Fictitious domain method for linear elasticity

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1 Introduction

The contribution deals with numerical realization of elliptic boundary value problems arising in linear elasticity by a fictitious domain method. Any fictitious domain formulation [2] extends the original problem defined in a domain ω to a new (fictitious) domain Ω with a simple geometry (e.g. a box) which contains $\overline{\omega}$. The main advantage consists in the fact that an uniform mesh can be constructed on $\overline{\Omega}$. Consequently, the stiffness matrix has a structure that enables us to use highly efficient multiplying procedures. We will apply multiplying procedures based on a correspondence between circulant matrices and the discrete Fourier transform (DFT).

The original fictitious domain method based on Lagrange multipliers [1] enforces boundary conditions by Lagrange multipliers defined on the boundary of the original domain γ. Therefore the fictitious domain solution has a singularity on γ that can result in an intrinsic error of the computed solution. Our modified version [3] uses an auxiliary curve Γ located outside of $\overline{\omega}$, on which we introduce a new control variable in order to satisfy the boundary conditions on γ. In this case the singularity is moved away from $\overline{\omega}$ so that the computed solution is smoother in ω. We have illustrated experimentally in [3] that the discretization error is significantly smaller in the second case and corresponding rate of convergence is higher.

2 Formulation of the problem

Let us consider an elastic body represented by a bounded domain $\omega \subset \mathbb{R}^2$ with the sufficiently smooth boundary γ consisting of two disjoint parts γ_u and γ_p, $\gamma = \overline{\gamma}_u \cup \overline{\gamma}_p$ (see Figure 4.1). The zero displacements are prescribed on γ_u while surface tractions of density $\boldsymbol{p} \in (L^2(\gamma_p))^2$ act on γ_p. Finally we suppose that the body ω is subject to volume forces of density $\boldsymbol{f}_{|\omega}$, $\boldsymbol{f} \in (L^2_{loc}(\mathbb{R}^2))^2$. We seek a displacement field $\boldsymbol{u}$ in ω satisfying the *equilibrium equation* and the *Dirichlet and Neumann boundary conditions*:

$$\left.\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} \quad \text{in} \quad \omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on} \quad \gamma_u, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} &= \boldsymbol{p} \quad \text{on} \quad \gamma_p, \end{aligned}\right\} \tag{1}$$

where $\boldsymbol{\sigma}(\boldsymbol{u})$ is the stress tensor in ω and $\boldsymbol{\nu}$ stands for the unit outward normal vector to γ. The stress tensor is related to the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) := 1/2(\nabla\boldsymbol{u} + \nabla^\top\boldsymbol{u})$ by the Hooke law for linear isotropic materials:

$$\boldsymbol{\sigma}(\boldsymbol{u}) := c_1 \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} + 2c_2\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \omega,$$

where "tr" denotes the trace of matrices, $\boldsymbol{I} \in \mathbb{R}^{2\times 2}$ is the identity matrix and $c_1, c_2 > 0$ are the Lamè constants.

Denote

$$\mathbb{V}(\omega) = \{\boldsymbol{v} \in (H^1(\omega))^2 |\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \gamma_u\}.$$

The *weak formulation* of (1) reads as follows:

$$\textit{Find } \boldsymbol{u} \in \mathbb{V}(\omega) \textit{ such that } \ a_\omega(\boldsymbol{u}, \boldsymbol{v}) = f_\omega(\boldsymbol{v}) + (\boldsymbol{p}, \boldsymbol{v})_{\gamma_p} \quad \forall \boldsymbol{v} \in \mathbb{V}(\omega), \tag{2}$$

where

$$a_\omega(\boldsymbol{u}, \boldsymbol{v}) = \int_\omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x}, \quad f_\omega(\boldsymbol{v}) = \int_\omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$$

and $(\cdot, \cdot)_{\gamma_p}$ is the scalar product in $(L^2(\gamma_p))^2$.

