Chapter 1 & 2

FORMULATION OF LINEAR PROGRAMMING PROBLEM

INTRODUCTION TO LINEAR PROGRAMMING
Linear Programming is a problem solving approach that has been developed to help managers to make decisions.

Linear Programming is a mathematical technique for determining the optimum allocation of resources and obtaining a particular objective when there are alternative uses of the resources, money, manpower, material, machine and other facilities.

THE NATURE OF LINEAR PROGRAMMING PROBLEM
Two of the most common are:
1. The product-mix problem
2. The blending Problem

In the product- mix problem there are two or more product also called candidates or activities competing for limited resources. The problem is to find out which products to include in production plan and in what quantities these should be produced or sold in order to maximize profit, market share or sales revenue.

The blending problem involves the determination of the best blend of available ingredients to form a certain quantity of a product under strict specifications. The best blend means the least cost blend of the required inputs.

FORMULATION OF THE LINEAR PROGRAMMING MODEL
Three components are:
1. The decision variable
2. The environment (uncontrollable) parameters
3. The result (dependent) variable
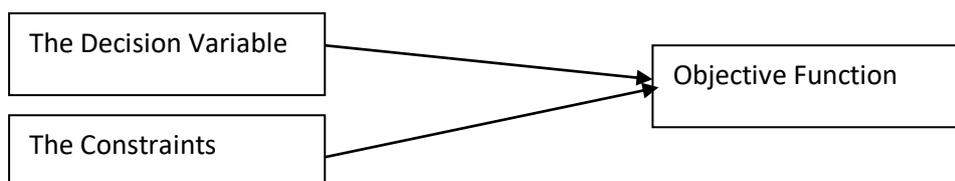
Linear Programming Model is composed of the same components

TERMINOLOGY USED IN LINEAR PROGRAMMING PROBLEM
1. Components of LP Problem: Every LPP is composed of a. Decision Variable, b. Objective Function, c. Constraints.

2. Optimization: Linear Programming attempts to either maximise or minimize the values of the objective function.

3. Profit of Cost Coefficient: The coefficient of the variable in the objective function express the rate at which the value of the objective function increases or decreases by including in the solution one unit of each of the decision variable.

4. Constraints: The maximisation (or minimisation) is performed subject to a set of constraints. Therefore LP can be defined as a constrained optimisation problem. They reflect the limitations of the resources.

5. Input-Output coefficients: The coefficient of constraint variables are called the Input-Output Coefficients. They indicate the rate at which a given resource is unitized or depleted. They appear on the left side of the constraints.

6. Capacities: The capacities or availability of the various resources are given on the right hand side of the constraints.

THE MATHEMATICAL EXPRESSION OF THE LP MODEL

The general LP Model can be expressed in mathematical terms as shown below:

Let

O_{ij} = Input-Output Coefficient

C_j = Cost (Profit) Coefficient

b_i = Capacities (Right Hand Side)

X_j = Decision Variables

Find a vector (x_1, x_2, x_3 ....x_n) that minimise or maximise a linear objective function F(x)

where $F(x) = c_1x_1 + c_2x_2 + ...+ c_nx_n$

subject to linear constraints

$a_1x_1 + a_2x_2 + ...+ a_nx_n = b_2$

$a_1x_1 + a_2x_2 + ...+ a_nx_n \leq b_2$

....

....

....

....

$am_1x_1 + am_2x_2 + ...+ am_nx_n \leq b_2$

and non-negativity constraints

$x_1 \geq 0$, $x_2 \geq 0$, ..., $x_n \geq 0$

FORMULATION OF LPP
STEPS

1. Identify decision variables
2. Write objective function
3. Formulate constraints

EXAMPLE 1. (PRODUCTION ALLOCATION PROBLEM)

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:

Machine	Time per unit (Minutes)			Machine Capacity (minutes/day)
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs. 4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical (L.P.) model that will maximise the daily profit.

Formulation of Linear Programming Model

Step 1

From the study of the situation find the key-decision to be made. In the given situation key decision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

Step 2

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) of products 1, 2 and 3 manufactured daily be x_1, x_2 and x_3 units respectively.

Step 3

Express the feasible alternatives mathematically in terms of variable. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of x_1, x_2 and x_3 units respectively.

where $\quad$ x_1, x_2 and $x_3 \geq 0$.

since negative production has no meaning and is not feasible.

Step 4

Mention the objective function quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit.

i.e., $\quad$ $Z = 4x_1 + 3x_2 + 6x_3$

Step 5

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equations/inequalities in terms of variables.

