FICTITIOUS DOMAIN FORMULATION OF UNILATERAL PROBLEMS

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Abstract: The fictitious domain method for the solution of variational inequalities with the Signorini boundary conditions is analyzed.

Keywords: *Unilateral problem, Fictitious domain method, Semismooth Newton method*

1. Introduction

Fictitious domain methods (FDM) belong to a class of methods for the numerical solution of large scale linear algebraic systems arising from finite element discretizations of elliptic boundary value problems. Their idea is simple: the original problem defined in a domain ω is replaced by a new one formulated in a larger domain $\Omega \supset \omega$ with a simple shape (a box, e.g.). The new problem is chosen in such a way that its solution restricted to ω coincides with the solution of the original problem. Since Ω has a simple shape, one can use specific partitions for constructing finite element spaces. We confine ourselves to the so-called non-fitted meshes when the partition of Ω does not respect the geometry of ω. In this case uniform meshes represent a natural choice and the resulting stiffness matrix does not depend on ω. In addition, it has a structure enabling us to use fast solvers. There are several ways how to define the problem in Ω with the property mentioned above. One of them is the method of boundary Lagrange multipliers which has been previously used for solving Dirichlet and Neumann boundary value problems. This approach however suffers from a serious drawback: the solution is only from $H^{3/2-\eta}(\Omega)$, $\eta > 0$, due to a generally non-zero jump of the normal derivative across γ (the boundary of ω). If non-fitted meshes are used, then this singularity appears inside of some elements of the used partition, namely those ones the interior of which is cut by γ. Consequently, the theoretical rate of convergence of approximate solutions in the $H^1(\Omega)$-norm can not exceed 1/2. In addition, the biggest error is concentrated around γ which explains also a slower convergence in the $H^1(\omega)$-norm of solutions restricted to ω. To improve the accuracy in ω, the authors proposed in [1] a new variant of FDM. Instead of Lagrange multipliers on γ they used control variables defined on a close curve Γ in Ω having a positive distance from ω and enforcing the Dirichlet condition on γ to be satisfied. The solution is still singular in Ω but the singularity is shifted from γ to Γ and as a result, convergence in ω became faster. The aim of the paper [2] is twofold: first to introduce a fictitious domain formulation of unilateral boundary value problems and secondly, to propose its „smooth" variant in the spirit mentioned above. We focus on a simple scalar variational inequality with Signorini type conditions on γ, but a similar approach can be used for contact problems, e.g. Our fictitious domain formulation consists of an elliptic equation in Ω completed by an equation on γ for the projection operator onto a convex set which represents the equivalent expression of the unilateral conditions prescribed there. Similarly to the Dirichlet problem we shall consider two cases, namely: (i) *boundary Lagrange multipliers on* γ; (ii) *control variables on* Γ, where Γ is a close curve exterior to ω. The reason for considering (ii) is the same as above, namely to smooth our fictitious domain solution in a vicinity of ω. In both cases the auxiliary boundary variables enforce the satisfaction of the unilateral conditions. We prove the existence and uniqueness of the solution in (i). On the other hand, (ii) is more involved. The existence analysis is based on approximate controllability type results. We show that using square integrable controls on Γ we are able to satisfy the unilateral conditions either exactly provided that an appropriate small source term δ is added on γ or with an arbitrary accuracy if this term is neglected. A typical finite element discretization gives a large system of linear equations completed with a small system of piecewise linear equations which arise from a discretization of the unilateral conditions. The resulting algebraic problem is numerically solved by a semismooth Newton method. Each linearized step leads to a non-symmetric, saddle-point type system which can be solved very efficiently by a projected Schur complement method [1].

2. Setting of the problem

We shall consider the following unilateral problem in a bounded domain $\omega \subset \mathbb{R}^2$ with the Lipschitz continuous boundary γ:

$$\left.\begin{aligned} -\Delta u + u = f \quad & \text{in } \omega, \\ u \geq g,\ \frac{\partial u}{\partial n_\gamma} \geq 0,\ \frac{\partial u}{\partial n_\gamma}(u-g) = 0 \quad & \text{on } \gamma, \end{aligned}\right\} \tag{1}$$

where $f \in L^2_{loc}(\mathbb{R}^2)$, $g \in H^{1/2}(\gamma)$ are given functions and $\dfrac{\partial}{\partial n_\gamma}$ denotes the normal derivative of a function on γ.