Let us consider a box Ω such that $\overline{\omega} \subset \Omega$ and construct a closed curve Γ surrounding ω (see Figure 4.1). Instead of (2), we propose to solve the following *fictitious domain formulation* of (1) in Ω:

$$\left.\begin{array}{l}
\textit{Find } (\hat{\boldsymbol{u}}, \boldsymbol{\lambda}) \in (H^1_{per}(\Omega))^2 \times \boldsymbol{\Lambda}(\Gamma) \textit{ such that} \\
a_\Omega(\hat{\boldsymbol{u}}, \boldsymbol{v}) + b_\Gamma(\boldsymbol{\lambda}, \boldsymbol{v}) = f_\Omega(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in (H^1_{per}(\Omega))^2, \\
b_{\gamma_u}(\boldsymbol{\mu}_u, \hat{\boldsymbol{u}}) = 0 \quad \forall \boldsymbol{\mu}_u \in \boldsymbol{\Lambda}(\gamma_u), \\
b_{\gamma_p}(\boldsymbol{\mu}_p, \boldsymbol{\sigma}(\hat{\boldsymbol{u}})\boldsymbol{\nu}) = b_{\gamma_p}(\boldsymbol{\mu}_p, \boldsymbol{p}) \quad \forall \boldsymbol{\mu}_p \in \boldsymbol{\Lambda}(\gamma_p),
\end{array}\right\} \tag{3}$$

where $H^1_{per}(\Omega)$ is the space of periodic functions from $H^1(\Omega)$; $\boldsymbol{\Lambda}(\Gamma) := (H^{-1/2}(\Gamma))^2$, $\boldsymbol{\Lambda}(\gamma_u) := (H^{-1/2}(\gamma_u))^2$, $\boldsymbol{\Lambda}(\gamma_p) := (H^{1/2}(\gamma_p))^2$ and b_Γ, b_{γ_u}, b_{γ_p} are the respective duality pairings between these spaces and their duals. It is readily seen that $\hat{\boldsymbol{u}}_{|\omega}$ solves (2).

3 Algebraic solvers

A discretization of (3) based on a mixed finite element method leads typically to the following algebraic *saddle-point* problem: find a pair $(u, \lambda) \in \mathbb{R}^{2n} \times \mathbb{R}^{2m}$ such that

$$\left(\begin{array}{c|c} A & B_\Gamma^\top \\ \hline B_{\gamma_u} & 0 \\ C_{\gamma_p} & 0 \end{array}\right) \left(\begin{array}{c} u \\ \lambda \end{array}\right) = \left(\begin{array}{c} f \\ \hline 0 \\ p \end{array}\right), \tag{4}$$

where $A \in \mathbb{R}^{2n \times 2n}$ is the stiffness matrix, $B_\Gamma \in \mathbb{R}^{2m \times 2n}$ and $B_{\gamma_u} \in \mathbb{R}^{2m_u \times 2n}$ are the Dirichlet trace matrices on Γ and γ_u, respectively, $C_{\gamma_p} \in \mathbb{R}^{2m_p \times 2n}$ is the Neumann trace matrix (representing the trace of $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu}$) on γ_p, $f \in \mathbb{R}^{2n}$, $p \in \mathbb{R}^{2m_p}$ and $m = m_u + m_p$.

The system (4) can be solved by the algorithm presented in [3] that combines the Schur complement reduction with the null-space method. It requires a multiplying procedure to perform the matrix-vector products $A^\dagger y$, where $A^\dagger$ is a generalized inverse to A and $y \in \mathbb{R}^{2n}$. Let us note that A is singular due to the presence of $H^1_{per}(\Omega)$ in (3). On the other hand, the periodic boundary condition on $\partial\Omega$ leads to a block circulant structure of A that enables us to handle the spectral decomposition of blocks of A by the DFT. Therefore one can evaluate $A^\dagger y$ by the FFT-algorithm without necessity to assemble and store A.

We introduce the main ideas of our multiplying procedure. First note that the differential operator in (1) reads as follows:

$$\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) = \left(\begin{array}{c|c} (c_1 + 2c_2)\dfrac{\partial^2 u_1}{\partial x_1^2} + c_2 \dfrac{\partial^2 u_1}{\partial x_2^2} + & (c_1 + c_2)\dfrac{\partial^2 u_2}{\partial x_1 \partial x_2} \\ \hline (c_1 + c_2)\dfrac{\partial^2 u_1}{\partial x_1 \partial x_2} + & c_2 \dfrac{\partial^2 u_2}{\partial x_1^2} + (c_1 + 2c_2)\dfrac{\partial^2 u_2}{\partial x_2^2} \end{array}\right),$$

where $\boldsymbol{u} = (u_1, u_2)$. Let us consider equidistant partitions of the sides of $\Omega := (0, l_1) \times (0, l_2)$ into n_1, n_2 segments with stepsizes $h_1 = l_1/n_1$, $h_2 = l_2/n_2$, respectively. Thus, Ω is partitioned into $n := n_1 n_2$ rectangles. On such a partition we define the finite element subspace of $H^1_{per}(\Omega)$ formed by piecewise bilinear functions. Then the stiffness matrix A takes the form:

