An algorithm for 3D contact problems with orthotropic friction

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1 Introduction

Contact problems represent a special branch of mechanics of solids whose goal is to find an equilibrium state of deformable bodies being in a mutual contact. Due to non-penetration and friction conditions, problems we have to solve are highly non-linear. For linearly elastic materials obeying a Hook law for small deformations, a linearization of the non-penetration conditions leads to a convex set of kinematically admissible displacements (geometrical nonlinearity). Another non-linearity originates from the presence of friction. In the simplest case with an à-priori given slip bound (Tresca model), the mathematical model is represented by a variational inequality of the second kind. This model is however too simple since the non-penetration and friction phenomena are decoupled. For this reason more realistic models of friction have to be used and the Coulomb friction law is the classical one. The slip bound prescribed in Tresca model is now replaced by the product of a coefficient of friction $\mathcal{F}$ and the norm of the normal contact force. The coupling of unilateral and friction conditions leads to the so-called implicit variational inequality (in terms of displacements) or to a quasivariational inequality (in terms of contact stresses). Due to material or contact surface properties it may happen that the effect of friction is directionally dependent. A discretization of 3D contact problems with orthotropic Coulomb friction characterized by two coefficients of friction $\mathcal{F}_1$ and $\mathcal{F}_2$ in two mutually orthogonal directions was presented in [4]. The scalable algorithm for this problem was developed in [3] while the main ideas may be found in [1].

2 Formulation and algorithm

Let us consider two elastic bodies represented by two non-overlapping domains $\Omega^k \subset \mathbb{R}^3$ with the boundaries $\partial\Omega^k$, $k = 1, 2$. Each boundary consists of three non-empty disjoint parts Γ_u^k, Γ_p^k, and Γ_c^k open in $\partial\Omega^k$, so that $\partial\Omega^k = \overline{\Gamma}_u^k \cup \overline{\Gamma}_p^k \cup \overline{\Gamma}_c^k$. The zero displacements are prescribed on Γ_u^k while surface tractions act on Γ_p^k. On the *contact interface* given by Γ_c^1 and Γ_c^2 we consider contact conditions: the non-penetration of the bodies, the transmission of the contact stresses, and the effect of orthotropic Coulomb friction. Finally we suppose that each body Ω^k is subject to volume forces.

Our algorithm is based on the fixed-point approach in which the solution to the original problem is defined as a fixed-point of an auxiliary mapping acting on the contact interface. To find fixed-point we use the method of successive approximations whose individual iterative steps are given by contact problems with orthotropic Tresca model of friction.

The finite element approximation of the auxiliary problems combined with the TFETI domain

decomposition method [2] leads to the following algebraic minimization problem:

$$\text{minimize} \qquad \frac{1}{2}u^\top Ku - u^\top f + \sum_{i=1}^{m_c} g_i \|\mathcal{F}_i(T_{1,i}u, T_{2,i}u)^\top\|_2, \tag{1}$$

$$\text{subject to} \qquad B_E u = 0,\ Nu \le d, \tag{2}$$

where $K = \mathrm{diag}(K_1, \ldots, K_s)$ is a symmetric positive semidefinite block-diagonal stiffness matrix of order n, $f \in \mathbb{R}^n$ is the load vector, B_E is an $m \times n$ full rank "gluing" matrix, N denotes an $m_c \times n$ full rank matrix describing together with $d \in \mathbb{R}^{m_c}$ the non-penetration condition, $T_{1,i}, T_{2,i}$ are rows of $m_c \times n$ full rank matrices T_1, T_2, respectively, $\mathcal{F}_i \in \mathbb{R}^{2\times 2}$ are the value of the coefficient of friction, and g_i denote discrete slip bound values at contact nodes.

Even though (1)-(2) is the minimization problem with the unique solution, it is not suitable for direct numerical solution. The reasons are that K is typically singular, the summation term in (1) is non-differentiable, and the feasible set in (2) is in general so complex that the projection into it can hardly be effectively computed. In order to overcome these difficulties, one can apply the duality theory of convex programming [1].

