbol{t}(\boldsymbol{x})^\top \boldsymbol{\sigma}^1(\boldsymbol{x}) \boldsymbol{\nu}(\boldsymbol{x})$ are the relative tangential contact displacement and the tangential contact stress at $\boldsymbol{x} \in \Gamma_c^1$, respectively.

Finally we require the transmission of the contact stresses:

$$\boldsymbol{\sigma}^1(\boldsymbol{x}) \boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{\sigma}^2(\boldsymbol{\chi}(\boldsymbol{x})) \boldsymbol{\nu}(\boldsymbol{x}) \quad \text{on } \Gamma_c^1, \quad (4)$$

Note that the satisfaction of (4) is inherently hidden in the weak form of the problem. Therefore, the discrete version of the problem does not include this condition explicitly. The formulation of the contact problem with Coulomb friction consists in finding a pair $\boldsymbol{u} := (\boldsymbol{u}^1, \boldsymbol{u}^2)$ of the displacement fields $\boldsymbol{u}^k$ in Ω^k, $k = 1, 2$, satisfying (1), (2), (3), and (4). Let us note that this problem results in an *implicit* variational inequality of the elliptic type. One of possible ways how to obtain the solution is the fixed–point approach. To this end we replace the slip bounds in (3) by an à–priori given non–negative function $g \in L_+^2(\Gamma_c^1)$, $g \geq 0$, so that for $\boldsymbol{x} \in \Gamma_c^1$

$$\begin{array}{l} |\sigma_t(\boldsymbol{x})| \leq g(\boldsymbol{x}) \\ |\sigma_t(\boldsymbol{x})| < g(\boldsymbol{x}) \quad \Longrightarrow \quad u_t(\boldsymbol{x}) = \boldsymbol{0} \\ |\sigma_t(\boldsymbol{x})| = g(\boldsymbol{x}) \quad \Longrightarrow \quad \exists c_t \geq 0 : u_t(\boldsymbol{x}) = -c_t \sigma_t(\boldsymbol{x}). \end{array} \quad (5)$$

Now, (1), (2), (5), and (4) describes the (auxiliary) contact problem with Tresca friction. It is well-known that this problem can be represented by the variational inequality of the second type for which there is the unique solution for each $g \in L_+^2(\Gamma_c^1)$.

Let us consider the mapping $\Psi : L_+^2(\Gamma_c^1) \mapsto L_+^2(\Gamma_c^1)$ defined by

$$\Psi(g) = -\mathcal{F}\sigma_\nu(g)$$

where $\sigma_\nu(g) := \sigma_\nu(\boldsymbol{u}(g))$ is the normal contact stress corresponding to the solution of the contact problem with Tresca friction with given g. It is easily seen that the fixed point g^* of Ψ defines the solution to the contact problem with Coulomb friction, i.e., the point g^* such that $\Psi(g^*) = g^*$. The natural way how to compute fixed points is the method of successive approximations. It is based on the following iterative scheme:

$$\textit{Initialize: } g^{(0)}, \quad g^{(k)} = \Psi(g^{(k-1)}), \quad k = 1, 2, \ldots. \quad (6)$$

These iterations converge when Ψ is contractive that is guaranteed by sufficiently small coefficient of friction $\mathcal{F}$. Finally, note that each evaluation of the mapping Ψ in (6) requires to solve the contact problem with Tresca friction.

Remark 1 When concerning the discrete problem with Tresca friction it is known that there exists a unique solution viz [3]. In case of the Coulomb friction disrete problem the solution exists, but the uniqueness is guaranteed only for $\mathcal{F}$ sufficiently small viz [7] that we consider in this work. The sufficient smallness is mesh dependent. Some of these problems concerning the non–unique solutions are discussed in [8] using continuation in contact problems with Coulomb friction.

3.   Discrete Problems

We describe the finite element approximation based on the linear or bilinear finite elements. Let $\mathcal{N}^k = \{\boldsymbol{x}_i^k : i = 1, \ldots, n^k\}$ be the set of the finite element nodes lying on $\overline{\Omega^k} \setminus \overline{\Gamma_u^k}$, $k = 1, 2$. By $\boldsymbol{u}^k \in \mathbb{R}^{2n^k}$ we denote the (unknown) vector of the nodal displacements corresponding to $\mathcal{N}^k$ and $\boldsymbol{u} = ((\boldsymbol{u}^1)^\top, (\boldsymbol{u}^2)^\top)^\top$. Let $\mathcal{N}_{cont}^k \subset \mathcal{N}^k$ be the subset of the contact nodes of Ω^k, i.e., the nodes lying on $\overline{\Gamma_c^k} \setminus \overline{\Gamma_u^k}$, and let m^k be their number. Moreover, we denote $n = n^1 + n^2$ and $m = m^1$. The stiffness matrix and the load vector corresponding to Ω^k are denoted by $\boldsymbol{K}^k \in \mathbb{R}^{2n^k \times 2n^k}$ and $\boldsymbol{f}^k \in \mathbb{R}^{2n^k}$, respectively, and $\boldsymbol{K} = diag(\boldsymbol{K}^1, \boldsymbol{K}^2)$, $\boldsymbol{f} = ((\boldsymbol{f}^1)^\top, (\boldsymbol{f}^2)^\top)^\top$. With the contact nodes of Ω^1 we associate two matrices $\boldsymbol{N}^1, \boldsymbol{T}^1 \in \mathbb{R}^{m \times n^1}$ so that the ith row of $\boldsymbol{N}^1$ and $\boldsymbol{T}^1$ contains at most two non-zero entries given by components of $\boldsymbol{\nu}(\boldsymbol{x}_i^1)$ and $\boldsymbol{t}(\boldsymbol{x}_i^1)$, respectively, $\boldsymbol{x}_i^1 \in \mathcal{N}_{cont}^1$. With the contact nodes of Ω^2 we associate two matrices $\boldsymbol{N}^2, \boldsymbol{T}^2 \in \mathbb{R}^{m \times n^2}$ as follows: we We identify two different nodes $\boldsymbol{x}_p^2, \boldsymbol{x}_q^2 \in \mathcal{N}_{cont}^2$ nearest to $\boldsymbol{\chi}(\boldsymbol{x}_i^1)$, $\boldsymbol{x}_i^1 \in \mathcal{N}_{cont}^1$ and we compute the relative distance $\delta_i = \|\boldsymbol{\chi}(\boldsymbol{x}_i^1) - \boldsymbol{x}_p^2\| / \|\boldsymbol{x}_q^2 - \boldsymbol{x}_p^2\|$. The ith row of $\boldsymbol{N}^2$ and $\boldsymbol{T}^2$ contains at most four non–zero entries given by the components of $-(1 - \delta_i)\boldsymbol{\nu}(\boldsymbol{x}_i^1)$, $-\delta_i \boldsymbol{\nu}(\boldsymbol{x}_i^1)$ and $-(1 - \delta_i)\boldsymbol{t}(\boldsymbol{x}_i^1)$, $-\delta_i \boldsymbol{t}(\boldsymbol{x}_i^1)$, respectively. Moreover, we introduce $\boldsymbol{N} = (\boldsymbol{N}^1, \boldsymbol{N}^2)$ and $\boldsymbol{T} = (\boldsymbol{T}^1, \boldsymbol{T}^2)$. By $\boldsymbol{d} \in \mathbb{R}^m$ we denote the vector whose entries are given by $d_i = d(\boldsymbol{x}_i^1)$. Finally, $\boldsymbol{\mathcal{F}}$ stands for the diagonal matrix $\boldsymbol{\mathcal{F}} := diag(\mathcal{F}_1, \ldots, \mathcal{F}_m) \in \mathbb{R}^{m \times m}$, where $\mathcal{F}_i = \mathcal{F}(\boldsymbol{x}_i^1)$, $\boldsymbol{x}_i^1 \in \mathcal{N}_{cont}^1$. Note that the relative tangential displacement at the contact node $\boldsymbol{x}_i^1$ is given by $u_{t,i} := u_t(\boldsymbol{x}_i^1) = (\boldsymbol{T}\boldsymbol{u})_i$.

