Interior point method for 3D contact problems with friction

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1 Introduction

We consider the problem

$$\min_{x\in\Omega} \; q(x) \tag{1}$$

with $q(x) = \frac{1}{2}\, x^\top Ax - x^\top b$ and $\Omega = \{x \in \mathbb{R}^{3m} : \; x_i \geq l_i, \; x_{i+m}^2 + x_{i+2m}^2 \leq g_i^2, \; i = 1, \ldots, m\}$, where $n = 3m$, $A \in \mathbb{R}^{n\times n}$ is symmetric, positive definite, $b \in \mathbb{R}^n$, and $l_i \in \mathbb{R}$, $g_i \in \mathbb{R}_+$, $i = 1, \ldots, m$. Such minimizations arise, e.g., from the finite element approximation of contact problems of linear elasticity with friction in three space dimensions (3D) [8]. The problem (1) can be solved by a general method,nevertheless, one can expect that an algorithm taking into account the structure of (1) will be more efficient. The specialized algorithm for solving (1) called KPRGP (KKT-Proportioning with Reduced Gradient Projections) was proposed and analyzed in [9, 10, 5]. It combines the conjugate gradient method with the gradient projections in a *feasible* active set strategy. This idea generalizes naturally the algorithm of Dostál and Schöberl [4, 6] developed originally for simple bound problems. The common feature of these algorithms is the same convergence rate enabling to achieve optimal convergence results of the domain decomposition methods for 3D contact problems without [3, 4] and with [5, 2] friction. However, their practical behavior may be different due to the difference in the finite termination property. After finding indices of simple bounds active in the solution, the simple bound problem reduces to a linear one, for which the conjugate gradient method completes the iterative process. Unfortunately, this property does not hold for the quadratic inequalities $x_{i+m}^2 + x_{i+2m}^2 \leq g_i^2$. Here, the algorithm seeks also positions of the pairs (x_{i+m}, x_{i+2m}) lying on the curved boundaries of the active circles. Therefore, "zig-zag iterations" may be generated so that short conjugate gradient sequences alternate with projective steps changing the active set. In order to overcome this drawback, one can recommend to use a *strictly feasible* algorithm that does not work with any active set. Its typical representative is an *interior-point* method.

We generalize the *path-following* (PF) variant of the interior point method that was proposed in [13, 7] for solving linear programming problems. The main idea consists in applying the Newton iterations to solve equations in the (modified) system of the Karush-Kuhn-Tucker (KKT) conditions to (1). Since some unknowns in the KKT system are constrained by simple bounds, the Newton steps are damped. Another key ingredient is the centering that keeps iterations deeper in the feasible region so that longer steps may be performed. The most expensive part of each iteration is the solution of an indefinite linear system. To this end, we apply methods based on the Schur complement reduction so that reduced linear systems are solved by the conjugate gradient method. As the reduced matrices are typically ill-conditioned, preconditioners are needed. Our preconditioners are optimal in the sense that condition numbers of the preconditioned matrices are bounded by a constant multiple of the condition number of A (although the condition numbers of the matrices without preconditioning converge to infinity). The total efficiency is increased by a precision control terminating adaptively the inner conjugate gradient iterations. See [11] for more details.

2 Algorithm

Let x^* denote the solution to (1). Let us introduce the Lagrangian to (1) by

$$L(x,\nu) = q(x) + \sum_{i=1}^{m} \nu_i (l_i - x_i) + \sum_{i=1}^{m} \nu_{i+m}(x_{i+m}^2 + x_{i+2m}^2 - g_i^2),$$

where $\nu = (\nu_1, \ldots, \nu_{2m})^\top$ is the Lagrange multiplier to the constraints in Ω. The solution x^* is the first component of the saddle-point $(x^*, \nu^*) \in \mathbb{R}^n \times \mathbb{R}^{2m}$ to L satisfying the KKT conditions:

$$\nabla_x L(x,\nu) = 0,\ \nabla_\nu L(x,\nu) \le 0,\ \nu \ge 0,\ \nu^\top \nabla_\nu L(x,\nu) = 0.$$

