An algorithm for solving 3D contact problems with friction

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The contribution deals with the numerical solving of contact problems with friction for 3D elastic bodies. The algorithm uses the splitting technique based on the Gauss-Seidel iterations that leads into two constrained quadratic programming problems (QPP) in each iterative step. The first QPP contains simple inequality bounds so that existing fast algorithms can be used directly. The second QPP contains quadratic constraints. A new algorithm based on an active set strategy is proposed for solving the second QPP. Numerical experiments illustrate the efficiency of the whole computational process.

1 Introduction

The mathematical model of contact problems with *Coulomb friction* leads to a *quasivariational inequality* [4, 5]. It is well-known that their solutions can be defined as fixed points of a certain mapping and that the *method of successive approximations* is a natural tool for the numerical realization [3]. Each iterative step is represented by an auxiliary contact problem with *given friction* described by a *variational inequality* of the second kind. The efficiency of the whole computational process depends (among others) on the realization of contact problems with given friction. In the contribution, we present a new algorithm for solving these auxiliary problems for 3D bodies.

Our method is based on the *dual formulation* of contact problems with given friction, i.e. the formulation in terms of normal and tangential contact stresses. A discrete variant of the dual formulation is represented by a quadratic programming problem (QPP) with two types of constraints. The constraints on the normal contact stresses are simple inequality bounds while the constraints on the tangential contact stresses are quadratic inequalities (the reason is that the tangential contact stress in each contact node can be represented by a vector with two components that are constrained by the isotropic friction). In order to separate the constraints, we use the Gauss-Seidel iterations so that two QPP are solved in each iterative step. The first QPP is constrained only by simple inequality bounds and therefore it can be solved by an existing fast algorithm with proportioning and gradient projections [1]. The second QPP is constrained only by quadratic inequalities that have the following simple interpretation: the vector whose components are the tangential contact stresses belongs to a circle in R^2 with the center at the origin and a given radius. In our previous paper [3], we have proposed an algorithm based on a piecewise linear approximation of the circle defined by an intersection of squares rotated of a constant angle. This idea enables to transform the quadratic inequality constraints to the simple inequality bounds but, unfortunately, it increases considerably the size of the QPP. Here, we shall propose a new algortihm for solving the QPP treating directly with the quadratic inequality constraints [6]. Our algorithm consists of an active set strategy combined with the conjugate gradients.

2 Discrete dual formulation of contact problems with given friction

The discrete dual formulation of contact problems with given friction reads as follows:

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$$(\mathcal{D}) \quad \begin{cases} \min & \frac{1}{2}\lambda^\top \mathbf{Q}\lambda - \lambda^\top \mathbf{h} \\ \text{s.t.} & \lambda_\nu \in \mathbf{\Lambda}_\nu,\ (\lambda_{t1}, \lambda_{t2}) \in \mathbf{\Lambda}_t,\ \lambda = \left(\lambda_\nu^\top, \lambda_{t1}^\top, \lambda_{t2}^\top\right)^\top \end{cases}$$

with

$$\begin{aligned} \mathbf{\Lambda}_\nu &= \{\lambda_\nu \in R^m |\ (\lambda_\nu)_i \geq 0,\ i = 1, \ldots, m\}, \\ \mathbf{\Lambda}_t &= \left\{(\lambda_{t1}, \lambda_{t2}) \in R^m \times R^m |\ (\lambda_{t1})_i^2 + (\lambda_{t2})_i^2 \leq g_i^2,\ i = 1, \ldots, m\right\}, \end{aligned}$$

where $\mathbf{Q} \in R^{3m \times 3m}$ is the positive definite dual Hessian, $\mathbf{h} \in R^{3m}$ and $g_i > 0$ are values of the slip bound at contact nodes. We search for the normal contact stresses λ_ν and the tangential contact stresses $\lambda_t = (\lambda_{t1}^\top, \lambda_{t2}^\top)^\top$; see [3] for more details.

Let us introduce a new notation for the natural block structure of the Hessian $\mathbf{Q}$ so that

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{\nu\nu} & \mathbf{Q}_{\nu t} \\ \mathbf{Q}_{t\nu} & \mathbf{Q}_{tt} \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} \mathbf{h}_\nu \\ \mathbf{h}_t \end{pmatrix}.$$

Exploiting this partition of $\mathbf{Q}$, we can consider the constrained block Gauss-Seidel method:

ALGORITHM GS

Initialize: $\lambda_t^{(0)},\ i := 0$

repeat

$i := i + 1$

$\lambda_\nu^{(i)} := \arg\min\{\frac{1}{2}\,\lambda_\nu^\top \mathbf{Q}_{\nu\nu}\lambda_\nu - \lambda_\nu^\top(\mathbf{h}_\nu - \mathbf{Q}_{\nu t}\lambda_t^{(i-1)}),\ \text{s.t. } \lambda_\nu \in \mathbf{\Lambda}_\nu\}$

$\lambda_t^{(i)} := \arg\min\{\frac{1}{2}\,\lambda_t^\top \mathbf{Q}_{tt}\lambda_t - \lambda_t^\top(\mathbf{h}_t - \mathbf{Q}_{t\nu}\lambda_\nu^{(i)}),\ \text{s.t. } \lambda_t \in \mathbf{\Lambda}_t\}$

until $\|\lambda^{(i)} - \lambda^{(i-1)}\|_{R^{3m}} \leq$ *tol*

It is well-known that the Gauss-Seidel method converges to the solution of $(\mathcal{D})$; see [2]. While the first QPP in each iterative step of ALGORITHM GS can be solved by the algorithm of [1], an algorithm for solving the second QPP shall be proposed in the next section.