The discrete problems use two Lagrange multipliers $\boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t \in \mathbb{R}^m$ that approximates $-\sigma_\nu$ and $-\sigma_t$ at the contact nodes, respectively. The discrete contact problem with Coulomb friction reads as follows:

$$\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} + \boldsymbol{N}^\top \boldsymbol{\lambda}_\nu + \boldsymbol{T}^\top \boldsymbol{\lambda}_t = \boldsymbol{0}, \qquad (7)$$

$$\boldsymbol{N}\boldsymbol{u} - \boldsymbol{d} \leq \boldsymbol{0}, \ \boldsymbol{\lambda}_\nu \geq \boldsymbol{0}, \ \boldsymbol{\lambda}_\nu^\top (\boldsymbol{N}\boldsymbol{u} - \boldsymbol{d}) = \boldsymbol{0}, \quad (8)$$

$$\begin{array}{l} |\lambda_{t,i}| \leq \mathcal{F}_i \lambda_{\nu,i} \\ |\lambda_{t,i}| < \mathcal{F}_i \lambda_{\nu,i} \Rightarrow u_{t,i} = 0 \qquad\qquad\qquad (9) \\ |\lambda_{t,i}| = \mathcal{F}_i \lambda_{\nu,i} \Rightarrow \exists c_t \geq 0 : \ u_{t,i} = c_t \lambda_{t,i} \end{array}$$

for $i = 1, \ldots, m$.

In order to obtain the discrete contact problem with Tresca friction, we replace $\mathcal{F}_i \lambda_{\nu,i}$ in (9) by the entries g_i of $\boldsymbol{g} \in \mathbb{R}^m$, $\boldsymbol{g} \geq \boldsymbol{0}$, that approximate values of g from (5) at the nodes of $\mathcal{N}_{cont}^1$.

Next we use the equivalent formulation of the previous problems as the systems of non–smooth equations. To this end we introduce the projection mappings

$$\boldsymbol{P}_{\mathbb{R}_+^m} : \mathbb{R}^m \mapsto \mathbb{R}_+^m, \quad \boldsymbol{P}_{\boldsymbol{\Lambda}(\boldsymbol{r})} : \mathbb{R}^m \mapsto \boldsymbol{\Lambda}(\boldsymbol{r})$$

with $\mathbb{R}_+^m := \{\boldsymbol{\mu} \in \mathbb{R}^m : \boldsymbol{\mu} \geq \boldsymbol{0}\}$, $\boldsymbol{\Lambda}(\boldsymbol{r}) := \{\boldsymbol{\mu} \in \mathbb{R}^m : |\boldsymbol{\mu}| \leq \boldsymbol{r}\}$, respectively, where the inequalities and the absolute value are understood componentwise and $\boldsymbol{r} \in \mathbb{R}_+^m$. The definitions of the components of $\boldsymbol{P}_{\mathbb{R}_+^m}$, $\boldsymbol{P}_{\boldsymbol{\Lambda}(\boldsymbol{r})}$ are based on the max–function in $\mathbb{R}^1$:

$$(\boldsymbol{P}_{\mathbb{R}_+^m})_i(\boldsymbol{\mu}) = \max\{0, \mu_i\}, \qquad\qquad (10)$$

$$(\boldsymbol{P}_{\boldsymbol{\Lambda}(\boldsymbol{r})})_i(\boldsymbol{\mu}) = \max\{0, \mu_i + r_i\} - \max\{0, \mu_i - r_i\} - r_i. \ (11)$$

Denote $\boldsymbol{y} := (\boldsymbol{u}^\top, \boldsymbol{\lambda}_\nu^\top, \boldsymbol{\lambda}_t^\top)^\top \in \mathbb{R}^{2n+2m}$ and consider a parameter $\rho > 0$. The discrete contact problem with Coulomb friction is equivalent to the equation

$$\boldsymbol{G}(\boldsymbol{y}) = \boldsymbol{0} \qquad\qquad (12)$$

where $\boldsymbol{G} : \mathbb{R}^{2n+2m} \mapsto \mathbb{R}^{2n+2m}$ is defined by

$$\boldsymbol{G}(\boldsymbol{y}) := \left(\begin{array}{c} \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} + \boldsymbol{N}^\top \boldsymbol{\lambda}_\nu + \boldsymbol{T}^\top \boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\mathbb{R}_+^m}(\boldsymbol{\lambda}_\nu + \rho(\boldsymbol{N}\boldsymbol{u} - \boldsymbol{d})) \\ \boldsymbol{\lambda}_t - \boldsymbol{P}_{\boldsymbol{\Lambda}(\boldsymbol{\mathcal{F}}\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t + \rho\boldsymbol{T}\boldsymbol{u}) \end{array} \right).$$

