# INEXACT SSNM FOR SOLVING FRICTIONAL CONTACT PROBLEMS

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**Abstract**: We use the semi-smooth Newton method to the solution of 2D contact problems with friction. The primal-dual algorithm for problems with the Tresca friction law is reformulated as the dual one. The conjugate gradient method is used for inexact solving of inner linear systems. Numerical experiments illustrate the performance of the inexact algorithm.

## 1 Introduction

We start with the algebraic counterpart of the elliptic PDEs describing the contact of two (or more) elastic bodies with Tresca friction. The problem arising from a finite element approximation reads as follows:

$$
\begin{aligned}
&\textit{Find } (u^*, \lambda_\nu^*, \lambda_\tau^*) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \textit{ such that} \\
&Ku + N^\top \lambda_\nu + T^\top \lambda_\tau - f = 0, && (1)\\
&Nu - d \le 0,\ \lambda_\nu \ge 0,\ \lambda_\nu^\top (Nu - d) = 0, && (2)\\
&\left.\begin{array}{l} |\lambda_{\tau,i}| \le g_i \\ |\lambda_{\tau,i}| < g_i \Rightarrow (Tu)_i = 0 \\ |\lambda_{\tau,i}| = g_i \Rightarrow \exists c_i \ge 0 :\ (Tu)_i = c_i \lambda_{\tau,i} \end{array}\right\} \quad i \in \mathcal{M}, && (3)
\end{aligned}
$$

where $\mathcal{M} = \{1, \ldots, m\}$ is the index set, $K \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, $N, T \in \mathbb{R}^{m\times n}$ have full row-rank, $f \in \mathbb{R}^n$, $d \in \mathbb{R}^m_+$, and $g_i$ are entries of $g \in \mathbb{R}^m_+$. The formulation (1)-(3) describes the algebraic *primal-dual* contact problem with Tresca friction. The primal unknown $u^*$ approximates displacements, while the dual unknowns $\lambda_\nu^*$, $\lambda_\tau^*$ approximate the (negative) normal, tangential contact stresses, respectively.

The semi-smooth Newton method (SSNM) uses the primal-dual formulation of contact problems reformulated by non-smooth functions as proposed already in [1]. Later on, it was recognized that the SSNM may be interpreted as a primal-dual active set method [4]. This approach is widely used for solving contact problems in two (2D) as well as three (3D) space dimensions with different friction laws; see e.g. [5]. The standard convergence analysis uses the slant differentiability concept [4] leading to the local superlinear convergence rate. This convergence result assumes exact solutions of inner linear systems that is, however, unrealistic for large-scale problems. The globally convergent variant of the method is analyzed in [3]. Here, we present the inexact implementation of the SSNM leading to the highest computational efficiency.

# 2 Dual variant of the SSNM

Let $P_{\Lambda_\nu} : \mathbb{R}^m \mapsto \Lambda_\nu$ and $P_{\Lambda_\tau} : \mathbb{R}^m \mapsto \Lambda_\tau$ be the projections onto $\Lambda_\nu = \mathbb{R}^m_+$ and $\Lambda_\tau = \{\lambda_\tau \in \mathbb{R}^m : |\lambda_{\tau,i}| \leq g_i,\ i \in \mathcal{M}\}$ defined by the max-function as follows:

$$P_{\Lambda_\nu,i}(\lambda_\nu) = \max\{0, \lambda_{\nu,i}\}, \tag{4}$$

$$P_{\Lambda_\tau,i}(\lambda_\tau) = \max\{0, \lambda_{\tau,i} + g_i\} - \max\{0, \lambda_{\tau,i} - g_i\} - g_i, \tag{5}$$

respectively. Let us introduce the function $G : \mathbb{R}^{n+2m} \mapsto \mathbb{R}^{n+2m}$ with $y = (u^\top, \lambda_\nu^\top, \lambda_\tau^\top)^\top$ given by

$$G(y) = \begin{pmatrix} Ku + N^\top\lambda_\nu + T^\top\lambda_\tau - f \\ \lambda_\nu - P_{\Lambda_\nu}(\lambda_\nu + \rho(Nu - d)) \\ \lambda_\tau - P_{\Lambda_\tau}(\lambda_\tau + \rho Tu) \end{pmatrix}, \tag{6}$$

where $\rho > 0$ is an arbitrary but fixed parameter. It is easy to verify that (1)-(3) and the equation

$$G(y) = 0, \tag{7}$$

have the same solution $y^* = (u^{*\top}, \lambda_\nu^{*\top}, \lambda_\tau^{*\top})^\top$. The function $G$ is nonsmooth due to the presence of the max-function. Fortunately, it is semi-smooth in the sense of [**?**] so that the SSNM can be used. We will present the dual variant of the SSNM.

