Quadratic programming with separable convex constraints and solving of 3D contact problems with friction

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Abstract. *The contribution deals with solving of contact problems with Coulomb friction for a system of 3D elastic bodies. The iterative method of successive approximations is used in order to find a fixed point of a certain mapping that defines the solution. In each iterative step, an auxiliary problem with given friction is solved that is discretized by the FETI method. Then the duality theory of convex optimization is used in order to obtain the constrained quadratic programming problem that, in contrast to the 2D case, is subject to quadratic inequality constraints. The solution is computed (among others) by a newly developed algorithm of constrained quadratic programming. Numerical experiments demonstrate experiences with the algorithm.*

1 Introduction

The FETI method was proposed [5] for parallel solution of problems described by elliptic partial differential equations. The key idea is elimination of the primal variables so that the original problem is reduced to a small, relatively well conditioned quadratic programming problem (QPP) in terms of the Lagrange multipliers. Then an iterative solver is used to compute the solution.

In context of 2D contact problems with friction, the FETI procedure leads to a sequence of QPPs constrained by simple inequality bounds [3, 7] so that the fast algorithm with proportioning and gradient projection [4] can be used. The situation is not so easy in 3D since the QPPs are subject to two types of constraints. The first one, representing nonnegativity of the normal contact stress, are again simple inequality bounds while the second one, representing the effect of isotropic friction, are quadratic inequalities. In our recent papers [8, 11], we have used linear approximations of quadratic inequalities transforming them into simple inequality bounds so that the fast algorithm mentioned above can be used again. Unfortunately, this procedure increases considerably the size of the QPPs if we require a sufficiently accurate approximation of quadratic inequalities. In order to overcome this drawback, we have developed a new algorithm of quadratic programming that treats the quadratic inequalities directly [10].

In this contribution, we shall present our experiences with the algorithm for solving the contact problems with Coulomb friction. The Coulomb's law of friction is treated by the iterative method of successive approximations. The problem is precoditioned by the "natural" corse grid so that, moreover, equality constraints are imposed and the iterative augmented Lagrangian method is used in order to satisfy them. These two iterative procedures are connected in the outer loop. The inner loop uses the restarted conjugate gradient method that solves the QPPs constrained by simple inequality bounds and quadratic constraints. The performance of the whole computational process is demonstrated on model examples.

2 Formulation of the problems

Let us consider a system of elastic bodies that occupy in the reference configuration bounded domains $\Omega^p \subset R^3$, $p = 1, 2, \dots, s$, with sufficiently smooth boundaries Γ^p that are split into three disjoint parts Γ^p_u, Γ^p_t and Γ^p_c so that $\Gamma^p = \overline{\Gamma^p_u} \cup \overline{\Gamma^p_t} \cup \overline{\Gamma^p_c}$. Let us suppose that zero displacements are prescribed on Γ^p_u and that the surface tractions of density $\mathbf{t}^p \in (L^2(\Gamma^p_t))^3$ act on Γ^p_t. Along Γ^p_c the body Ω^p may get into unilateral contact with some other of the bodies. Finally we suppose that the bodies Ω^p are subject to volume forces of density $\mathbf{f}^p \in (L^2(\Omega^p))^3$.

To describe non-penetration of the bodies, we shall use linearized non-penetration condition that is defined by a mapping $\chi : \Gamma_c \longrightarrow \Gamma_c$, $\Gamma_c = \bigcup_{p=1}^s \Gamma^p_c$, which assigns to each $\mathbf{x} \in \Gamma^p_c$ some nearby point $\chi(\mathbf{x}) \in \Gamma^q_c$, $p \neq q$. Let $\mathbf{v}^p(\mathbf{x}), \mathbf{v}^q(\chi(\mathbf{x}))$ denote the displacement vectors at $\mathbf{x}, \chi(\mathbf{x})$, respectively. Assuming the small displacements, the *non-penetration condition* reads

$$v^p_n(\mathbf{x}) \equiv (\mathbf{v}^p(\mathbf{x}) - \mathbf{v}^q(\chi(\mathbf{x}))) \cdot \mathbf{n}^p(\mathbf{x}) \leq \delta^p(\mathbf{x}),$$

where $\delta^p(\mathbf{x}) = (\chi(\mathbf{x}) - \mathbf{x}) \cdot \mathbf{n}^p(\mathbf{x})$ is the initial gap and $\mathbf{n}^p(\mathbf{x})$ is the critical direction defined by $\mathbf{n}^p(\mathbf{x}) = (\chi(\mathbf{x}) - \mathbf{x})/\|\chi(\mathbf{x}) - \mathbf{x}\|$ or, if $\chi(\mathbf{x}) = \mathbf{x}$, by the outer unit normal vector to Γ^p_c.