To regularize the non-differentiability we use the following idea based on the Cauchy-Schwarz inequality in $\mathbb{R}^2$:

$$\max_{\|\mathcal{F}_i^{-1}\lambda_{T,i}\|_2 \le g_i} (T_{1,i}u, T_{2,i}u)\lambda_{T,i} = g_i \|\mathcal{F}_i(T_{1,i}u, T_{2,i}u)^\top\|_2, \tag{3}$$

where $\lambda_{T,i} \in \mathbb{R}^2$ plays the role of Lagrange multipliers. We will denote $\lambda_{T,i} = (\lambda_{T_1,i}, \lambda_{T_2,i})^\top$. It is easily seen that the constraints on $\lambda_{T,i}$ in (3) are the ellipsoidal inequalities.

In the dual formulation of (1)-(2) we use three types of Lagrange multipliers: $\lambda_E \in \mathbb{R}^m$ and $\lambda_N \in \mathbb{R}^{m_c}$ are associated with the equality and the inequality constraints in (2), while $\lambda_{T_1}, \lambda_{T_2} \in \mathbb{R}^{m_c}$ regularize the non-differentiability via (3). To simplify the notation we denote

$$\lambda = \begin{pmatrix} \lambda_E \\ \lambda_N \\ \lambda_{T_1} \\ \lambda_{T_2} \end{pmatrix}, \quad B = \begin{pmatrix} B_E \\ N \\ T_1 \\ T_2 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ d \\ 0 \\ 0 \end{pmatrix}.$$

The Lagrangian associated with the problem (1)-(2) reads as

$$L(u, \lambda) = \frac{1}{2}u^\top Ku - u^\top f + \lambda^\top(Bu - c), \quad (u, \lambda) \in \mathbb{R}^n \times \Lambda(g),$$

and the set of the Lagrange multipliers is given by

$$\Lambda(g) = \{\lambda \in \mathbb{R}^{m+3m_c} : \lambda_{N,i} \ge 0, \|\mathcal{F}_i^{-1}\lambda_{T,i}\|_2^2 \le g_i^2, i = 1, \ldots, m_c\}.$$

It is well known [1] that (1)-(2) is equivalent to the *saddle-point problem* that is the problem of finding $(\bar{u}, \bar{\lambda}) \in \mathbb{R}^n \times \Lambda(g)$ such that

$$L(\bar{u}, \bar{\lambda}) = \min_{u \in \mathbb{R}^n} \max_{\lambda \in \Lambda(g)} L(u, \lambda).$$

As L is convex in the first variable, $\bar{u}$ can be eliminated by

$$\bar{u} = K^\dagger(f - B\bar{\lambda}) + R\bar{\alpha},$$

where $K^\dagger \in \mathbb{R}^{n\times n}$ is a generalized inverse to K, $R \in \mathbb{R}^{n\times l}$ is a matrix whose columns span the null-space $KerK$, l denotes the defect of K, and $\bar{\alpha} \in \mathbb{R}^l$ is an appropriate vector. In advance,

$$f - B^\top \bar{\lambda} \in Im\, K.$$

Therefore, (1)-(2) leads to the dual problem:

$$\text{minimize } \frac{1}{2}\lambda^\top F\lambda - \lambda^\top \widetilde{h}, \quad \text{subject to } \lambda \in \Lambda(g),\ G\lambda = e,$$

where

$$F = BK^\dagger B^\top, \quad \widetilde{h} = BK^\dagger f - c, \quad G = R^\top B, \quad e = R^\top f.$$

After homogenization, using orthogonal projectors, and penalization, we arrive at the following problem:

$$\text{minimize } \frac{1}{2}\lambda^\top (PFP + \rho Q)\lambda - \lambda^\top Ph, \quad \text{subject to } \lambda \in \Lambda(g),\ G\lambda = 0, \tag{4}$$

where $\rho > 0$ is arbitrary and $Q = G^\top (GG^\top)^{-1} G$, $P = I - Q$ denote the orthogonal projectors on $Im\, G^\top$ and $Ker\, G$, respectively.