First of all, we introduce notation. Let $q : \mathbb{R}^{2m} \mapsto \mathbb{R}$ be the quadratic cost function defined by

$$q(\lambda) = \frac{1}{2}\lambda^\top A\lambda - \lambda^\top b, \tag{8}$$

where $\lambda = (\lambda_\nu^\top, \lambda_\tau^\top)^\top$, $A = BK^{-1}B^\top$ with $B = \left(N^\top, T^\top\right)^\top$ is symmetric and positive definite, $b = BK^{-1}f - c$, and $c = \left(d^\top, 0^\top\right)^\top$. The gradient $r : \mathbb{R}^{2m} \mapsto \mathbb{R}$ to $q$ at $\lambda \in \mathbb{R}^{2m}$ is given by

$$r(\lambda) = A\lambda - b. \tag{9}$$

Denote $\lambda^* = (\lambda_\nu^{*\top}, \lambda_\tau^{*\top})^\top$, $\Lambda = \Lambda_\nu \times \Lambda_\tau$, and introduce the the projection onto $\Lambda$ by $P_\Lambda : \mathbb{R}^{2m} \mapsto \Lambda$ given by $P_\Lambda = (P_{\Lambda_\nu}^\top, P_{\Lambda_\tau}^\top)^\top$. The *reduced gradient* $\widetilde{r}_\alpha : \Lambda \mapsto \mathbb{R}$ to $q$ for $\alpha > 0$ is defined by:

$$\widetilde{r}_\alpha(\lambda) = \frac{1}{\alpha}(\lambda - P_\Lambda(\lambda - \alpha r(\lambda))). \tag{10}$$

It is well-known [2] that $\widetilde{r}_\alpha$ is the optimality criterion to the problem $\min_{\lambda\in\Lambda} q(\lambda)$ in the sense that $\lambda^* \in \Lambda$ solves this problem iff $\widetilde{r}_\alpha(\lambda^*) = 0$. Therefore, the reduced

gradient will be used as the stopping criterion.

ALGORITHM SSNM Given $\lambda^0 \in \mathbb{R}^{2m}$, $\varepsilon \geq 0$, and $\rho > 0$. For $k \geq 0$, compute:

(*Step 1*) If $\|\widetilde{r}_\rho(P_\Lambda(\lambda^k))\| \leq \varepsilon$, return $\lambda = P_\Lambda(\lambda^k)$, else go to step *Step 2*.

(*Step 2*) Assembly the active/inactive sets at $\lambda^k$:

$$\begin{aligned}
\mathcal{A}_\nu &= \{i \in \mathcal{M} : \lambda_i^k - \rho r_i^k \geq 0\}, \\
\mathcal{I}_\nu &= \mathcal{M} \setminus \mathcal{A}_\nu, \\
\mathcal{I}_\tau^+ &= \{i \in \mathcal{M} : g_i < \lambda_{i+m}^k - \rho r_{i+m}^k\}, \\
\mathcal{I}_\tau^- &= \{i \in \mathcal{M} : \lambda_{i+m}^k - \rho r_{i+m}^k < -g_i\}, \\
\mathcal{A}_\tau &= \mathcal{M} \setminus (I_\tau^+ \cup \mathcal{I}_\tau^-).
\end{aligned}$$

(*Step 3*) Find $\lambda^{k+1}$ so that

$$\lambda^{k+1} = \arg\min q(\lambda) \;\text{ s.t. }\; \lambda_{\nu,\mathcal{I}_\nu} = 0,\; \lambda_{\tau,\mathcal{I}_\tau^+} = g_{\mathcal{I}_\tau^+},\; \lambda_{\tau,\mathcal{I}_\tau^-} = -g_{\mathcal{I}_\tau^-}. \tag{11}$$

Note that $\rho$ can be discarded from $\mathcal{A}_\nu$ and $\mathcal{I}_\nu$, when the inner subproblems in *Step 3* are solved exactly (and $\lambda^0 = 0$, e.g.), since either $\lambda_i^k = 0$ or $r_i^k = 0$. A similar observation is valid also for $\mathcal{A}_\tau$, $\mathcal{I}_\tau^+$, and $\mathcal{I}_\tau^-$ provided that $\lambda^k$ is sufficiently close to $\lambda^*$ and $g$ is sufficiently large.