We start with an auxiliary contact problem with given friction. To this end we introduce the space of *virtual displacements* V and its closed convex subset of *kinematically admissible* displacements K by

$$\begin{aligned} V &= \{\mathbf{v} = (\mathbf{v}^1, \dots, \mathbf{v}^s) \in \prod_{p=1}^s (H^1(\Omega^p))^3 : \mathbf{v}^p = 0 \text{ on } \Gamma^p_u\}, \\ K &= \{\mathbf{v} \in V : v^p_n(\mathbf{x}) \leq \delta^p(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma^p_c\}. \end{aligned}$$

Let us assume that the normal contact stress $T_n \in L^\infty(\Gamma_c)$, $T_n \geq 0$, is known *apriori* so that one can evaluate the slip bound g on Γ_c by $g = FT_n$, where $F = F^p > 0$ is a coefficient of friction on Γ^p_c. Denote $g^p = g|_{\Gamma^p_c}$.

The variational formulation of the contact problem with *given* friction reads

$$\min \mathcal{J}(\mathbf{v}) \quad \text{subject to} \quad \mathbf{v} \in K, \tag{1}$$

where

$$\mathcal{J}(\mathbf{v}) = \tfrac{1}{2} a(\mathbf{v}, \mathbf{v}) - b(\mathbf{v}) + j(\mathbf{v})$$

is the total potential energy functional with the bilinear form a representing the inner energy of the bodies and with the linear form b representing the work of the applied forces $\mathbf{t}^p$ and $\mathbf{f}^p$, respectively. The sublinear functional j represents the work of friction forces

$$j(\mathbf{v}) = \sum_{p=1}^s \int_{\Gamma^p_c} g^p \|\mathbf{v}^p_t\| \, d\Gamma, \tag{2}$$

where $\mathbf{v}^p_t$ is the projection of the displacement $\mathbf{v}^p$ on the plane tangential to the critical direction $\mathbf{n}^p$. Let us introduce unit tangential vectors $\mathbf{t}^p_1, \mathbf{t}^p_2$ such that the triplet $\mathcal{B} = \{\mathbf{n}^p, \mathbf{t}^p_1, \mathbf{t}^p_2\}$ is an orthonormal basis in R^3 for almost all $\mathbf{x} \in \Gamma^p_c$ and denote $v^p_{t_1} = \mathbf{v}^p \cdot \mathbf{t}^p_1$, $v^p_{t_2} = \mathbf{v}^p \cdot \mathbf{t}^p_2$. Then $\mathbf{v}^p_t = (0, v^p_{t_1}, v^p_{t_2})$ with respect to the basis $\mathcal{B}$ so that the norm appearing in j reduces to the Euclidean norm in R^2. More details about the formulation of contact problems can be found in [9].

Let us point out that the solution $\mathbf{u} \equiv \mathbf{u}(g)$ of (1) depends on the particular choice of $g \in L^\infty(\Gamma_c)$, $g \geq 0$. We can define a mapping Φ which associates with every g the product $FT_n(\mathbf{u}(g))$, where

$T_n(\mathbf{u}(g)) \geq 0$ is the normal contact stress related to $\mathbf{u}(g)$. The classical Coulomb's law of friction corresponds to the fixed point of Φ which is defined by $g = FT_n(\mathbf{u}(g))$. To find it, we can use the *method of successive approximations* which starts from a given $g^{(0)}$ and generates the iterations $g^{(l)}$ by

(MSA)
$$g^{(l+1)} = \Phi(g^{(l)}),\ l = 1, 2, \ldots.$$

This iterative process converges provided Φ is *contractive*, that is guaranteed for sufficiently small F (see [6]).

