Continuation of the static contact problem with Coulomb friction

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1 Discrete static contact problems with Coulomb friction

Let $\Omega \subset \mathbb{R}^2$ be a linearly elastic body supported by a rigid foundation along the contact boundary Γ_C. On Γ_N and Γ_D, Neumann and Dirichlet boundary conditions are prescribed. We consider the static contact problems with Coulomb friction, see e.g. [1]. In particular, we will investigate a discrete version of this problem, see e.g. [2, 3]. This may be understood as a *FEM-approximation* of the continuous mechanical problem.

Let integers n and p define the degrees of freedom of the body Ω and the number of contact nodes on Γ_C, $n \geq 2p$. Let $\mathbf{f} \in \mathbb{R}^n$ and $\mathcal{F}$ be the given distributed volume force and the friction coefficient. We seek for

- nodal displacement field $\mathbf{u} \in \mathbb{R}^n$
- nodal normal and tangential stress components $\boldsymbol{\lambda}_\nu \in \mathbb{R}^p$ and $\boldsymbol{\lambda}_t \in \mathbb{R}^p$

such that $(\mathbf{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) \in \mathbb{R}^n \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_t(\mathcal{F}, -\boldsymbol{\lambda}_\nu)$,

$$(\mathbb{A}\mathbf{u}, \mathbf{v})_n = (\mathbf{f}, \mathbf{v})_n + (\boldsymbol{\lambda}_\nu, \mathbb{N}\mathbf{v})_p + (\boldsymbol{\lambda}_t, \mathbb{T}\mathbf{v})_p \quad \forall\, \mathbf{v} \in \mathbb{R}^n, \tag{1}$$

$$(\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu, \mathbb{N}\mathbf{u})_p + (\boldsymbol{\mu}_t - \boldsymbol{\lambda}_t, \mathbb{T}\mathbf{u})_p \geq 0 \quad \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) \in \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu)\,. \tag{2}$$

Here, $\mathbb{A} \in \mathbb{R}^{n\times n}$ is a positive definite stiffness matrix. The full-rank matrices $\mathbb{N} \in \mathbb{R}^{p\times n}$ and $\mathbb{T} \in \mathbb{R}^{p\times n}$ associate $\mathbf{u} \in \mathbb{R}^n$ with its normal and tangential component at the contact nodes. The convex sets of Lagrange multipliers are

$$\boldsymbol{\Lambda}_\nu = \mathbb{R}^p_-\,, \quad \boldsymbol{\Lambda}_t(\mathcal{F}, -\boldsymbol{\lambda}_\nu) = \{\boldsymbol{\mu}_t \in \mathbb{R}^p : |\mu_{t,i}| \leq -\mathcal{F}\lambda_{\nu,i}\,, \quad \forall\, i = 1, \ldots, p\}\,. \tag{3}$$

It is worth noticing that the second set in (3) depends on the solution component λ_ν.

Let $r > 0$ be a fixed parameter. The variational inequality (2) is equivalent to the equations

$$\boldsymbol{\lambda}_\nu = P_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\mathbb{N}\mathbf{u})\,, \quad \boldsymbol{\lambda}_t = P_{\boldsymbol{\Lambda}_t(\mathcal{F}, -\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t - r\mathbb{T}\mathbf{u})\,, \tag{4}$$

see e.g. [4, 5]. Here $P_{\boldsymbol{\Lambda}_\nu}$ and $P_{\boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu)}$ are the orthogonal projections of $\mathbb{R}^p$ onto $\boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu)$, see (3).

Under generic assumptions, there exists a solution of (1)&(2) for any data $\mathbf{f} \in \mathbb{R}^n$ and $\mathcal{F} > 0$. If $\mathcal{F}$ is sufficiently small, the solution is unique. See e.g. [2, 7].

2 Continuation of the static solutions

Solving (1)&(2) for $(\mathbf{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) \in \mathbb{R}^n \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_t(\mathcal{F}, -\boldsymbol{\lambda}_\nu)$ is equivalent to finding *roots* of a nonlinear mapping (1)&(4).