Let $z := -\nabla_\nu L(x,\nu)$ be the new variable and let us define the function $F : \mathbb{R}^{n+4m} \mapsto \mathbb{R}^{n+4m}$ by $F(v) = (\nabla_x L(x,\nu)^\top, (\nabla_\nu L(x,\nu) + z)^\top, e^\top NZ)^\top$, where $v = (x^\top, \nu^\top, z^\top)^\top \in \mathbb{R}^{n+4m}$, $N = diag(\nu)$, $Z = diag(z)$, and $e \in \mathbb{R}^{2m}$ is the vector of all ones. The solution x^* is the first component of the solution $v^* = (x^{*\top}, \nu^{*\top}, z^{*\top})^\top$ to

$$F(v) = 0,\ \nu \ge 0,\ z \ge 0. \tag{2}$$

To derive our path-following algorithm, we replace (2) by

$$F(v) = (0^\top, 0^\top, \tau e^\top)^\top,\ \nu > 0,\ z > 0, \tag{3}$$

where $\tau > 0$. Solutions v^τ to (3) define in $\mathbb{R}^{n+4m}$ a curve $\mathcal{C}(\tau)$ called the *central path*. This curve leads to v^*, when τ tends to zero. We combine the Newton method applied to the equation in (3) with changes of τ so that the iterations lay in the neighborhood of $\mathcal{C}(\tau)$ defined by

$$\begin{aligned} \mathcal{N}(\gamma,\beta) \;=\; & \{v = (x^\top, \nu^\top, z^\top)^\top \in \mathbb{R}^{n+4m} :\ \|\nabla_x L(x,\nu)\| \le \beta\vartheta, \\ & \|\nabla_\nu L(x,\nu) + z\| \le \beta\vartheta,\ \nu \ge 0,\ z \ge 0,\ \nu_i z_i \ge \gamma\vartheta,\ i = 1, \ldots, 2m\}, \end{aligned} \tag{4}$$

where $\beta \ge 0$, $\gamma \in (0,1]$, and $\vartheta = \vartheta(v) = \nu^\top z/(2m)$. In the kth iteration, we modify $\tau = \tau_k$ by the product of $\vartheta_k = \vartheta(v^{(k)})$ with the centering parameter σ_k lying in a positive interval. The algorithm uses also the *Armijo-type condition* (6) ensuring that the sequence $\{\vartheta_k\}$ is decreasing. Below, we introduce restrictions on the values of β and σ that are required by the convergence analysis [11]. By $J = J(v)$ in (5), we denote the Jacobi matrix to F at v.

ALGORITHM PF: Given $\gamma \in (0,1]$, $\beta \ge 1$, $0 < \sigma_{\min} \le \sigma_{\max} \le 1/2$, $\omega \in (0,1)$, and $\epsilon \ge 0$. Let $v^{(0)} \in \mathcal{N}(\gamma,\beta)$ and set $k := 0$.

(1°) Choose $\sigma_k \in [\sigma_{\min}, \sigma_{\max}]$.

(2°) If $\nu^{(k)} > 0$ and $z^{(k)} > 0$, solve

$$J(v^{(k)})\Delta v^{(k+1)} = -F(v^{(k)}) + (0^\top, 0^\top, \sigma_k \vartheta_k e^\top)^\top, \tag{5}$$

else set $\Delta v^{(k+1)} = 0$.

(3°) Set

$$v^{(k+1)} = v^{(k)} + \alpha_k \Delta v^{(k+1)}$$

using the largest $\alpha_k \in (0,1]$ satisfying $v^{(k+1)} \in \mathcal{N}(\gamma,\beta)$ and

$$\vartheta_{k+1} \le (1 - \alpha_k \omega(1 - \sigma_k))\vartheta_k. \tag{6}$$

(4°) Return $\bar{v} = v^{(k+1)}$, when

$$err^{(k)} := \|v^{(k+1)} - v^{(k)}\| / \|v^{(k+1)}\| \le \epsilon,$$

else set $k := k + 1$ and go to step (1°).

3 Preconditioning and experiments

The Jacobi matrix $J = J(v)$ reads as follows:

$$J = \begin{pmatrix} J_{11} & J_{12} & 0 \\ J_{21} & 0 & I \\ 0 & Z & N \end{pmatrix},$$

where $J_{11} = A + diag(0, 2N_2, 2N_2)$, $J_{12} = J_{21}^\top = diag(-I, 2(X_2\ X_3)^\top)$, $N_2 = diag(\nu_{m+1}, \ldots, \nu_{2m})$, and $X_k = diag(x_{(k-1)m+1}, \ldots, x_{km})$, $k = 2, 3$. The computational efficiency consists in the way how the ill-conditioned (inner) linear systems with the matrix J are solved. We use two solution methods both based on the Schur complement reduction. The first one is based on the elimination of the 2nd and 3rd unknown that results in the positive definite matrix J_{SC} preconditioned by P_{SC}:

$$J_{SC} = J_{11} + J_{12}D_{22}^{-1}J_{21}, \quad P_{SC} = D_{11} + J_{12}D_{22}^{-1}J_{21},$$

where $D_{22} = N^{-1}Z$, $D_{11} = diag(J_{11})$. The second one is based on the elimination of the 3rd unknown leading to the indefinite matrix J_{AM} preconditioned by P_{AM}:

$$J_{AM} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & -D_{22} \end{pmatrix}, \quad P_{AM} = \begin{pmatrix} D_{11} & J_{12} \\ J_{21} & -D_{22} \end{pmatrix}.$$

The following result is proved in [11]; see also [12, 1].