Here, constraints are on the machine capacities and can be mathematically expressed as

$$2x_1 + 3x_2 + 2x_3 \leq 440$$
$$4x_1 + 0x_2 + 3x_3 \leq 470$$
$$2x_1 + 5x_2 + 0x_3 \leq 430$$

EXAMPLE 2: PRODUCT MIX PROBLEM

A factory manufactures two products A and B. To manufacture one unit of A, 1.5 machine hours and 2.5 labour hours are required. To manufacture product B, 2.5 machine hours and 1.5 labour hours are required. In a month, 300 machine hours and 240 labour hours are available.

Profit per unit for A is Rs. 50 and for B is Rs. 40. Formulate as LPP.

Solution:

Products	Resource/unit	
	Machine	**Labour**
A	1.5	2.5
B	2.5	1.5
Availability	300 hrs	240 hrs

There will be two constraints. One for machine hours availability and for labour hours availability.

Decision variables

X_1 = Number of units of A manufactured per month.

X_2 = Number of units of B manufactured per month.

The objective function:

$$\text{Max } Z = 50x_1 + 40x_2$$

Subjective Constraints

For machine hours

$$1.5x_1 + 2.5x_2 \leq 300$$

For labour hours

$$2.5x_1 + 1.5x_2 \leq 240$$

Non negativity

$$x_1, x_2 \geq 0$$

EXAMPLE: 3

A company produces three products A, B, C.

For manufacturing three raw materials P, Q and R are used.

Profit per unit

A - Rs. 5, B - Rs. 3, C - Rs. 4

Resource requirements/unit

Raw Material Product	P	Q	R
A	-	20	50
B	20	30	-
C	30	20	40

Maximum raw material availability:

P - 80 units;   Q - 100 units;   R - 150 units. Formulate LPP.

Solution:

Decision variables:

x_1 = Number of units of A

x_2 = Number of units of B

x_3 = Number of units of C

Objective Function

Since Profit per unit is given, objective function is maximisation

$$\text{Max } Z = 5x_1 + 3x_2 + 4x_3$$

Constraints:

For P:

$$0x_1 + 20x_2 + 30x_3 \leq 80$$

For Q:

$$20x_1 + 30x_2 + 20x_3 \leq 100$$

For R:

$$50x_1 + 0x_2 + 40x_3 \leq 150$$

(for B, R is not required)

$$X_1, X_2, X_3 \geq 0$$

EXAMPLE 4: PORTFOLIO SELECTION (INVESTMENT DECISIONS)

An investor is considering investing in two securities 'A' and 'B'. The risk and return associated with these securities is different.

Security 'A' gives a return of 9% and has a risk factor of 5 on a scale of zero to 10. Security 'B' gives return of 15% but has risk factor of 8.

Total amount to be invested is Rs. 5, 00, 000/- Total minimum returns on the investment should be 12%. Maximum combined risk should not be more than 6. Formulate as LPP.

Solution:

Decision Variables:

$$X_1 = \text{Amount invested in Security A}$$
$$X_2 = \text{Amount invested in Security B}$$

Objective Function:

The objective is to maximise the return on total investment.

$\therefore$ Max $Z = 0.09 \, X_1 + 0.015 \, X_2$ ((% = 0.09, 15% = 0.15)

Constraints:

1. Related to Total Investment:

$X_1 + X_2 = 5, 00, 000$

2. Related to Risk:

$5X_1 + 8X_2 = (6 \text{ X } 5, 00, 000)$

$5X_1 + 8X_2 = 30, 00, 000$

3. Related to Returns:

$0.09X_1 + 0.15X_2 = (0.12 \text{ X } 5, 00, 000)$

$\therefore 0.09X_1 + 0.15X_2 = 60, 000$

4. Non-negativity

$X_1, X_2 \geq 0$

EXAMPLE 5: INSPECTION PROBLEM

A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1, 500 pieces must be inspected in an 8-hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92%.

Wages of grade I inspector are Rs. 5 per hour while those of grade II inspector are Rs. 4 per hour. Any error made by an inspector costs Rs. 3 to the company. If there are, in all, 10 grade I inspectors and 15 grade II inspectors in the company, find the optimal assignment of inspectors that minimise the daily inspection cost.

Solution:

Let x_1 and x_2 denote the number of grade I and grade II inspectors that may be assigned the job of quality control inspection.

The objective is to minimise the daily cost of inspection. Now the company has to incur two types of costs; wages paid to the inspectors and the cost of their inspection errors. The cost of grade I inspector/hour is

Rs. $(5 + 3 \text{ X } 0.04 \text{ X } 20) = $ Rs. 7.40.

Similarly, cost of grade II inspector/hour is

Rs. $(4 + 3 \text{ X } 0.08 \text{ X } 14) = $ Rs. 7.36.