The static Coulomb friction model depends on parameters. For example, we may prescribe a smooth *loading path* $\alpha \in \mathbb{R} \longmapsto \mathbf{f}(\alpha) \in \mathbb{R}^n$ and ask for a *continuous* response of the body. Then the above mention roots depend on the parameter α. We will define

$$\mathbf{z} \equiv \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda}_\nu \\ \boldsymbol{\lambda}_t \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+2p+1} \longmapsto \mathcal{H}(\mathbf{z}) \equiv \begin{pmatrix} \mathbb{A}\mathbf{u} - \mathbf{f}(\alpha) - \mathbb{N}^\top \boldsymbol{\lambda}_\nu - \mathbb{T}^\top \boldsymbol{\lambda}_t \\ \boldsymbol{\lambda}_\nu - P_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\mathbb{N}\mathbf{u}) \\ \boldsymbol{\lambda}_t - P_{\boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t - r\mathbb{T}\mathbf{u}) \end{pmatrix} \in \mathbb{R}^{n+2p}. \tag{5}$$

The mapping $\mathcal{H} : \mathbb{R}^{n+2p+1} \to \mathbb{R}^{n+2p}$ is *continuous*, *piecewise smooth*, see [7]. Hence, the set $\mathcal{H}(\mathbf{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t, \alpha) = \mathbf{0} \in \mathbb{R}^{n+2p}$ defines generically a continuous, piecewise smooth *curve* in $\mathbb{R}^{n+2p+1}$. The objective is to trace the curves (5) numerically using *path-following* (i.e. *continuation*) techniques. Note that the standard continuation techniques require the curve to be smooth. The idea is:

1. Continue the smooth pieces by a classical path-following software, see e.g. [6].
2. Join the smooth pieces *continuously*, preserving the *orientation*.

For details, see [7, 8].

3 Case study: $n = 1320$, $p = 30$

For the geometry of the example, see Figure 1: It is understood that each nodal mesh point has two degrees of freedom for the vertical and horizontal displacement. The indicated *surface traction* depend on a scalar parameter α; we omit the particular formulae. The contact boundary Γ_C is approximated by $p = 30$ points. The contact data $\boldsymbol{\lambda}_\nu$, $\boldsymbol{\lambda}_t$, $\mathbf{u}_\nu$, $\mathbf{u}_t$ are changed with α. A snapshot as $\alpha = 3.6$ is shown in Figure 2.

We consider continuation of the curve (5) in the parameter range $-0.5 \le \alpha \le 1.5$, starting at $\alpha = -0.5$. The curve is continuous, piecewise smooth. Hence, the curve is smooth up to *transition points*. There were detected 14 transition points on the path: E.g., at the six-th transition point which is related to $\alpha = 0.28019791259766$, the contact nodal point $i = 13$ changes its classification from *no contact* to *contact, slip*. At the seven-th transition point which is related to $\alpha = 0.42934036865234$, the contact nodal point $i = 3$ changes its classification from *contact, slip* to *contact, stick*. At the eight-th transition point which is related to $\alpha = 0.60403706054688$, the contact nodal point $i = 14$ changes its classification from *no contact* to *contact, slip*.

In fact, if we know transition points, we can cheaply compute the solution for any given α, see Figure 3.

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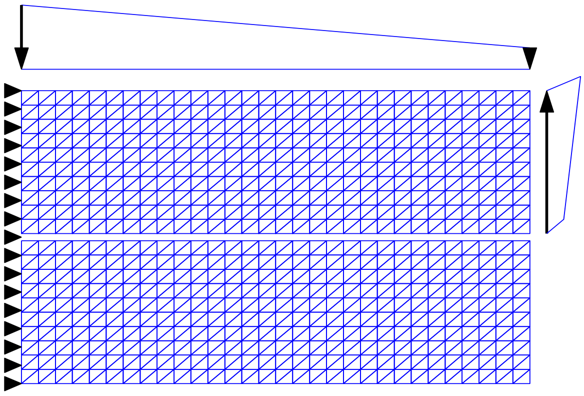

Figure 1: FEM approximation: Case Study $n = 1320$, $p = 30$; the mesh on the rectangular domain Ω. The loading is due to the surface traction.