The discrete contact problem with Tresca friction is equivalent to the equation

$$\boldsymbol{G}_{\boldsymbol{g}}(\boldsymbol{y}) = \boldsymbol{0} \qquad\qquad (13)$$

where $\boldsymbol{G}_{\boldsymbol{g}} : \mathbb{R}^{2n+2m} \mapsto \mathbb{R}^{2n+2m}$ is defined by

$$\boldsymbol{G}_{\boldsymbol{g}}(\boldsymbol{y}) := \left(\begin{array}{c} \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} + \boldsymbol{N}^\top \boldsymbol{\lambda}_\nu + \boldsymbol{T}^\top \boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\mathbb{R}_+^m}(\boldsymbol{\lambda}_\nu + \rho(\boldsymbol{N}\boldsymbol{u} - \boldsymbol{d})) \\ \boldsymbol{\lambda}_t - \boldsymbol{P}_{\boldsymbol{\Lambda}(\boldsymbol{g})}(\boldsymbol{\lambda}_t + \rho\boldsymbol{T}\boldsymbol{u}) \end{array} \right).$$

Note that both $\boldsymbol{G}$ and $\boldsymbol{G}_{\boldsymbol{g}}$ are piecewise smooth functions with the non-smooth points determined by the max–function in (10) and (11).

4.   Algorithms

There are two principal possibilities how to derive an algorithm: (i) solving directly the discrete contact problem with Coulomb friction; (ii) solving the discrete contact problem with Tresca friction and combine it with the method of successive approximations (6). In both cases we will use a variant of the Newton method that is based on the iterative scheme:

$$\boldsymbol{J}_{\boldsymbol{F}}(\boldsymbol{y}^{(k-1)})\boldsymbol{y}^{(k)} = \boldsymbol{J}_{\boldsymbol{F}}(\boldsymbol{y}^{(k-1)})\boldsymbol{y}^{(k-1)} - \boldsymbol{F}(\boldsymbol{y}^{(k-1)}) \quad (14)$$

for $k = 1, 2, \ldots$, where $\boldsymbol{F} : \mathbb{R}^{2n+2m} \mapsto \mathbb{R}^{2n+2m}$ is a sufficiently smooth function and $\boldsymbol{J_F}(\boldsymbol{y})$ approximates the Jacobi matrix to $\boldsymbol{F}$ at $\boldsymbol{y} \in \mathbb{R}^{2n+2m}$. It is well known that the sequence $\{\boldsymbol{y}^{(k)}\}$ generated by (14) converges superlinearly to the solution of $\boldsymbol{F}(\boldsymbol{y}) = \boldsymbol{0}$, if $\boldsymbol{F}$ is slantly differentiable and if the initial iteration $\boldsymbol{y}^{(0)} \in \mathbb{R}^{2n+2m}$ is sufficiently close to the solution, see [2]. The slanting function $\boldsymbol{F}^o$, see [21], is used as $\boldsymbol{J_F}$ and (14) is called the semi-smooth Newton method (SSNM). As the max-function is slantly differentiable, we can apply the SSNM for solving (12) and (13).

We will show how to derive the slanting function $\boldsymbol{G}^o$, $\boldsymbol{G_g^o}$ to $\boldsymbol{G}$, $\boldsymbol{G_g}$, respectively. The idea is based on introducing the active and inactive sets that determine the smooth pieces of $\boldsymbol{G}$ or $\boldsymbol{G_g}$. These smooth pieces are differentiable so that the standard differential rules can be used. Such implementation of the SSNM is equivalent to an active set algorithm. The distinction between $\boldsymbol{G}^o$ and $\boldsymbol{G_g^o}$ consists in the fact that $\boldsymbol{G_g^o}$ may be symmetrized.

Before discussing both algorithms we introduce notations. Let $\mathcal{M} = \{1, 2, \ldots, m\}$ be the set of all indices and let $\boldsymbol{y} = (\boldsymbol{u}^\top, \boldsymbol{\lambda}_\nu^\top, \boldsymbol{\lambda}_t^\top)^\top \in \mathbb{R}^{2n+2m}$ be given. The active set $\mathcal{A}_\nu := \mathcal{A}_\nu(\boldsymbol{y})$ corresponding to the non-penetration condition is defined by

$$\mathcal{A}_\nu(\boldsymbol{y}) = \{i \in \mathcal{M} : \lambda_{\nu,i} + \rho(\mathbf{N}\mathbf{u} - \mathbf{d})_i > 0\}. \quad (15)$$

The respective inactive set is its complement $\mathcal{I}_\nu := \mathcal{I}_\nu(\boldsymbol{y}) = \mathcal{M} \setminus \mathcal{A}_\nu(\boldsymbol{y})$. For $\mathcal{S} \subseteq \mathcal{M}$ we introduce the diagonal matrix $\boldsymbol{D}_\mathcal{S} = diag(s_1, \ldots, s_m) \in \mathbb{R}^{m \times m}$ with

$$s_i = \begin{cases} 1 & \text{for } i \in \mathcal{S}, \\ 0 & \text{for } i \notin \mathcal{S}. \end{cases}$$

4.1. Non–symmetric case

First we propose the algorithm for direct solving discrete contact problems with Coulomb friction. We introduce two inactive sets $\mathcal{I}_t^+ := \mathcal{I}_t^+(\boldsymbol{y})$, $\mathcal{I}_t^- := \mathcal{I}_t^-(\boldsymbol{y})$ corresponding to the conditions of Coulomb friction:

$$\mathcal{I}_t^+(\boldsymbol{y}) = \{i \in \mathcal{M} : \lambda_{t,i} + \rho(\mathbf{T}\mathbf{u})_i > \mathcal{F}_i \lambda_{\nu,i}\}, \quad (16)$$
$$\mathcal{I}_t^-(\boldsymbol{y}) = \{i \in \mathcal{M} : \lambda_{t,i} + \rho(\mathbf{T}\mathbf{u})_i < -\mathcal{F}_i \lambda_{\nu,i}\}. (17)$$