3 Domain decomposition and discretization

We divide the bodies Ω^p into tetrahedral finite elements $\mathcal{T}$ with a maximum diameter h and assume that the partitions are regular and consistent with the decompositions of $\partial\Omega^p$ into Γ_u^p, Γ_t^p and Γ_c^p. Moreover, we restrict ourselves to the geometrical conforming situation where the intersection between the boundaries of any two different bodies $\partial\Omega^p \cap \partial\Omega^q$, $p \neq q$, is either empty, a vertex, an entire edge, or an entire face.

Let the domains Ω^p be decomposed into nonoverlapping subdomains $\Omega^{p,i}$, $i = 1, \ldots, n_p$, each of which is the union of finite elements of $\mathcal{T}$. In $\Omega^{p,i}$, we introduce the finite element space $V_h^{p,i}$ by

$$V_h^{p,i} = \{\mathbf{v}^{p,i} \in (C(\Omega^{p,i}))^3 : \mathbf{v}^{p,i}|_{\mathcal{T}} \in (P_1(\mathcal{T}))^3 \text{ for all } \mathcal{T} \subset \Omega^{p,i}, \mathbf{v}^{p,i}|_{\partial\Omega^{p,i}\cap\Gamma_u^p} = 0\},$$

where $P_m(\mathcal{T})$ denotes the set of all polynomials on $\mathcal{T}$ of degree $\leq m$. Finally, let us introduce the product space $V_h = \prod_{p=1}^{s} \prod_{i=1}^{n_p} V_h^{p,i}$.

Replacing V by V_h and using the *gluing condition* $\mathbf{v}^{p,i}(\mathbf{x}) = \mathbf{v}^{p,j}(\mathbf{x})$ for any $\mathbf{x}$ in the interface $\partial\Omega^{p,i} \cap \partial\Omega^{p,j}$, $i \neq j$, we can rewrite the approximate contact problem with the given friction (1) into the algebraic form

$$\begin{aligned} &\min\ \frac{1}{2}\mathbf{u}^\top\mathbf{K}\mathbf{u} - \mathbf{u}^\top\mathbf{f} + \sum_{k=1}^{m} g_k \|((\mathbf{T}_1\mathbf{u})_k, (\mathbf{T}_2\mathbf{u})_k)\| \\ &\text{s.t.}\quad \mathbf{N}\mathbf{u} \leq \mathbf{d},\ \ \mathbf{B}_E\mathbf{u} = \mathbf{0}. \end{aligned} \tag{3}$$

Here, $\mathbf{K}$ denotes the positive semidefinite block diagonal stiffness matrix, $\mathbf{f}$ is the vector of nodal forces, $\mathbf{N}, \mathbf{d}$ describe the discretized non-penetration condition and $\mathbf{B}_E$ describes the gluing condition. The summation term in the minimized functional arises using numerical quadrature in (2), where $\mathbf{T}_1$, $\mathbf{T}_2$ describe projections of displacements at the nodes lying on Γ_c to the tangential planes and g_k are the values of the slip bound $\mathbf{g}$.

Let us point out that the problem (3) is non-differentiable due to the R^2-norms appearing in the summation term. Therefore we shall introduce two kinds of Lagrange multipliers $\boldsymbol{\lambda}_t = (\boldsymbol{\lambda}_{t_1}^\top, \boldsymbol{\lambda}_{t_2}^\top)^\top$ and $\boldsymbol{\lambda}_c = (\boldsymbol{\lambda}_I^\top, \boldsymbol{\lambda}_E^\top)^\top$. While the first one removes the non-differentiability, the second one accounts for the constraints in (3). Denote

$$\mathbf{B} = \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \mathbf{N} \\ \mathbf{B}_E \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{d} \\ \mathbf{0} \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_{t_1} \\ \boldsymbol{\lambda}_{t_2} \\ \boldsymbol{\lambda}_I \\ \boldsymbol{\lambda}_E \end{bmatrix}$$

and introduce the Lagrange multiplier set

$$\Lambda(\mathbf{g}) = \{\boldsymbol{\lambda} : \|(\lambda_{t_1,k}, \lambda_{t_2,k})\| \leq g_k, \lambda_{I,k} \geq 0\ \forall k\}.$$