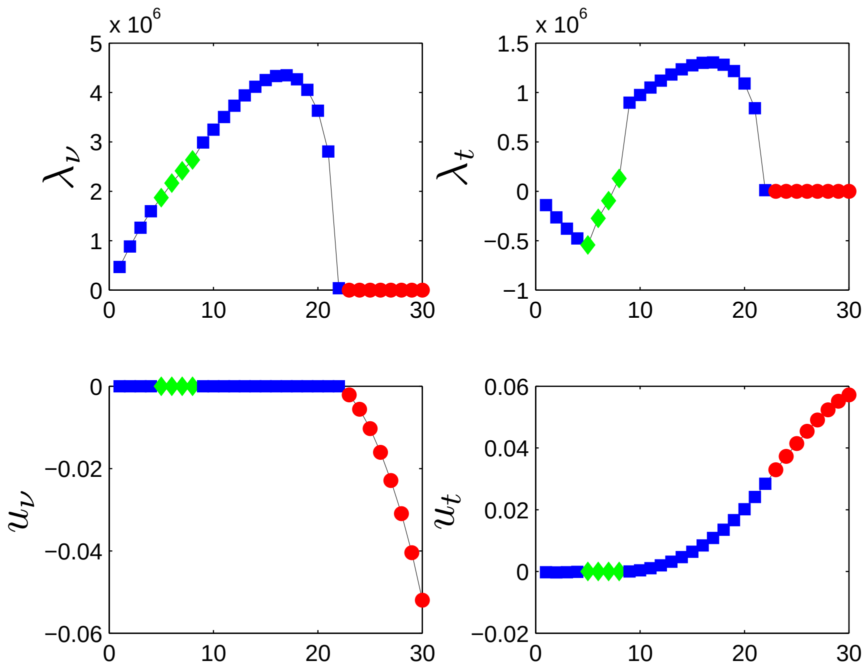

Figure 2: Contact data $\boldsymbol{\lambda}_\nu$, $\boldsymbol{\lambda}_t$, $\mathbf{u}_\nu$, $\mathbf{u}_t$ at the contact points $i = 1, 2, \ldots, 30$ for $\alpha = 3.6$. Contact classification: circle ... *no contact*, diamond ... *contact-stick*, square ... *contact-slip*. Here, $\mathbf{u}_\nu$ and $\mathbf{u}_t$ are the normal and tangential displacement components at particular contact points.

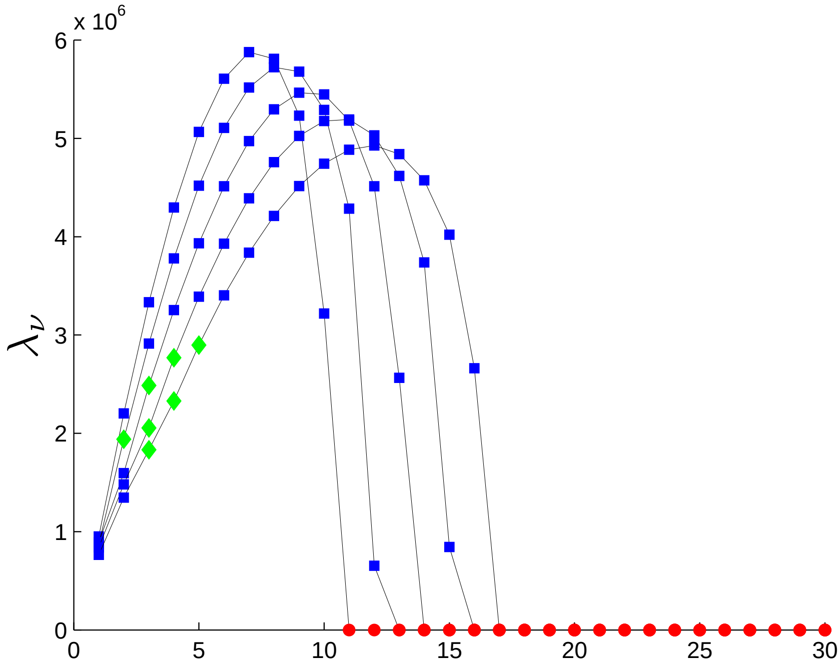

Figure 3: Profiles of the normal stress components: Contact points $i = 1, 2, \ldots, 30$ vs $\boldsymbol{\lambda}_\nu$ for selected parameter values $\alpha = -0.5, 0, 0.5, 1, 1.5$. Contact classification: circle ... *no contact*, diamond ... *contact-stick*, square ... *contact-slip*.

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