$$A = \left(\begin{array}{c|c} (c_1 + 2c_2)A_1 \otimes M_2 + c_2 M_1 \otimes A_2 & -(c_1 + c_2)B_1 \otimes B_2 \\ \hline -(c_1 + c_2)B_1 \otimes B_2 & c_2 A_1 \otimes M_2 + (c_1 + 2c_2)M_1 \otimes A_2 \end{array} \right), \tag{5}$$

where $A_k, M_k, B_k \in \mathbb{R}^{n_k \times n_k}$ are the circulants with the first columns $a_k, m_k, b_k \in \mathbb{R}^{n_k}$, $a_k = \frac{1}{h_k}(2, -1, 0, \ldots, 0, -1)^\top$, $m_k = \frac{h_k}{6}(4, 1, 0, \ldots, 0, 1)^\top$, $b_k = \frac{1}{2}(0, -1, 0, \ldots, 0, 1)^\top$, $k = 1, 2$, respectively, and $\otimes$ stands for the Kronecker product. It is well-known that the eigenvalues of any circulant can be obtained by the DFT of its first column while the eigenvectors are the columns of the inverse to the DFT matrix [2]. Introducing notation X_k for the DFT matrix of order n_k, we can write $A_k = X_k^{-1} D_{A_k} X_k$, $M_k = X_k^{-1} D_{M_k} X_k$, $B_k = X_k^{-1} D_{B_k} X_k$, where D_{A_k}, D_{M_k}, D_{B_k}, $k = 1, 2$, are the respective diagonal matrices of eigenvalues. Substituting into (5), we obtain:

$$A = \left(\begin{array}{c|c} X_1^{-1} \otimes X_2^{-1} & 0 \\ \hline 0 & X_1^{-1} \otimes X_2^{-1} \end{array} \right) \left(\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array} \right) \left(\begin{array}{c|c} X_1 \otimes X_2 & 0 \\ \hline 0 & X_1 \otimes X_2 \end{array} \right), \tag{6}$$

where $D_{11} = (c_1 + 2c_2) D_{A_1} \otimes D_{M_2} + c_2 D_{M_1} \otimes D_{A_2}$, $D_{22} = c_2 D_{A_1} \otimes D_{M_2} + (c_1 + 2c_2) D_{M_1} \otimes D_{A_2}$, $D_{12} = (c_1 + c_2) D_{B_1} \otimes D_{B_2}$, $D_{21} = D_{12}$. Denote D the second matrix on the right hand-side of (6). The generalized inverse $A^\dagger$ may be obtained replacing D by $D^\dagger$ in (6). Let us note that the actions of $D^\dagger$ can be easily performed using the following factorization of D:

$$D = \left(\begin{array}{c|c} I & 0 \\ \hline D_{21} D_{11}^\dagger & I \end{array} \right) \left(\begin{array}{c|c} D_{11} & 0 \\ \hline 0 & D_{22} - D_{21} D_{11}^\dagger D_{12} \end{array} \right) \left(\begin{array}{c|c} I & D_{11}^\dagger D_{12} \\ \hline 0 & I \end{array} \right), \tag{7}$$

where $D_{11}^\dagger = \mathrm{diag}(\tilde{d}_1, \ldots, \tilde{d}_n)$ with $\tilde{d}_i = 1/d_i$, if $d_i \neq 0$, and $\tilde{d}_i = 0$, if $d_i = 0$. Taking into account the fact that all blocks in (7) are diagonal, we obtain the following result.

Lemma 3.1 *Let n_1 and n_2 be powers of two. Then the matrix-vector product $A^\dagger v$, $v \in \mathbb{R}^{2n}$, can be evaluated by the total complexity $\mathcal{O}(4n \log_2 n + 4n)$.*

4 Numerical experiments

Let ω be given by the interior of the circle (see Figure 4.1):