Theorem 1 *(i) The eigenvalues λ of $P_{AM}^{-1}J_{AM}$ are positive. The eigenvalue $\lambda = 1$ is of the multiplicity $2m$ and the remaining n eigenvalues are the same as the eigenvalues of $P_{SC}^{-1}J_{SC}$.*
(ii) All eigenvalues of $P_{SC}^{-1}J_{SC}$ lay in the interval $[a_{\min}d_{\max}^{-1}, a_{\max}d_{\min}^{-1}]$, where $a_{\min}$, $d_{\min}$ and $a_{\max}$, $d_{\max}$ are the smallest and the largest eigenvalues of A, $D = diag(A)$, respectively.
(iii) The condition numbers $\kappa(P_{SC}^{-1}J_{SC})$ and $\kappa(P_{AM}^{-1}J_{AM})$ are bounded by the product of the condition numbers $\kappa(A)$ and $\kappa(D)$.

Numerical experiments illustrate behavior of ALGORITHM PF for the contact problem with Tresca friction [8]. All computations are performed in Matlab on PC Core i7(2.8GHz) with 4GB RAM. In Table 1, we report the number of iterations *iter*, the number n_A of matrix-vector multiplications by A, and the solution time in seconds. In labeling of columns PF(r_{tol}, c_{fact}), we introduce the parameters that control the inner (conjugate gradient) precisions [11]. To compare the efficiency with KPRGP, we report its solution characteristics. In Figure 1, we depict the iteration history of the condition numbers that is in agreement with Theorem 1.

	PF$(0.3, 0.99)$		PF$(0.1, 0.9)$		PF$(0.01, 0.99)$		KPRGP	
n/m	$iter/n_A$	*time*	$iter/n_A$	*time*	$iter/n_A$	*time*	n_A	*time*
10890/990	19/132	24.91	17/134	24.98	16/136	25.24	203	30.64
18252/1404	16/95	44.43	13/75	36.49	16/141	60.47	230	78.13
28350/1890	19/128	117.44	16/120	109.72	16/133	118.97	254	177.31
41616/2448	16/102	184.47	18/135	232.27	17/157	260.74	259	344.97

Table 1: Tresca friction, inner solver based on J_{AM}.

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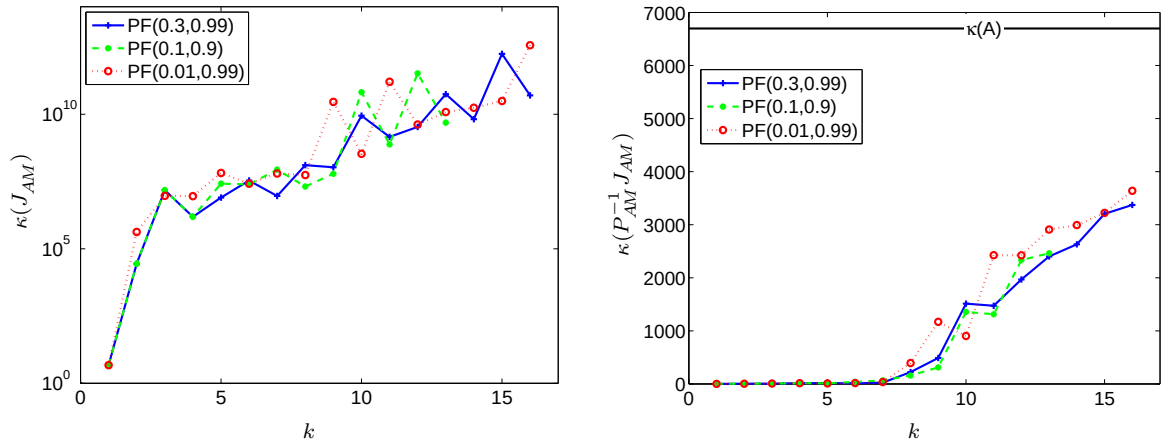

Figure 1: The preconditioning effect (for J_{AM}).

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