# 3 Inexact implementation

The computational efficiency of the SSNM depends on a way how the inner subproblems are implemented. We propose to accept inexact solutions to (11), denoted again by $\lambda^{k+1}$, that are computed by few CGM iterations. It is referred by

$$\lambda^{k+1} = \mathrm{CGM}(A, b, \mathcal{A}, \lambda^{k+1,0}, tol^{k+1}),$$

where $\mathcal{A} = \mathcal{A}_\nu \cup \{i + m |\ i \in \mathcal{A}_\tau\}$, $\lambda^{k+1,0}$ is the initial CGM iteration, and $tol^{k+1}$ denotes the stopping tolerance. The implementation ideas are summarized by ALGORITHM ISSNM, where $err^k = \|\widetilde{r}_\rho(P_\Lambda(\lambda^k))\|$ stands for the precision achieved on the outer level. The value $tol^{k+1}$ in *Step 3.1* respects $err^k$ but, when the progress is not sufficient, it improves the previous tolerance $tol^k$. The inner initialization $\lambda^{k+1,0}$ in *Step 3.2* is chosen by the previous iteration $\lambda^k$ and by the constraints in (11).

ALGORITHM ISSNM Given $\lambda^0 \in \mathbb{R}^{2m}$, $\varepsilon \geq 0$, $\rho > 0$, and $r_{tol}, c_{fact} \in (0, 1)$.

Set $err^0 = \|\widetilde{r}_\rho(P_\Lambda(\lambda^0))\|$, $tol^0 = r_{tol}/c_{fact}$, and $k = 0$.

(*Step 1*) If $err^k \leq \varepsilon$ , return $\lambda = P_\Lambda(\lambda^k)$, else go to step *Step 2*.

(*Step 2*) Assembly the active/inactive sets at $\lambda^k$.

(*Step 3.1*) $tol^{k+1} = \min\{r_{tol} \times err^k/err^0, c_{fact} \times tol^k\}$

(*Step 3.2*) $\lambda_{\mathcal{A}}^{k+1,0} = \lambda_{\mathcal{A}}^k$, $\lambda_{\nu,\mathcal{I}_\nu}^{k+1,0} = 0$, $\lambda_{\tau,\mathcal{I}_\tau^+}^{k+1,0} = g_{\mathcal{I}_\tau^+}$, $\lambda_{\tau,\mathcal{I}_\tau^-}^{k+1,0} = -g_{\mathcal{I}_\tau^-}$

(*Step 3.3*) $\lambda^{k+1} = \mathrm{CGM}(A, b, \mathcal{A}, \lambda^{k+1,0}, tol^{k+1})$

(*Step 3.4*) $err^{k+1} = \|\widetilde{r}_\rho(P_\Lambda(\lambda^{k+1}))\|$, $k = k + 1$, and go to *Step 1*.

# 4 Numerical experiments

Table 1 shows how ALGORITHM ISSNM behaves with respect to the value $\rho = \beta \times \sigma_{\max}^{-1}$, where $\sigma_{\max}$ denotes the largest eigenvalue of $A$. We observe that the dependence on $\rho$ is weak.

Table 1: ISSNM with $\rho = \beta \times \sigma_{\max}^{-1}$

| $\beta$ | 0.05 | 1 | 1.9 | 20 | 100 |
|---|---|---|---|---|---|
| $n/m$ | $iter/n_A$ | $iter/n_A$ | $iter/n_A$ | $iter/n_A$ | $iter/n_A$ |
| 1320/60 | 07/35 | 07/35 | 07/35 | 07/36 | 10/48 |
| 11160/180 | 09/49 | 09/49 | 09/49 | 08/41 | 10/52 |
| 30600/300 | 09/48 | 09/48 | 09/48 | 08/43 | 10/55 |
| 59640/420 | 09/49 | 09/49 | 09/49 | 10/59 | 11/62 |
| 98280/540 | 09/51 | 09/51 | 09/51 | 09/45 | 12/72 |
| 146520/660 | 09/53 | 10/57 | 10/57 | 10/59 | 10/54 |
| 204360/780 | 10/59 | 10/59 | 10/59 | 10/61 | 10/56 |

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