$$\omega = \{(x, y) \in \mathbb{R}^2 |\ (x - 0.5)^2 + (y - 0.5)^2 < 0.3^2\}$$

and $\Omega = (0, 1) \times (0, 1)$. The right hand-side in (1) are chosen as $\boldsymbol{f} = -\mathrm{div}\, \boldsymbol{\sigma}(\hat{\boldsymbol{u}})$, $\boldsymbol{p} = \boldsymbol{\sigma}(\hat{\boldsymbol{u}})\boldsymbol{\nu}$, where $\hat{\boldsymbol{u}}(x, y) = (0.1 \ln(x + y + 1), 0.1xy)$, $(x, y) \in \mathbb{R}^2$. The approximation of $H^1_{per}(\Omega)$ in (3) has been described in the previous section while $\boldsymbol{\Lambda}(\gamma_u)$, $\boldsymbol{\Lambda}(\gamma_p)$ and $\boldsymbol{\Lambda}(\Gamma)$ are replaced by their subspaces of piecewise constant functions on partitions of polygonal approximations of γ_u, γ_p and Γ, respectively. The stepsizes H on γ_u, γ_p and Γ are chosen to guarantee the requirement $\dim \boldsymbol{\Lambda}(\gamma_u) + \dim \boldsymbol{\Lambda}(\gamma_p) = \dim \boldsymbol{\Lambda}(\Gamma)$. The auxiliary boundary Γ is constructed by shifting γ four h units in the direction of the outward normal vector with $h := h_1 = h_2$. The original and deformed geometries are depicted in Figure 4.2 and the difference between the exact and computed displacements is shown in Figure 4.3 for $h = 1/256$.

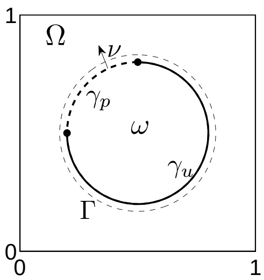

Figure 4.1: Geometry of ω.

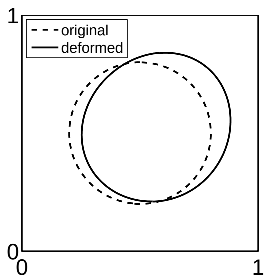

Figure 4.2: Original and deformed geometry.

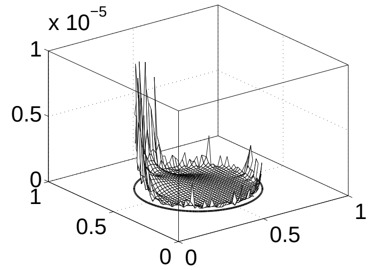

Figure 4.3: Differ. $|\hat{\boldsymbol{u}}_h - \hat{\boldsymbol{u}}|$ in ω.

In Table 4.1, we report the number of primal ($2n$) and control ($2m$) variables, the number of BiCGSTAB iterations, the computational time and the relative errors in the following norms:

$$\mathrm{Err}_{(L_2(\omega))^2} = \frac{\|\hat{\boldsymbol{u}}_h - \hat{\boldsymbol{u}}\|_{(L_2(\omega))^2}}{\|\hat{\boldsymbol{u}}\|_{(L_2(\omega))^2}}, \ \mathrm{Err}_{(H^1(\omega))^2} = \frac{\|\hat{\boldsymbol{u}}_h - \hat{\boldsymbol{u}}\|_{(H^1(\omega))^2}}{\|\hat{\boldsymbol{u}}\|_{(H^1(\omega))^2}}, \ \mathrm{Err}_{(L_2(\gamma))^2} = \frac{\|\hat{\boldsymbol{u}}_h - \hat{\boldsymbol{u}}\|_{(L_2(\gamma))^2}}{\|\hat{\boldsymbol{u}}\|_{(L_2(\gamma))^2}}.$$

From the computed errors, we determine the convergence rates of the fictitious domain solution in the $(L_2(\omega))^2$, $(H^1(\omega))^2$ and $(L_2(\gamma))^2$-norm, respectively. We consider partitions with the non-constant ratio of stepsizes $H/h = |\log_2(h)|$ found experimentally which leads to a smooth behavior of the approximations of control variables as $H \to 0+$.

Table 4.1: Results of the FD approach (3).

Step h	$2n/2m$	Iters.	C.time[s]	$\mathrm{Err}_{(L^2(\omega))^2}$	$\mathrm{Err}_{(H^1(\omega))^2}$	$\mathrm{Err}_{(L^2(\gamma))^2}$
1/64	8450/44	20	0.2808	4.2348e-004	5.2662e-001	9.7813e-004
1/128	33282/68	19	0.39	1.7261e-004	3.3539e-001	3.4267e-004
1/256	132098/124	34	2.371	3.8171e-005	1.5851e-001	1.4673e-004
1/512	526338/212	46	16.26	1.0374e-005	8.2440e-002	2.9814e-005
1/1024	2101250/384	77	109	4.7117e-006	5.5679e-002	1.1683e-005
Convergence rates:				1.7036	0.8508	1.6298

Acknowledgement: This work has been supported by the grant 101/08/0574 of the Grant Agency of the Czech Republic, by the grant IAA100750802 of the Grant Agency of the Czech Academy of Sciences and by the Research Project MSM6198910027 of the Czech Ministry of Education.

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