The respective active set is the complement $\mathcal{A}_t := \mathcal{A}_t(\boldsymbol{y}) = \mathcal{M} \setminus (\mathcal{I}_t^+(\boldsymbol{y}) \cup \mathcal{I}_t^-(\boldsymbol{y}))$. Due to the definition of the projections (10) and (11), we get the following expression for $\boldsymbol{G}$ at $\boldsymbol{y}$:

$$\boldsymbol{G}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} + \boldsymbol{N}^\top\boldsymbol{\lambda}_\nu + \boldsymbol{T}^\top\boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_\nu - \boldsymbol{D}_{\mathcal{A}_\nu}(\boldsymbol{\lambda}_\nu + \rho(\boldsymbol{N}\boldsymbol{u} - \boldsymbol{d})) \\ \boldsymbol{\lambda}_t - \boldsymbol{D}_{\mathcal{A}_t}(\boldsymbol{\lambda}_t + \rho\boldsymbol{T}\boldsymbol{u}) + \mathcal{F}(\boldsymbol{D}_{\mathcal{I}_t^-} - \boldsymbol{D}_{\mathcal{I}_t^+})\boldsymbol{\lambda}_\nu \end{pmatrix}.$$

After differentiating the right-hand side, we arrive at the slanting function:

$$\boldsymbol{G}^o(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{K} & \boldsymbol{N}^\top & \boldsymbol{T}^\top \\ -\rho\boldsymbol{D}_{\mathcal{A}_\nu}\boldsymbol{N} & \boldsymbol{D}_{\mathcal{I}_\nu} & \boldsymbol{0} \\ -\rho\boldsymbol{D}_{\mathcal{A}_t}\boldsymbol{T} & \mathcal{F}(\boldsymbol{D}_{\mathcal{I}_t^-} - \boldsymbol{D}_{\mathcal{I}_t^+}) & \boldsymbol{D}_{\mathcal{I}_t^+ \cup \mathcal{I}_t^-} \end{pmatrix}$$

and

$$\boldsymbol{G}^o(\boldsymbol{y})\boldsymbol{y} - \boldsymbol{G}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{f} \\ -\rho\boldsymbol{D}_{\mathcal{A}_\nu}\boldsymbol{d} \\ \boldsymbol{0} \end{pmatrix}.$$

We get the active-set implementation of the SSNM.

Algorithm *ActiveSetCoulomb*

(0.) Set $k := 1$, $\rho > 0$, $\varepsilon_u > 0$, $\boldsymbol{u}^{(0)} \in \mathbb{R}^{2n}$, $\boldsymbol{\lambda}_\nu^{(0)}, \boldsymbol{\lambda}_t^{(0)} \in \mathbb{R}^m$.

(1.) At $\boldsymbol{y} = ((\boldsymbol{u}^{(k-1)})^\top, (\boldsymbol{\lambda}_\nu^{(k-1)})^\top, (\boldsymbol{\lambda}_t^{(k-1)})^\top)^\top$ define the active and inactive sets:

$$\mathcal{A}_\nu = \mathcal{A}_\nu(\boldsymbol{y}), \ \mathcal{I}_\nu = \mathcal{I}_\nu(\boldsymbol{y}),$$
$$\mathcal{A}_t = \mathcal{A}_t(\boldsymbol{y}), \ \mathcal{I}_t^+ = \mathcal{I}_t^+(\boldsymbol{y}), \ \mathcal{I}_t^- = \mathcal{I}_t^-(\boldsymbol{y}).$$

(2.) Solve:

$$\begin{pmatrix} \boldsymbol{K} & \boldsymbol{N}^\top & \boldsymbol{T}^\top \\ \boldsymbol{D}_{\mathcal{A}_\nu}\boldsymbol{N} & \boldsymbol{D}_{\mathcal{I}_\nu} & \boldsymbol{0} \\ \boldsymbol{D}_{\mathcal{A}_t}\boldsymbol{T} & \mathcal{F}(\boldsymbol{D}_{\mathcal{I}_t^-} - \boldsymbol{D}_{\mathcal{I}_t^+}) & \boldsymbol{D}_{\mathcal{I}_t^+ \cup \mathcal{I}_t^-} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}^{(k)} \\ \boldsymbol{\lambda}_\nu^{(k)} \\ \boldsymbol{\lambda}_t^{(k)} \end{pmatrix}$$
$$= \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{D}_{\mathcal{A}_\nu}\boldsymbol{d} \\ \boldsymbol{0} \end{pmatrix}.$$

(3.) Set $err^{(k)} := \|\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(k-1)}\|/\|\boldsymbol{u}^{(k)}\|$. If $err^{(k)} \leq \varepsilon_u$, return $\boldsymbol{u} := \boldsymbol{u}^{(k)}$, $\boldsymbol{\lambda}_\nu := \boldsymbol{\lambda}_\nu^{(k)}$, and $\boldsymbol{\lambda}_t := \boldsymbol{\lambda}_t^{(k)}$.

(4.) Set $k := k + 1$ and go to step (1.).

The most expensive part of each iteration is solving the block linear systems in step (2.). These systems correspond to the finite element approximation of the linear elasticity problem with the following mixed conditions on Γ_c^1: the relative normal contact displacements, the zero relative normal contact stresses, and the zero relative tangential contact displacements at the contact nodes with indices of $\mathcal{A}_\nu$, $\mathcal{I}_\nu$, and $\mathcal{A}_t$, respectively; the achievement of the slip bounds $\lambda_{t,i} = -\mathcal{F}_i\lambda_{\nu,i}$ and $\lambda_{t,i} = \mathcal{F}_i\lambda_{\nu,i}$ at the contact nodes with indices of $\mathcal{I}_t^-$ and $\mathcal{I}_t^+$, respectively. Note that ρ is discarded from the system in step (2.). Indeed, if this system is solved exactly, then ρ would not play any role in the definitions of the active and inactive sets (except of the first iteration).