It is well known that (3) is equivalent to the saddle-point problem

$$\text{Find } (\mathbf{u},\boldsymbol{\lambda}) \quad \text{s.t.} \quad \mathcal{L}(\mathbf{u},\boldsymbol{\lambda}) = \sup_{\boldsymbol{\mu}\in\Lambda(\mathbf{g})} \min_{\mathbf{v}} \mathcal{L}(\mathbf{v},\boldsymbol{\mu}), \tag{4}$$

where $\mathcal{L}$ is the Lagrangian to (3) defined by

$$\mathcal{L}(\mathbf{u},\boldsymbol{\lambda}) = \frac{1}{2}\mathbf{u}^\top\mathbf{K}\mathbf{u} - \mathbf{u}^\top\mathbf{f} + \boldsymbol{\lambda}^\top(\mathbf{B}\mathbf{u} - \mathbf{c}).$$

From minima in (4), we obtain

$$\mathbf{K}\mathbf{u} - \mathbf{f} + \mathbf{B}^\top\boldsymbol{\lambda} = \mathbf{0}. \tag{5}$$

This equality holds iff $\mathbf{f} - \mathbf{B}^\top\boldsymbol{\lambda} \in \mathit{Im}\,\mathbf{K}$ or, equivalently, iff $\mathbf{f} - \mathbf{B}^\top\boldsymbol{\lambda} \perp \mathit{Ker}\,\mathbf{K}$. Using a full rank matrix $\mathbf{R}$ whose columns span the kernel of $\mathbf{K}$, we can write

$$\mathbf{R}^\top(\mathbf{f} - \mathbf{B}^\top\boldsymbol{\lambda}) = \mathbf{0}. \tag{6}$$

If (6) is fulfiled, then $\mathbf{u}$ satisfying (5) reads as follows

$$\mathbf{u} = \mathbf{K}^\dagger(\mathbf{f} - \mathbf{B}^\top\boldsymbol{\lambda}) + \mathbf{R}\boldsymbol{\alpha},$$

where $\mathbf{K}^\dagger$ denotes a generalized inverse to $\mathbf{K}$. The later formula can be used to eliminate the first unknown $\mathbf{u}$ from (4). After simple algebraic manipulations and changing the sign, we obtain the minimization problem in terms of $\boldsymbol{\lambda}$ as

$$\begin{aligned} &\min \ \frac{1}{2}\boldsymbol{\lambda}^\top\mathbf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top\widetilde{\mathbf{h}} \\ &\text{s.t.} \quad \boldsymbol{\lambda} \in \Lambda(\mathbf{g}), \ \ \mathbf{G}\boldsymbol{\lambda} = \mathbf{e}, \end{aligned} \tag{7}$$

where $\mathbf{F} = \mathbf{B}\mathbf{K}^\dagger\mathbf{B}^\top$ is positive definite, $\widetilde{\mathbf{h}} = \mathbf{B}\mathbf{K}^\dagger\mathbf{f} - \mathbf{c}$, $\mathbf{G} = \mathbf{R}^\top\mathbf{B}^\top$ has full row-rank and $\mathbf{e} = \mathbf{R}^\top\mathbf{f}$.

The problem (7) can be adapted by using orthogonal projectors as proposed in [5]. To this end we define the projectors on $\mathit{Im}\,\mathbf{G}^\top$ and on $\mathit{Ker}\,\mathbf{G}$ as

$$\mathbf{Q} = \mathbf{G}^\top(\mathbf{G}\mathbf{G}^\top)^{-1}\mathbf{G} \quad \text{and} \quad \mathbf{P} = \mathbf{I} - \mathbf{Q},$$

respectively. Since $R^m = \mathit{Im}\,\mathbf{G}^\top \oplus \mathit{Ker}\,\mathbf{G}$, the solution $\boldsymbol{\lambda}$ to (7) can be uniquely decomposed by

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_Q + \boldsymbol{\lambda}_P \tag{8}$$

so that $\boldsymbol{\lambda}_Q = \mathbf{Q}\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_P = \mathbf{P}\boldsymbol{\lambda}$. Because of $\mathbf{G}\boldsymbol{\lambda}_P = \mathbf{0}$, the equality constraints in (7) imply $\mathbf{G}\boldsymbol{\lambda} = \mathbf{G}\boldsymbol{\lambda}_Q = \mathbf{e}$ and therefore $\boldsymbol{\lambda}_Q$ is known apriori by

$$\boldsymbol{\lambda}_Q = \mathbf{G}^\top(\mathbf{G}\mathbf{G}^\top)^{-1}\mathbf{e}.$$