3 Quadratic programming with quadratic constraints

Let us consider the following problem:

$$(QPQ) \begin{cases} \min & f(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}^\top \mathbf{A}\mathbf{x} - \mathbf{x}^\top \mathbf{b} \\ \text{s. t.} & \mathbf{x} \in \Omega = \{\mathbf{x} \in R^{2m} : x_i^2 + x_{2i}^2 \leq g_i^2, i \in \mathcal{N}\}, \end{cases}$$

where $\mathbf{A} \in R^{2m \times 2m}$ is a symmetric positive definite matrix, $\mathbf{b} \in R^{2m}$, g_i are positive values, $\mathcal{N} = \{1, \ldots, m\}$ and x_i denotes the i-th entry of a vector $\mathbf{x} \in R^{2m}$.

Let us define for given $\mathbf{x} \in \Omega$ the *active set* $\mathcal{A}(\mathbf{x})$ so that

$$\mathcal{A}(\mathbf{x}) = \{i \in \mathcal{N} : x_i^2 + x_{2i}^2 = g_i^2\}.$$

Let us introduce the *residual* $\mathbf{r}$ so that $\mathbf{r} = \mathbf{A}\mathbf{x} - \mathbf{b}$ and the *free residual* $\varphi = \varphi(\mathbf{x})$ so that

$$\varphi_i = r_i, \varphi_{2i} = r_{2i} \ \text{ for } i \notin \mathcal{A}(\mathbf{x}),\ \varphi_i = \varphi_{2i} = 0 \ \text{ for } i \in \mathcal{A}(\mathbf{x}).$$

The algorithm for solving the problem (QPQ) reads as follows:

ALGORITHM QPQ Let $\mathbf{x}^0 \in \Omega$. For $k \geq 0$, find $\mathbf{x}^{k+1}$ by the following rules:
(a) If $\varphi(\mathbf{x}^k) \neq \mathbf{o}$, find $\mathbf{x}^{k+1} \in \Omega$ such that $f(\mathbf{x}^{k+1}) < f(\mathbf{x}^k)$ and either

$$\mathcal{A}(\mathbf{x}^k) \subset \mathcal{A}(\mathbf{x}^{k+1}) \text{ and } \varphi(\mathbf{x}^{k+1}) \neq \mathbf{o}$$

or

$$\mathcal{A}(\mathbf{x}^k) \subseteq \mathcal{A}(\mathbf{x}^{k+1}) \text{ and } \varphi(\mathbf{x}^{k+1}) = \mathbf{o}.$$

(b) If $\varphi(\mathbf{x}^k) = \mathbf{o}$, find $\mathbf{x}^{k+1} \in \Omega$ such that $f(\mathbf{x}^{k+1}) < f(\mathbf{x}^k)$ and

$$\mathcal{A}(\mathbf{x}^k) \supset \mathcal{A}(\mathbf{x}^{k+1}).$$

The step (a) can be realized using the unconstrained minimization for the reduced problem (QPQ). We shall assume that the conjugate gradients are used analogously as in the algorithm for the QPP with inequality bounds [1]. The realization of the step (b) can be based on an analyse of the Karush-Kuhn-Tucker (KKT) conditions to the problem (QPQ). Before giving their appropriate form, we introduce notations

$$\mathbf{x}_i = (x_i, x_{2i})^\top \in R^2, \quad \mathbf{r}_i = (r_i, r_{2i})^\top \in R^2.$$

Lemma 3.1 *The vector $\bar{\mathbf{x}} \in \Omega$ is the solution to (QPQ) iff for $i \in \mathcal{N}$*

$$\bar{\mathbf{r}}_i = \mathbf{o} \qquad \textit{for} \quad \|\bar{\mathbf{x}}_i\| < g_i, \tag{1}$$

$$\bar{\mathbf{r}}_i + \frac{\|\bar{\mathbf{r}}_i\|}{g_i}\bar{\mathbf{x}}_i = \mathbf{o} \qquad \textit{for} \quad \|\bar{\mathbf{x}}_i\| = g_i, \tag{2}$$

where $\|\cdot\|$ denotes the Euclidean norm in R^2.

The conditions (2) are called the *KKT boundary conditions*. We can interpret them geometrically so that the half-line of the direction $\bar{\mathbf{r}}_i$ begining at the end point of $\bar{\mathbf{x}}_i$ goes trough the center of the circle representing the i-th qudratic constraint; see Fig. 1.