4.2. Symmetric case

To propose the algorithm for solving the discrete contact problem with Tresca friction we define the inactive sets $\mathcal{I}_t^+ := \mathcal{I}_t^+(\boldsymbol{y})$, $\mathcal{I}_t^- := \mathcal{I}_t^-(\boldsymbol{y})$ as follows:

$$\mathcal{I}_t^+(\boldsymbol{y}) = \{i \in \mathcal{M} : \lambda_{t,i} + \rho(\mathbf{T}\mathbf{u})_i > g_i\}, \quad (18)$$
$$\mathcal{I}_t^-(\boldsymbol{y}) = \{i \in \mathcal{M} : \lambda_{t,i} + \rho(\mathbf{T}\mathbf{u})_i < -g_i\}. \quad (19)$$

The respective active set is again the complement. Using (10) and (11), we get

$$\boldsymbol{G_g}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} + \boldsymbol{N}^\top\boldsymbol{\lambda}_\nu + \boldsymbol{T}^\top\boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_\nu - \boldsymbol{D}_{\mathcal{A}_\nu}(\boldsymbol{\lambda}_\nu + \rho(\boldsymbol{N}\boldsymbol{u} - \boldsymbol{d})) \\ \boldsymbol{\lambda}_t - \boldsymbol{D}_{\mathcal{A}_t}(\boldsymbol{\lambda}_t + \rho\boldsymbol{T}\boldsymbol{u}) + (\boldsymbol{D}_{\mathcal{I}_t^-} - \boldsymbol{D}_{\mathcal{I}_t^+})\boldsymbol{g} \end{pmatrix}.$$

The slanting function G_g^o to G_g reads as follows:

$$G_g^o(y) = \begin{pmatrix} K & N^\top & T^\top \\ -\rho D_{\mathcal{A}_\nu} N & D_{\mathcal{I}_\nu} & 0 \\ -\rho D_{\mathcal{A}_t} T & 0 & D_{\mathcal{I}_t^+ \cup \mathcal{I}_t^-} \end{pmatrix}$$

and

$$G_g^o(y)y - G_g(y) = \begin{pmatrix} f \\ -\rho D_{\mathcal{A}_\nu} d \\ -(D_{\mathcal{I}_t^-} - D_{\mathcal{I}_t^+})g \end{pmatrix}.$$

The implementation of the SSNM uses the fact that components of the solutions of the inner linear systems corresponding to the indices of $\mathcal{I}_\nu$, $\mathcal{I}_t^+$ and $\mathcal{I}_t^-$ may be found without any complication. Eliminating them, we arrive at the reduced system with the symmetric saddle–point matrix for the remaining components.

Algorithm *ActiveSetTresca*

This algorithm differs from the previous one *ActiveSetCoulomb* only in step (2.):

(2.) Solve:

$$\begin{pmatrix} K & N_{\mathcal{A}_\nu}^\top & T_{\mathcal{A}_t}^\top \\ N_{\mathcal{A}_\nu} & 0 & 0 \\ T_{\mathcal{A}_t} & 0 & 0 \end{pmatrix} \begin{pmatrix} u^{(k)} \\ \lambda_{\nu,\mathcal{A}_\nu}^{(k)} \\ \lambda_{t,\mathcal{A}_t}^{(k)} \end{pmatrix} =$$

$$= \begin{pmatrix} f - T_{\mathcal{I}_t^+}^\top g_{\mathcal{I}_t^+} + T_{\mathcal{I}_t^-}^\top g_{\mathcal{I}_t^-} \\ d_{\mathcal{A}_\nu} \\ 0 \end{pmatrix},$$

and set $\lambda_{\nu,\mathcal{I}_\nu}^{(k)} = 0$, $\lambda_{t,\mathcal{I}_t^+}^{(k)} = g_{\mathcal{I}_t^+}$, $\lambda_{t,\mathcal{I}_t^-}^{(k)} = -g_{\mathcal{I}_t^-}$.

Here, we use the index set as the multiindex. In the case of a matrix, it represents its submatrix composed by rows with indices of the index set. The system in step (2.) corresponds again to the finite element approximation of a mixed problem on Γ_c^1. Comparing with that one in the previous section, the only distinction consists in the interpretation of conditions corresponding to the indices of $\mathcal{I}_t^-$ and $\mathcal{I}_t^+$.

5. Implementation

In this section, we interpret step (2.) of algorithm *ActiveSetTresca* as the minimization of a strictly quadratic objective function in terms of λ. Therefore, the conjugate gradient method can be used as the inner solver. First of all we introduce notation.

Let us combine the active and inactive sets defined by (15), (18) and (19) in $\mathcal{A}$ and $\mathcal{I}$ for $y = (u^\top, \lambda_\nu^\top, \lambda_t^\top)^\top$ as follows: $\mathcal{A} = \{i | i \in \mathcal{A}_\nu\} \cup \{i+m | i \in \mathcal{A}_t\}$ and $\mathcal{I} = \{1, 2, \ldots, 2m\} \setminus \mathcal{A}$. Denote

$$B = \begin{pmatrix} N \\ T \end{pmatrix}, \quad c = \begin{pmatrix} d \\ 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda_\nu \\ \lambda_t \end{pmatrix},$$

and

$$q(\lambda) = \frac{1}{2}\lambda^\top A \lambda - \lambda^\top b,$$

where $A = BK^{-1}B^\top$, $b = BK^{-1}f - c$. The gradient to q at λ reads as

$$\nabla q(\lambda) = A\lambda - b.$$

The linear system in step (2.) of algorithm *ActiveSetTresca* takes the form

$$\begin{pmatrix} K & B_{\mathcal{A}}^\top \\ B_{\mathcal{A}} & 0 \end{pmatrix} \begin{pmatrix} u^{(k)} \\ \lambda_{\mathcal{A}}^{(k)} \end{pmatrix} = \begin{pmatrix} \widehat{f} \\ c_{\mathcal{A}} \end{pmatrix}, \qquad (20)$$

where $\widehat{f} = f - T_{\mathcal{I}_t^+}^\top g_{\mathcal{I}_t^+} + T_{\mathcal{I}_t^-}^\top g_{\mathcal{I}_t^-}$. The remaining components of $\lambda^{(k)}$ are given by

$$\lambda_{\nu,\mathcal{I}_\nu}^{(k)} = 0, \ \lambda_{t,\mathcal{I}_t^+}^{(k)} = g_{\mathcal{I}_t^+}, \ \lambda_{t,\mathcal{I}_t^-}^{(k)} = -g_{\mathcal{I}_t^-}. \qquad (21)$$