Using the decomposition (8) in the minimized function of (7), we obtain

$$\begin{aligned} \frac{1}{2}\boldsymbol{\lambda}^\top\mathbf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top\widetilde{\mathbf{h}} &= \frac{1}{2}\boldsymbol{\lambda}_Q^\top\mathbf{F}\boldsymbol{\lambda}_Q - \boldsymbol{\lambda}_Q^\top\widetilde{\mathbf{h}} + \boldsymbol{\lambda}_P^\top\mathbf{F}\boldsymbol{\lambda}_Q + \frac{1}{2}\boldsymbol{\lambda}_P^\top\mathbf{F}\boldsymbol{\lambda}_P - \boldsymbol{\lambda}_P^\top\widetilde{\mathbf{h}} \\ &= \mathit{const.} + \frac{1}{2}\boldsymbol{\lambda}_P^\top\mathbf{F}\boldsymbol{\lambda}_P - \boldsymbol{\lambda}_P^\top(\widetilde{\mathbf{h}} - \mathbf{F}\boldsymbol{\lambda}_Q) \end{aligned}$$

Hence, (7) is equivalent with

$$\begin{aligned} &\min \ \frac{1}{2}\boldsymbol{\lambda}_P^\top\mathbf{F}\boldsymbol{\lambda}_P - \boldsymbol{\lambda}_P^\top\mathbf{h} \\ &\text{s.t.} \quad \boldsymbol{\lambda}_P + \boldsymbol{\lambda}_Q \in \Lambda(\mathbf{g}), \ \ \mathbf{G}\boldsymbol{\lambda}_P = \mathbf{0}, \end{aligned} \tag{9}$$

where $\mathbf{h} = \widetilde{\mathbf{h}} - \mathbf{F}\boldsymbol{\lambda}_Q$. Since the solution $\boldsymbol{\lambda}_P$ to (9) belongs to $Ker\mathbf{G}$, it satisfies $\boldsymbol{\lambda}_P = \mathbf{P}\boldsymbol{\lambda}_P$ and therefore it solves

$$\begin{aligned} &\min \frac{1}{2}\boldsymbol{\lambda}_P^\top \mathbf{PFP}\boldsymbol{\lambda}_P - \boldsymbol{\lambda}_P^\top \mathbf{Ph} \\ &\text{s.t.} \quad \boldsymbol{\lambda}_P + \boldsymbol{\lambda}_Q \in \Lambda(\mathbf{g}),\ \ \mathbf{G}\boldsymbol{\lambda}_P = \mathbf{0}. \end{aligned} \tag{10}$$

Let us point out that the solution to (10) is unique since the Hessian $\mathbf{PFP}$ is positive definite on $Ker\mathbf{G}$. It follows from the fact that, for all $\boldsymbol{\delta} \in Ker\mathbf{G} \setminus \{\mathbf{0}\}$, it holds

$$\boldsymbol{\delta}^\top \mathbf{PFP}\boldsymbol{\delta} = \boldsymbol{\delta}^\top \mathbf{F}\boldsymbol{\delta} > 0.$$

Therefore the problems (9) and (10) are equivalent in the sense that they have the same solution.

4 Algorithms

Let us denote the augmented Lagrangian to (10) by

$$\begin{aligned} L(\boldsymbol{\lambda}_P, \boldsymbol{\beta}, \rho) &= \frac{1}{2}\boldsymbol{\lambda}_P^\top \mathbf{PFP}\boldsymbol{\lambda}_P - \boldsymbol{\lambda}_P^\top \mathbf{Ph} + \boldsymbol{\beta}^\top \mathbf{G}\boldsymbol{\lambda}_P + \frac{\rho}{2}\|\mathbf{G}\boldsymbol{\lambda}_P\|^2_{(\mathbf{GG}^\top)^{-1}} \\ &= \frac{1}{2}\boldsymbol{\lambda}_P^\top (\mathbf{PFP} + \rho\mathbf{Q})\boldsymbol{\lambda}_P - \boldsymbol{\lambda}_P^\top \mathbf{Ph} + \boldsymbol{\beta}^\top \mathbf{G}\boldsymbol{\lambda}_P, \end{aligned}$$

where $\rho > 0$. It easy to show that (10) is equivalent to the saddle-point problem

$$\text{Find } (\boldsymbol{\lambda}_P, \boldsymbol{\beta}) \quad \text{s.t.} \quad L(\boldsymbol{\lambda}_P, \boldsymbol{\beta}, \rho) = \min_{\boldsymbol{\mu} \in \Lambda(\mathbf{g}) \setminus \{\boldsymbol{\lambda}_Q\}} \sup_{\boldsymbol{\delta}} L(\boldsymbol{\mu}, \boldsymbol{\delta}, \rho). \tag{11}$$

The iterative algorithm for solving (11) alternately minimize and maximize the augmented Lagrangian with respect to $\boldsymbol{\lambda}_P$ and $\boldsymbol{\beta}$, respectively.