If the KKT boundary conditions are not satisfied for $\mathbf{x}_i$, then we can find a fesible decrease direction so that it releases indece i from the active set. Let us denote by $\mathbf{x}_i^\perp$ and $\mathbf{r}_i^\perp$ the orthogonal vectors to $\mathbf{x}_i$ and $\mathbf{r}_i$, respectively; see Fig. 2. The vectors lying in the acute angle formed by $\mathbf{x}_i^\perp$ and $\mathbf{r}_i^\perp$ are feasible (since they are oriented into the circle) and decrease (since the angle with respect to the reduced gradient $\mathbf{r}_i$ is greater than $\pi/2$). The optimal feasible decrease direction is represented by the *unbalanced boundary residual*

$$\mathbf{v}_i = -\mathbf{r}_i - \frac{\|\mathbf{r}_i\|}{g_i}\mathbf{x}_i$$

that bisects the angle formed by $\mathbf{x}_i^\perp$ and $\mathbf{r}_i^\perp$. These observations can be used to realize the step (b) in ALGORITHM QPQ; see [6] for more details.

4 Numerical experiments

Let us consider the model problem from [3], i.e. the steel brick unilaterally supported by the rigid foundation, where the non-penetrability condition and the efect of a given friction is prescribed. The brick is partitioned into linear finite elements as in Fig. 3. Table 1 summarizes the CPU times (in the column ALG_GS) and compares them with the CPU times of the algorithms presented in [3] (the columns ALG_2, ALG_4 corresponds to the approximation of the circle by 2, 4 squares, respectively).

5 Conclusion

The numerical experiments illustrate the high efficiency of the whole computational process with respect to our previous methods. It enables to solve sufficiently fine discretizations of the contact problems in a short time.

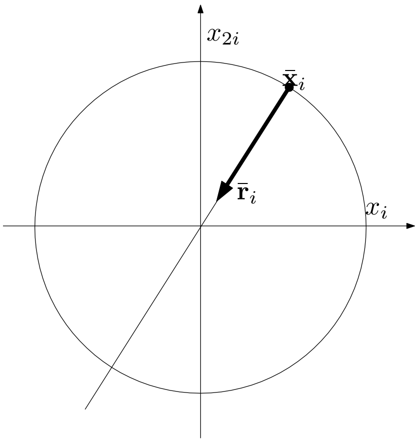

Fig. 1 The KKT boundary condition.

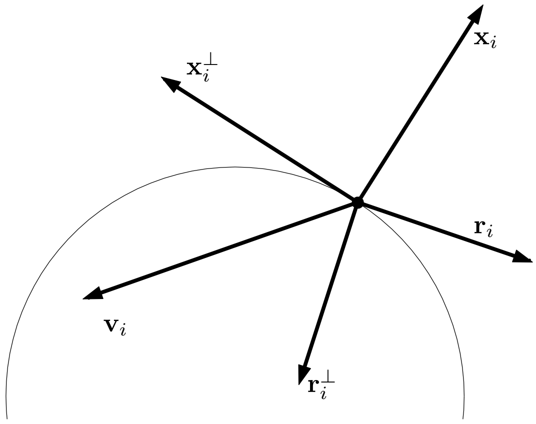

Fig. 2 The feasible decrease direction $\mathbf{v}_i$.

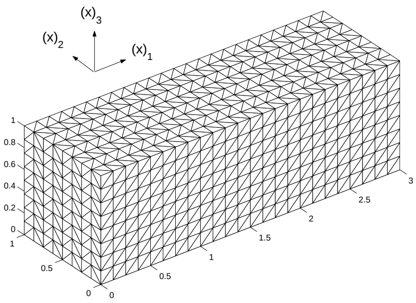

Fig. 3 The partition of the steel brick.

Table 1 CPU time (sec.); n denotes the number of nodes; m denotes the number of contact nodes.

n/m	ALG_GS	ALG_2	ALG_4
189/54	1	1	3
975/180	2	15	61
2793/378	18	101	548
6075/648	25	486	2114
11253/990	60	1542	7724
18759/1404	148	5004	20534

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References

[1] Z. Dostál, J. Schöberl, Minimizing quadratic functions over non-negative cone with the rate of convergence and finite termination, accepted in Comput. Opt. Appl. (2004).
[2] R. Glowinski, Numerical methods for nonlinear variational problems, (Springer, Berlin, 1984).
[3] J. Haslinger, R. Kučera and Z. Dostál, An algorithm for the numerical realization of 3D contact problems with Coulomb friction, J. Comput. Appl. Math. **164-165**, 387–408 (2004).
[4] I. Hlaváček, J. Haslinger, J. Nečas and J. Lovíšek, Numerical solution of variational inequalities, Springer Series in Applied Mathematical Sciences Vol. 66 (Springer, Berlin, 1988).
[5] N. Kikuchi and J. T. Oden, Contact Problems in Elasticity (SIAM, Philadelphia, 1988).
[6] R. Kučera, Quadratic programming with quadratic constraints, in preparation (2004).