Lemma 1 *(i) Let $\mathcal{A} = \emptyset$ at $y = y^{(k-1)}$. Then $\lambda^{(k)}$ is fully determined by (21) and*

$$u^{(k)} = K^{-1}\widehat{f} = K^{-1}(f - B^\top \lambda^{(k)}).$$

(ii) Let $\mathcal{A} \neq \emptyset$ at $y = y^{(k-1)}$. Then $\lambda_{\mathcal{A}}^{(k)}$ is the minimizer to the problem

$$\min \frac{1}{2}\lambda_{\mathcal{A}}^\top A_{\mathcal{A}\mathcal{A}} \lambda_{\mathcal{A}} - \lambda_{\mathcal{A}}^\top \widehat{b}_{\mathcal{A}}, \qquad (22)$$

where $\widehat{b}_{\mathcal{A}} = B_{\mathcal{A}} K^{-1} \widehat{f} - c_{\mathcal{A}}$. The remaining components of $\lambda^{(k)}$ are given by (21) and

$$u^{(k)} = K^{-1}(\widehat{f} - B_{\mathcal{A}}^\top \lambda_{\mathcal{A}}^{(k)}) = K^{-1}(f - B^\top \lambda^{(k)}).$$

Proof: The statement *(i)* is straightforward. Let us assume $\mathcal{A} \neq \emptyset$. The first block equation in (20) gives $u^{(k)} = K^{-1}(\widehat{f} - B_{\mathcal{A}}^\top \lambda_{\mathcal{A}}^{(k)})$ that enables us to eliminate the unknown $u^{(k)}$. The resulting linear system in terms of $\lambda^{(k)}$ reads as

$$A_{\mathcal{A}\mathcal{A}} \lambda_{\mathcal{A}}^{(k)} = \widehat{b}_{\mathcal{A}}. \qquad (23)$$

Note that $A_{\mathcal{A}\mathcal{A}}$ is the symmetric, positive definite so that (23) and the minimization problem (22) result in the same solution. The lemma is proved. $\square$

The next lemma shows how to define the active and inactive sets without the knowledge of $u^{(k)}$.

Lemma 2 *Let $\lambda = \lambda^{(k)}$. It holds:*

$$\begin{aligned} \mathcal{A}_\nu &= \{i \in \mathcal{M} : \lambda_{\nu,i} - \rho \nabla_i q(\lambda) > 0\}. \\ \mathcal{I}_t^+ &= \{i \in \mathcal{M} : \lambda_{t,i} - \rho \nabla_{i+m} q(\lambda) > g_i\}, \\ \mathcal{I}_t^- &= \{i \in \mathcal{M} : \lambda_{t,i} - \rho \nabla_{i+m} q(\lambda) < -g_i\}. \end{aligned}$$

Proof: Using $\boldsymbol{u}^{(k)} = \boldsymbol{K}^{-1}(\boldsymbol{f} - \boldsymbol{B}^\top \boldsymbol{\lambda}^{(k)})$, we get

$$\nabla q(\boldsymbol{\lambda}^{(k)}) = \boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{B}^\top \boldsymbol{\lambda}^{(k)} - \boldsymbol{B}\boldsymbol{K}^{-1}\boldsymbol{f} + \boldsymbol{c} =$$
$$= -\boldsymbol{B}\boldsymbol{u}^{(k)} + \boldsymbol{c} = \left(\begin{array}{c} -\boldsymbol{N}\boldsymbol{u}^{(k)} + \boldsymbol{d} \\ -\boldsymbol{T}\boldsymbol{u}^{(k)} \end{array} \right).$$

The lemma is proved. $\qquad\square$

Finally, we prove how to determine $\boldsymbol{\lambda}^{(k)}$ via the constrained minimization problem.

Lemma 3 *Let $\mathcal{A} \neq \emptyset$ at $\boldsymbol{y} = \boldsymbol{y}^{(k-1)}$. Then $\boldsymbol{\lambda}^{(k)}$ determined in step (2.) of algorithm ActiveSetTresca is the minimizer of the problem:*

$$\min q(\boldsymbol{\lambda}), \text{ subject to} \qquad (24)$$

$$\boldsymbol{\lambda}_{\nu,\mathcal{I}_\nu} = \boldsymbol{0}, \boldsymbol{\lambda}_{t,\mathcal{I}_t^+} = \boldsymbol{g}_{\mathcal{I}_t^+}, \boldsymbol{\lambda}_{t,\mathcal{I}_t^-} = -\boldsymbol{g}_{\mathcal{I}_t^-}. \qquad (25)$$

Proof: Let $\bar{\boldsymbol{\lambda}}$ be the solution to (24), (25). Then trivially $\bar{\boldsymbol{\lambda}}_\mathcal{I} = \boldsymbol{\lambda}_\mathcal{I}^{(k)}$. We derive:

$$\begin{aligned} q(\boldsymbol{\lambda}) &= \frac{1}{2}\boldsymbol{\lambda}_\mathcal{A}^\top \boldsymbol{A}_{\mathcal{A}\mathcal{A}}\boldsymbol{\lambda}_\mathcal{A} + \boldsymbol{\lambda}_\mathcal{A}^\top \boldsymbol{A}_{\mathcal{A}\mathcal{I}}\boldsymbol{\lambda}_\mathcal{I} + \frac{1}{2}\boldsymbol{\lambda}_\mathcal{I}^\top \boldsymbol{A}_{\mathcal{I}\mathcal{I}}\boldsymbol{\lambda}_\mathcal{I} \\ &\quad - \boldsymbol{\lambda}_\mathcal{A}^\top \boldsymbol{b}_\mathcal{A} - \boldsymbol{\lambda}_\mathcal{I}^\top \boldsymbol{b}_\mathcal{I} \\ &= \frac{1}{2}\boldsymbol{\lambda}_\mathcal{A}^\top \boldsymbol{A}_{\mathcal{A}\mathcal{A}}\boldsymbol{\lambda}_\mathcal{A} - \boldsymbol{\lambda}_\mathcal{A}^\top (\boldsymbol{b}_\mathcal{A} - \boldsymbol{A}_{\mathcal{A}\mathcal{I}}\boldsymbol{\lambda}_\mathcal{I}) \\ &\quad + \left(\frac{1}{2}\boldsymbol{\lambda}_\mathcal{I}^\top \boldsymbol{A}_{\mathcal{I}\mathcal{I}}\boldsymbol{\lambda}_\mathcal{I} - \boldsymbol{\lambda}_\mathcal{I}^\top \boldsymbol{b}_\mathcal{I} \right). \end{aligned}$$