$\therefore$ The objective function is

minimise $Z = 8(7.40x_1 + 7.36x_2) = 59.20 \, x_1 + 58.88x_2.$

Constraints are

on the number of grade I inspectors: $x_1 \leq 10$,

on the number of grade II inspectors: $x_2 \leq 15$

on the number of pieces to be inspected daily: $20 \times 8x_1 + 14 \times 8x_2 \geq 1500$

$$\text{or } 160x_1 + 112x_2 \geq 1500$$

where, $x_1, x_2 \geq 0$.

EXAMPLE 6: TRIM LOSS PROBLEM

A manufacturer of cylindrical containers receives tin sheets in widths of 30 cm and 60 cm respectively. For these containers the sheets are to be cut to three different widths of 15 cm, 21 cm and 27 cm respectively. The number of containers to be manufactured from these three widths are 400, 200 and 300 respectively. The bottom plates and top covers of the containers are purchased directly from the market. There is no limit on the lengths of standard tin sheets. Formulate the LPP for the production schedule that minimises the trim losses.

Solution:

Key decision is to determine how each of the two standard widths of tin sheets be cut to the require widths so that trim losses are minimum.

From the available widths of 30 cm and 60 cm, several combinations of the three required widths of 15 cm, 21 cm and 27 cm are possible.

Let x_{ij} represent these combinations. Each combination results in certain trim loss.

Constraints can be formulated as follows:

The possible cutting combinations (plans) for both types of sheets are shown in the table below:

Width (cm)	i = I (30 cm)			i = II (60 cm)					
	X_{11}	X_{12}	X_{13}	X_{21}	X_{22}	X_{23}	X_{24}	X_{25}	X_{26}
15	2	0	0	4	2	2	1	0	0
21	0	1	0	0	1	0	2	1	0
27	0	0	1	0	0	1	0	1	2
Trim Loss (cm)	0	9	3	0	9	3	3	12	6

Thus, the constraints are

$2x_{11} + 4x_{21} + 2x_{22} + 2x_{23} + x_{24} \geq 400$

$x_{12} + x_{22} + 2x_{24} + x_{25} \geq 200$

$x_{13} + x_{23} + x_{25} + x_{26} \geq 300$

Objective is to maximise the trim losses.

i.e., minimise $Z = 9x_{12} + 3x_{13} + 9x_{22} + 3x_{23} + 3x_{24} + 12x_{25} + 6x_{26}$

where $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26} \geq 0$.

EXAMPLE 7: MEDIA SELECTION

An advertising agency is planning to launch an ad campaign. Media under consideration are T.V., Radio & Newspaper. Each medium has different reach potential and different cost.

Minimum 10, 000, 000 households are to be reached through T.V. Expenditure on newspapers should not be more than Rs. 10, 00, 000. Total advertising budget is Rs. 20 million.

Following data is available:

Medium	Cost per Unit (Rs.)	Reach per unit (No. of households)
Television	2, 00, 000	20, 00, 000

Radio	80, 000	10, 00, 000
Newspaper	40, 000	2, 00, 000

Solution:

Decision Variables:

x_1 = Number of units of T.V. ads,

x_2 = Number of units of Radio ads,

x_3 = Number of units of Newspaper ads.

Objective function: (Maximise reach)

Max. Z = 20, 00, 000 x_1 + 10, 00, 000 x_2 + 2, 00, 000x_3

Subject to constraints:

20, 00, 000 x_1 ≥ 10, 000, 000 .... (for T.V.)

40, 000 x_3 ≤ 10, 00, 000 .... (for Newspaper)

2, 00, 000x_1 + 80, 000x_2 + 40, 000x_3 ≤ 20, 000, 000 .... (Ad. budget)

x_1, x_2, x_3 ≥ 0

∴ Simplifying constraints:

for T.V.	2 x_1 ≥ 10	∴ x_1 ≥ 5
for Newspaper	4 x_3 ≤ 100	∴ x_3 ≤ 25

Ad. Budget

20 x_1 + 8 x_2 + 4 x_3 ≤ 2000

5 x_1 + 2x_2 + x_3 ≤ 500

x_1, x_2, x_3 ≥ 0

EXAMPLE 8: DIET PROBLEM

Vitamins B_1 and B_2 are found in two foods F_1 and F_2. 1 unit of F_1 contains 3 units of B_1 and 4 units of B_2. 1 unit of F_2 contains 5 units of B_1 and 3 units of B_2 respectively.

Minimum daily prescribed consumption of B_1 & B_2 is 50 and 60 units respectively. Cost per unit of F_1 & F_2 is Rs. 6 & Rs. 3 respectively.

Formulate as LPP.