Let us choose a bounded domain Ω having a simple shape such that $\overline{\omega} \subset \Omega$ and construct a close curve $\Gamma \subset \Omega$ surrounding ω. Let P denote the projection of $L^2(\gamma)$ onto $L^2_+(\gamma)$. We define the following *fictitious domain formulation* of (1):

$$\left.\begin{array}{l} \textit{Find } (\hat{u}, \lambda) \in H^1_0(\Omega) \times H^{-1/2}(\Gamma) \textit{ such that} \\ (\hat{u}, v)_{1,\Omega} = (f, v)_{0,\Omega} + \langle \lambda, v \rangle_\Gamma \quad \forall v \in H^1_0(\Omega), \\ \dfrac{\partial}{\partial n_\gamma} \hat{u}_{|\omega} \in L^2(\gamma), \\ \dfrac{\partial}{\partial n_\gamma} \hat{u}_{|\omega} = P(\dfrac{\partial}{\partial n_\gamma} \hat{u}_{|\omega} - \rho(\hat{u}_{|\omega} - g)), \quad \rho > 0, \end{array}\right\} \tag{2}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ stands for the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

3. Discretization and numerical experiments

Let $H^1_0(\Omega)$, $L^2(\gamma)$, and $H^{-1/2}(\Gamma)$ be replaced in (2) by their finite dimensional approximations V_h, $\Lambda_H(\gamma)$, and $\Lambda_H(\Gamma)$, respectively, such that $\dim V_h = n$ and $\dim \Lambda_H(\gamma) = \dim \Lambda_H(\Gamma) = m$. We arrive at the following algebraic representation of (2):

$$\left.\begin{array}{l} \textit{Find } (\vec{u}, \vec{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m \textit{ such that} \\ A\vec{u} = \vec{f} + B_\Gamma^\top \vec{\lambda}, \\ C_\gamma \vec{u} = \max\{0, C_\gamma \vec{u} - \rho(B_\gamma \vec{u} - g)\}, \end{array}\right\} \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ denotes the stiffness matrix, $B_\gamma, B_\Gamma \in \mathbb{R}^{m\times n}$ are the Dirichlet trace matrices on γ, Γ, respectively, $C_\gamma \in \mathbb{R}^{m\times n}$ is the Neumann trace matrix on γ, and $\vec{f} \in \mathbb{R}^n$, $\vec{g} \in \mathbb{R}^m$. The problem (3) can be understood as one vector equation that, however, is non-smooth due to the presence of the max-function. Therefore, the semi-smooth variant of the Newton method is suitable for the numerical solution [2].

We consider the model example (2) with $\Omega = (0, 1) \times (0, 1)$ and $\omega = \{(x, y) \in \mathbb{R}^2 |\ (x - 0.5)^2/0.4^2 + (y - 0.5)^2/0.2^2 < 1\}$ for the exact solution $u_{ex}(x, y) = ((x - 0.5)^+)^3 + 0.5((y - 0.5)^+)^3$. The obstacle g is defined by $g_{|\gamma_1} = u_{ex|\gamma_1}$ on γ_1, $\gamma_1 = \gamma \setminus \overline{\gamma}_2$, $\gamma_2 = \{(x, y) \in \gamma |\ x < 0.5,\ y < 0.5\}$ and by $g(x, y) = \sin(-2\varphi)$ for $(x, y) \in \gamma_2$, where (φ, r) is the polar coordinate of the point $(x - 0.5, y - 0.5)$.

In tables below we report for each finite element step-size h, the number of the Newton iterations it_{O} and the matrix-vector multiplications it_{I} (by A), the time in seconds, and the errors in the norms of $L^2(\omega)$, $H^1(\omega)$, and $L^2(\gamma)$. The convergence rates are computed from these errors.

h	n/m	$it_{\mathrm{O}}/it_{\mathrm{I}}$	time	$\mathrm{err}_{L^2(\omega)}$	$\mathrm{err}_{H^1(\omega)}$	$\mathrm{err}_{L^2(\gamma)}$
1/128	16641/34	5/24	0.3	4.10e-2	3.33e+0	1.70e-2
1/256	66049/62	6/45	1.8	2.00e-2	2.33e+0	8.52e-3
1/512	263169/110	7/69	13.1	9.78e-3	1.62e+0	4.25e-3
1/1024	1050625/198	7/93	74.7	4.84e-3	1.14e+0	2.08e-3
1/2048	4198401/360	7/115	432.6	2.27e-3	7.85e-1	1.09e-3
1/4096	16785409/662	8/131	2328	1.13e-3	5.54e-1	7.21e-4
Convergence rates:				1.03	0.51	0.93

Tab. 1. Non-smooth fictitious domain formulation ($\gamma \equiv \Gamma$).

h	n/m	$it_{\mathrm{O}}/it_{\mathrm{I}}$	time	$\mathrm{err}_{L^2(\omega)}$	$\mathrm{err}_{H^1(\omega)}$	$\mathrm{err}_{L^2(\gamma)}$
1/128	16641/34	5/48	0.4	3.24e-4	2.95e-1	5.07e-4
1/256	66049/62	5/69	2.5	6.31e-5	1.30e-1	9.10e-5
1/512	263169/110	5/112	20.9	1.59e-5	6.54e-2	2.65e-5
1/1024	1050625/198	7/162	150.6	4.35e-6	3.42e-2	1.17e-5
1/2048	4198401/360	6/190	674.1	1.38e-6	1.92e-2	5.18e-6
1/4096	16785409/662	9/296	5000	8.07e-7	1.47e-2	2.28e-6
Convergence rates:				1.76	0.88	1.50

Tab. 2. Smooth fictitious domain formulation ($\gamma \not\equiv \Gamma$).

Acknowledgements

Supported by GACR 101/08/0574 and by the Czech Ministry of Education MSM6198910027.

References

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