Algorithm 1. Set $\boldsymbol{\beta}^{(0)} = \mathbf{0}$, $l = 0$.

repeat

$\boldsymbol{\lambda}_P^{(l+1)} \doteq \text{argmin } L(\boldsymbol{\mu}, \boldsymbol{\beta}^{(l)}, \rho), \ \text{s.t. } \boldsymbol{\mu} \in \Lambda(\mathbf{g}) \setminus \{\boldsymbol{\lambda}_Q\}$

$\boldsymbol{\beta}^{(l+1)} = \boldsymbol{\beta}^{(l)} + \rho\mathbf{G}\boldsymbol{\lambda}_P^{(l+1)}$

Update ρ and increase l by one.

until stopping criterion

Algorithms of this type have been intensively studied recently [2, 1] with the inner minimization represented by a QPP with simple inequality bounds [4]. For 3D contact problems with friction, the quadratic inequality constraints on $\boldsymbol{\lambda}_{P,t}$ are imposed so that the inner minimization must be realized by a different way. We shall use a newly developed algorithm for minimizing strictly quadratic functions with separable convex constraints [10].

The method of successive approximations (MSA) for solving the contact problem with Coulomb friction can be implemented so that Algorithm 1 is used in each iterative step to evaluate the mapping Φ. We shall present a more efficient version of this method, in which the iterative steps of (MSA) and the loop of Algorithm 1 are connected in one loop. The resulting algorithm can be viewed as the method of successive approximations with an *inexact* solving of the auxiliary problems with given friction.

Algorithm 2. Set $\boldsymbol{\beta}^{(0)} = \mathbf{0}$, $\boldsymbol{\lambda}_I^{(0)}$, $l := 0$.

repeat

$$\boldsymbol{\lambda}_P^{(l+1)} \doteq \text{argmin } L(\boldsymbol{\mu}, \boldsymbol{\beta}^{(l)}, \rho), \text{ s.t. } \boldsymbol{\mu} \in \Lambda(F\boldsymbol{\lambda}_I^{(l)}) \setminus \{\boldsymbol{\lambda}_Q\}$$

$$\boldsymbol{\beta}^{(l+1)} = \boldsymbol{\beta}^{(l)} + \rho \mathbf{G} \boldsymbol{\lambda}_P^{(l+1)}$$

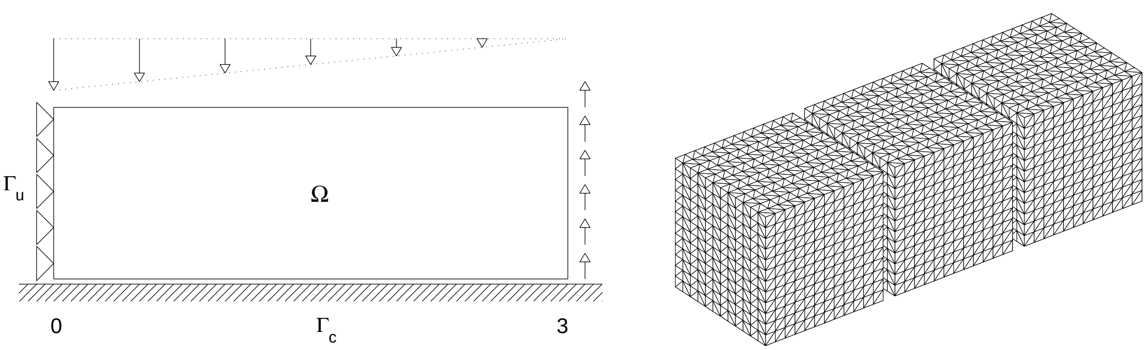
Update ρ and increase l by one.

until stopping criterion

We have used the fact that the Lagrange multiplier $\boldsymbol{\lambda}_I$ represents the normal contact stress so that $\mathbf{g} = F\boldsymbol{\lambda}_I^{(l)}$, $\boldsymbol{\lambda}_I^{(l)} = \boldsymbol{\lambda}_{P,I}^{(l)} + \boldsymbol{\lambda}_{Q,I}$ approximates the slip bound.