As (4) consists of the minimization of the quadratic objective function subject to separable convex inequalities and linear equality constraints, we use the recently proposed optimization algorithm based on the augmented Lagrangian method [3]. The important property of this algorithm is that the number of iterations needed to get a solution with a given accuracy is uniformly bounded (with respect to the scale of the problem) provided that the spectrum of the Hessian is confined in a given interval. The assumption on the spectrum is satisfied due to TFETI domain decomposition method.

3 Numerical experiments

We use the algorithm for solving contact problem with Coulomb friction with the geometry as in Figure 3.1. The upper body is made of steel while the lower one is made of aluminium. The applied surface tractions are seen in the figure, the volume forces are neglected. The coefficient of friction is given by $\mathcal{F}_i = \mathrm{diag}(0.3, 0.3)$ (isotropic case).

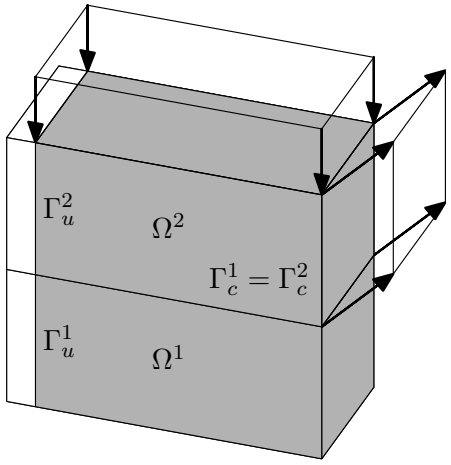

Figure 3.1: Geometry of the model problem

Each body Ω^k, $k = 1, 2$, is divided into the same number of sub-domains represented by bricks of the same size that are decomposed then into cubes (trilinear finite elements). By H and h we denote the decomposition parameter (diameter of bricks) and the discretization parameter (diameter of cubes), respectively. We apply the inexact implementation of the algorithm so that *iter* are connected iterates of the augmented Lagrangian algorithm and the method of successive approximations. By n_{PFP} we denote the number of matrix vector multiplications by the Hessian matrix. Finally, n, $n_d = m + 3m_c$, and l is the number of primal unknowns, dual unknowns, and rigid body modes. The results of our experiments are summarized in Table 3.1, where $rel_{\mathit{eff}} := n_{PFP}/n$ is the relative efficiency of the solver.

Table 3.1: Scalability and relative efficiency

s	$H/h = 2$	$H/h = 3$	$H/h = 4$	$H/h = 5$
4	(324/153/24)	(768/276/24)	(1500/435/24)	(2592/630/24)
	10/180	**10/269**	**11/356**	**11/470**
	0.5556	*0.3503*	*0.2373*	*0.1813*
32	(2592/1527/192)	(6144/2889/192)	(12000/4683/192)	(20736/6909/192)
	11/483	**11/657**	**11/665**	**12/847**
	0.1863	*0.1069*	*0.0554*	*0.0408*
108	(8748/5493/648)	(20736/10506/648)	(40500/17139/648)	(69984/25392/648)
	11/636	**11/878**	**13/906**	**14/1071**
	0.0727	*0.0423*	*0.0224*	*0.0153*
256	(20736/13419/1536)	(49152/25791/1536)	(96000/42195/1536)	(165888/62631/1536)
	12/737	**14/939**	**15/1173**	**16/1400**
	0.0355	*0.01910*	*0.0122*	*0.0084*
500	(40500/26673/3000)	(96000/51408/3000)	(187500/84243/3000)	(324000/125047/3000)
	14/812	**15/1039**	**17/1533**	**18/1776**
	0.0200	*0.0108*	*0.0081*	*0.0054*

[a] At each position $(n/n_d/l)$, $\boldsymbol{iter/n_{PFP}}$, and rel_{eff} are displayed

Acknowledgement: This work has been supported by the grant 101/08/0574 of the Grant Agency of the Czech Republic and by the Research Project MSM6198910027 of the Czech Ministry of Education.

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