The value of the last bracket is given by (25). Moreover, $\widehat{\boldsymbol{b}}_\mathcal{A} = \boldsymbol{b}_\mathcal{A} - \boldsymbol{A}_{\mathcal{A}\mathcal{I}}\boldsymbol{\lambda}_\mathcal{I}$ so that

$$q(\boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{\lambda}_\mathcal{A}^\top \boldsymbol{A}_{\mathcal{A}\mathcal{A}}\boldsymbol{\lambda}_\mathcal{A} - \boldsymbol{\lambda}_\mathcal{A}^\top \widehat{\boldsymbol{b}}_\mathcal{A} + const.$$

Therefore, $\bar{\boldsymbol{\lambda}}_\mathcal{A}$ is the solution to (22), i.e., $\bar{\boldsymbol{\lambda}}_\mathcal{A} = \boldsymbol{\lambda}_\mathcal{A}^{(k)}$. The lemma is proved. $\qquad\square$

Algorithm *ActiveSetTresca* (Dual Version)

(0.) Set $k := 1$, $\rho > 0$, $\varepsilon_\lambda > 0$, $\boldsymbol{\lambda}^{(0)} \in \mathbb{R}^{2m}$.

(1.) Define the active and inactive sets at $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(k-1)}$:

$$\begin{aligned} \mathcal{A}_\nu &= \{i \in \mathcal{M} : \lambda_{\nu,i} - \rho\nabla_i q(\boldsymbol{\lambda}) > 0\}, \\ \mathcal{I}_t^+ &= \{i \in \mathcal{M} : \lambda_{t,i} - \rho\nabla_{i+m} q(\boldsymbol{\lambda}) > g_i\}, \\ \mathcal{I}_t^- &= \{i \in \mathcal{M} : \lambda_{t,i} - \rho\nabla_{i+m} q(\boldsymbol{\lambda}) < -g_i\}, \\ \mathcal{I}_\nu &= \mathcal{M} \setminus \mathcal{A}_\nu, \ \mathcal{A}_t = \mathcal{M} \setminus (\mathcal{I}_t^+ \cup \mathcal{I}_t^-) \end{aligned}$$

and $\mathcal{A}, \mathcal{I}$.

(2.) If $\mathcal{A} = \emptyset$, set $\boldsymbol{\lambda}^{(k)}$ so that

$$\boldsymbol{\lambda}_{\nu,\mathcal{I}_\nu}^{(k)} = \boldsymbol{0}, \ \boldsymbol{\lambda}_{t,\mathcal{I}_t^+}^{(k)} = \boldsymbol{g}_{\mathcal{I}_t^+}, \ \boldsymbol{\lambda}_{t,\mathcal{I}_t^-}^{(k)} = -\boldsymbol{g}_{\mathcal{I}_t^-},$$

else find $\qquad \boldsymbol{\lambda}^{(k)} := \arg\min q(\boldsymbol{\lambda})$

s.t. $\boldsymbol{\lambda}_{\nu,\mathcal{I}_\nu} = \boldsymbol{0}, \ \boldsymbol{\lambda}_{t,\mathcal{I}_t^+} = \boldsymbol{g}_{\mathcal{I}_t^+}, \ \boldsymbol{\lambda}_{t,\mathcal{I}_t^-} = -\boldsymbol{g}_{\mathcal{I}_t^-}.$

(3.) Set $err^{(k)} := \|\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}^{(k-1)}\|/\|\boldsymbol{\lambda}^{(k)}\|$. If $err^{(k)} \leq \varepsilon_\lambda$, return

$$\boldsymbol{u} := \boldsymbol{K}^{-1}(\boldsymbol{f} - \boldsymbol{B}^\top \boldsymbol{\lambda}^{(k)}), \ \boldsymbol{\lambda} := \boldsymbol{\lambda}^{(k)}$$

and stop.

(4.) Set $k := k + 1$ and go to step (1.).

6.   Globalization Strategy

The standard result on convergence of the semi–smooth Newton method [2, 21] requires an initial iteration "sufficiently close" to the solution. Unfortunately, it is impractical from the point of view of computations. To overcome this drawback, it is proposed by Jungho Lee in [17] to find an initial guess of the solution by projecting an auxiliary solution of an appropriate unconstrained problem, i.e., the problem without the non–penetration and friction conditions.

In this section, we comment briefly an other *globalization strategy*. The main idea consists in inexact solving of inner linear systems using several steps of the conjugate gradient method. Indeed, our implementation of the algorithm *ActiveSetTresca* may be interpreted as the restarted conjugate gradient method, in which all inner iterations connected to one sequence $\{\boldsymbol{\lambda}^{(l)}\}$ give the decreasing sequence $\{q(\boldsymbol{\lambda}^{(l)})\}$. Moreover, we adopt from [3, 20] the criterion for the adaptive (inner) precision control. Therefore, we arrive at the final implementation of the inexact semi–smooth Newton method that is closely related to the MPRGP algorithm of [3]. The main distinction consists in definitions of the active and inactive sets that are given in our case by (15), (18) and (19). Nevertheless, the same convergence result may be proved as for the MPRGP that is valid for each initial iteration $\boldsymbol{\lambda}^0$ from an à–priori known feasible set $\mathbb{R}^m_+ \times \boldsymbol{\Lambda}(\boldsymbol{g})$.