5 Numerical experiments and conclusions

Let us consider a model brick $\Omega = (0,3) \times (0,1) \times (0,1)$ made of an elastic isotropic, homogeneous material characterized by the Young modulus $2.1 \cdot 10^{11}$ and the Poisson's ratio 0.28 (steel). The brick is unilaterally supported by a rigid foundation, where the non-penetration condition and the effect of Coulomb friction is considered. The applied surface tractions and the parts of the boundary Γ_u and Γ_c are seen in Figure 1.a. The volume forces vanish. The brick Ω is artificially decomposed onto three parts as seen in Figure 1.b so that the resulting problem has 12 rigid modes.



a b

Figure 1: a) The cross-section of the brick Ω. b) Domain decomposition and discretiztion.

The algorithm is terminated by means of the stopping criterion

$$ERR \leq \epsilon,$$

where

$$ERR \equiv \frac{\|\boldsymbol{\beta}^{(l+1)} - \boldsymbol{\beta}^{(l)}\|}{\|\boldsymbol{\beta}^{(l+1)}\| + 1} + \frac{\|F\boldsymbol{\lambda}_I^{(l+1)} - F\boldsymbol{\lambda}_I^{(l)}\|}{\|F\boldsymbol{\lambda}_I^{(l+1)}\| + 1}.$$

The inner minimization uses restarted conjugate gradient method with an adaptive terminating criterion [10]. If the norm of violation of the Karush-Kuhn-Tucker conditions to the inner problem is less or equel to $M \cdot ERR$, $M > 0$, then the inner loop is terminated.

We test experimentally efficiency of the algorithm for various types of problems:

Linear denotes the noncontact problem, in which the rigid foundation is shifted far enough so that the algebraic problem is equivalent with a linear system. The augmented Lagrangian algorithm converges after one iteration.

Nonpen omits the effect of friction, i.e. $F = 0$. The algebraic problem reduces to the QPP problem with simple inequality bounds and equality constraints. The behaviour of the algorithm is completly covered by the analysis of [1] so that the algorithm is scalable and converges in $\mathcal{O}(1)$ iterative steps.

Given denotes the contact problem with given friction $g = 5.25 \cdot 10^4$ solved by Algorithm 1.

Coulomb denotes the contact problem with the coefficient of Coulomb friction $F = 0.3$ that is solved by Algorithm 2.

Table 1 summarizes results of numerical experiments, where n denotes the number of primal unknowns (displacements) and m denotes the number of dual unknowns (stresses). The efficiency is assessed by n_{outer}/n_{inner}, where n_{outer} is the number of augmented Lagrangian iterations and n_{inner} is the total number of conjugate gradient steps. All numerical experiments are made by Matlab 7, Pentium(R)4, 3GHz, 512MB RAM.

Table 1: $\epsilon = 10^{-6}$, $M = 0.01$.

n	m	*Linear*	*Nonpen*	*Given*	*Coulomb*
243	81	1/26	3/108	6/240	7/404
1125	225	1/40	3/122	11/406	5/373
3087	441	1/47	3/274	16/651	6/591
6561	729	1/52	4/372	18/753	8/959
11979	1089	1/56	4/392	20/808	9/946
19773	1521	1/58	4/467	22/850	8/983

The numerical experiments are in agreement with the theoretical results in cases *Linear* and *Nonpen* [1]. The computational costs for solving the contact problems with the given friction increase since the algorithm of [10] have not the finite termination property due to the non-linear constraints. Finally, let us point out that computational requirements in solving physically more realistic contact problem with Coulomb friction are stabilized and even decreased.

6 Conclusion

We have presented results of numerical experiments of the algorithms for solving 3D contact problems that are based on a newly developed algorithm of quadratic programming with separable convex constraints [10]. In contrast to the implementation used in [12], we treate here the simple inequality bounds and the quadratic constraints together in the one loop. Therefore we can omit Gauss-Seidel splitting used in [12] and, moreover, the rate of convergence of the resulting quadratic programm can be proved. We prepare to present this result elsewhere.

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