Note that the MPRGP uses the reduced gradient that is defined by the steplength $\tilde{\alpha}$ whose value is bounded by a multiple of the inverse to the largest eigenvalue of $\boldsymbol{A}$. In our implementation, the role of $\tilde{\alpha}$ plays the parameter ρ from the definitions of the active and inactive sets (15), (18) and (19).

7.   Numerical Experiments

Numerical experiments have been computed by Matlab R2009a using computer with RAM 4GB, Intel Core i5 (2.53GHz) 460M.

Let us consider two plane elastic bodies

$$\Omega^1 = (0,3) \times (1,2) \quad \text{and} \quad \Omega^2 = (0,3) \times (0,1)$$

made of an isotropic, homogeneous material characterized by the Young modulus 2.119×10^{11} and the Poisson ratio 0.277 (steel). The decompositions of $\partial\Omega^1$ and $\partial\Omega^2$ are as follows:

$$\Gamma_u^1 = \{0\} \times (1,2), \quad \Gamma_u^2 = \{0\} \times (0,1),$$
$$\Gamma_c^1 = (0,3) \times \{1\}, \quad \Gamma_c^2 = (0,3) \times \{1\},$$
$$\Gamma_p^1 = \partial\Omega^1 \backslash \overline{\Gamma_u^1 \cup \Gamma_c^1}, \quad \Gamma_p^2 = \partial\Omega^2 \backslash \overline{\Gamma_u^2 \cup \Gamma_c^2}.$$

The given function g representing the slip bound is equal to 1.7×10^7. The volume forces vanish for both bodies. The non–vanishing surface tractions $\boldsymbol{p}^1 = (p_1^1, p_2^1)$ act on Γ_p^1 so that

$$p_1^1(s,2) = 0, \ p_2^1(s,2) = p_{2,L}^1 + p_{2,R}^1 s,$$

$s \in (0,3)$,

$$p_1^1(3,s) = 0, \ p_2^1(3,s) = p_{2,B}^1(2-s) + p_{2,U}^1(s-1),$$

$s \in (1,2)$, where $p_{2,L}^1 = -7 \times 10^7$, $p_{2,R}^1 = -1/3 \times 10^7$, $p_{2,B}^1 = 4 \times 10^6$, and $p_{2,U}^1 = 1.8 \times 10^7$.

In tables below, we compare the performance of various implementations of the algorithm *ActiveSetTresca* with the MPRGP for different numbers of dofs n and m. We report the number $n_{\mathbf{K}}$ of matrix-vector multiplications by $\mathbf{K}^{-1}$ and the number $k_{\max}$ of the outer (Newton) iterations in the case of the algorithm *ActiveSetTresca*. The final terminating precision (ε_λ) is set 10^{-6} in all cases.

In Table 1, we present results of numerical tests without the globalization strategy. The column labeled MPRGP is computed (by the MPRGP algorithm) with $\tilde\alpha = 2\alpha_{\max}^{-1}$, where $\alpha_{\max}$ denotes the largest eigenvalue of $\mathbf{A}$. The column labeled SNM1 is obtained by exact solving the inner linear systems using the fixed inner precision 10^{-12} and with $\rho = 1$. In the column SNM2, we show the results, when the inner linear systems were solved inexactly using the heuristic adaptive (inner) precision control as proposed in [21] and with $\rho = 1$ again. Finally in the column SNM3, we use the same adaptive precision control but we set $\rho = \alpha_{\max}^{-1}$. One can see that the last choice leads to the most efficient behavior of the algorithm *ActiveSetTresca*. In the column SNM4 of Table 2 we report the results of computations with the globalization strategy for $\rho = 20\alpha_{\max}^{-1}$. One can observe that the efficiency in this case is certain compromise between SSNM4 and MPRGP.

8. Conclusion

In the present contribution, we have investigated a two–body 2D contact problem with Coulomb and Tresca friction. Primarily, we have designed and implement an effective solver for these problems in Matlab

Tab. 1: Comparisons of MPRGP ($n_{\mathbf{K}}$) and *ActiveSetTresca* ($n_{\mathbf{K}}/k_{\max}$) without the globalization strategy.

n/m	MPRGP	SNM1	SNM2	SNM3
30600/150	181	712/12	19/7	19/7
36960/165	181	747/12	55/13	63/12
43920/180	193	826/13	57/12	37/11
51480/195	210	802/12	50/12	50/12
59640/210	213	828/12	35/11	35/11
68400/225	217	898/13	41/11	41/11
77760/240	221	889/12	46/12	46/12
87720/255	231	895/12	55/14	49/13
98280/270	224	981/13	42/11	42/11
109440/285	246	997/13	35/11	35/11
121200/300	248	1067/14	47/12	47/12
133560/315	265	1043/13	37/11	37/11
146520/330	255	1082/13	46/13	37/11
160080/345	263	1140/14	39/11	39/11
174240/360	266	1110/13	41/11	41/11
189000/375	288	1130/13	45/11	45/11
204360/390	281	1198/14	49/13	66/17
220320/405	296	1173/13	49/13	49/13
236880/420	291	1205/13	37/11	37/11
254040/435	298	1270/14	47/12	47/12
271800/450	302	1224/13	55/14	56/14

that is based on the semi–smooth Newton method and the active set strategy. Moreover, we have arrived at the globally convergent dual implementation of the algorithm. Our numerical experiments indicate better results than MPRGP.

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Tab. 2: Comparisons of MPRGP ($n_{\mathbf{K}}$) and *ActiveSetTresca* ($n_{\mathbf{K}}/k_{\max}$) with the globalization strategy.

n/m	MPRGP,	SNM4
30600/150	181	95/56
36960/165	181	100/57
43920/180	193	111/62
51480/195	210	113/67
59640/210	213	119/72
68400/225	217	139/81
77760/240	221	116/66
87720/255	231	117/69
98280/270	224	142/84
109440/285	246	127/74
121200/300	248	146/88
133560/315	265	142/82
146520/330	255	156/93
160080/345	263	145/85
174240/360	266	144/83
189000/375	288	147/86
204360/390	281	151/87
220320/405	296	151/86
236880/420	291	175/105
254040/435	298	159/94
271800/450	302	149/85

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