Solution:

Vitamins	Foods		Minimum Consumption
	F_1	F_2	
B_1	3	5	30
B_2	5	7	40

Decision Variables:

x_1 = No. of units of P_1 per day.

x_2 = No. of units of P_2 per day.

Objective function:

Min. Z = 100 x_1 + 150 x_2

Subject to constraints:

3x_1+ 5x_2 ≥ 30 (for N_1)

5x_1+ 7x_2 ≥ 40 (for N_2)

x_1, x_2 ≥ 0

EXAMPLE 9: BLENDING PROBLEM

A manager at an oil company wants to find optimal mix of two blending processes. Formulate LPP.

Data:

Process	Input (Crude Oil)		Output (Gasoline)	
	Grade A	Grade B	X	Y
P_1	6	4	6	9
P_2	5	6	5	5

Profit per operation: Process 1 (P_1) = Rs. 4, 000

Process 2 (P_2) = Rs. 5, 000

Maximum availability of crude oil: Grade A = 500 units

Grade B = 400 units

Minimum Demand for Gasoline: X = 300 units

Y = 200 units

Solution:

Decision Variables:

x_1 = No. of operations of P_1

x_2 = No. of operations of P_2

Objective Function:

Max. $Z = 4000 x_1 + 5000 x_2$

Subjective to constraints:

$6x_1 + 5x_2 \leq 500$

$4x_1 + 6x_2 \leq 400$

$6x_1 + 5x_2 \geq 300$

$9x_1 + 5x_2 \geq 200$

$x_1, x_2 \geq 0$

EXAMPLE 10: FARM PLANNING

A farmer has 200 acres of land. He produces three products X, Y & Z. Average yield per acre for X, Y & Z is 4000, 6000 and 2000 kg.

Selling price of X, Y & Z is Rs. 2, 1.5 & 4 per kg respectively. Each product needs fertilizers. Cost of fertilizer is Rs. 1 per kg. Per acre need for fertilizer for X, Y & Z is 200, 200 & 100 kg respectively. Labour requirements for X, Y & Z is 10, 12 & 10 man hours per acre. Cost of labour is Rs. 40 per man hour. Maximum availability of labour is 20, 000 man hours. Formulate as LPP to maximise profit.

Solution:

Decision variables:

The production/yield of three products X, Y & Z is given as per acre.

Hence,

x_1 = No. of acres allocated to X

x_2 = No. of acres allocated to Y

x_3 = No. of acres allocated to Z

Objective Function:

Profit = Revenue - Cost

Profit = Revenue - (Fertiliser Cost + Labour Cost)

Product	X	Y	Z
Revenue	2 (4000) x_1	1.5 (6000) x_2	4 (2000) x_3
(-) Less:			

Fertiliser Cost	1 (200) x_1	1 (200) x_2	1 (100) x_3
Labour Cost	40 (10) x_1	40 (12) x_2	40 (10) x_3
Profit	7400 x_1	8320 x_2	7500 x_3

$\therefore$ Objective function

$$\text{Max.} = 7400 x_1 + 8320 x_2 + 7500 x_3$$

Subject to constraints:

$$x_1 + x_2 + x_3 = 200 \qquad \text{(Total Land)}$$
$$10 x_1 + 12 x_2 + 10 x_3 \leq 20,000 \qquad \text{(Max Man hours)}$$
$$x_1, x_2, x_3 \geq 0$$

MERITS OF LPP

1. Helps management to make efficient use of resources.
2. Provides quality in decision making.
3. Excellent tools for adjusting to meet changing demands.
4. Fast determination of the solution if a computer is used.
5. Provides a natural sensitivity analysis.
6. Finds solution to problems with a very large or infinite number of possible solution.

DEMERITS OF LPP

1. Existence of non-linear equation: The primary requirements of Linear Programming is the objective function and constraint function should be linear. Practically linear relationship do not exist in all cases.

2. Interaction between variables: LP fails in a situation where non-linearity in the equation emerge due to joint interaction between some of the activities like total effectiveness.

3. Fractional Value: In LPP fractional values are permitted for the decision variable.

4. Knowledge of Coefficients of the equation: It may not be possible to state all coefficients in the objective function and constraints with certainty.

EXERCISES

1. Explain what is meant by decision variables, objective function and constraints in Linear Programming.

2. Give the mathematical formulation of the linear programming problems.

3. What are the components of LPP? What is the significance of non-negativity restriction?

4. State the limitations of LPP.

5. Give the assumptions and advantages of LPP.

6. An investor wants to identify how much to invest in two funds, one equity and one debt. Total amount available is Rs. 5, 00, 000. Not more than Rs. 3, 00, 000 should be invested in a single fund. Returns expected are 30% in equity and 8% in debt. Minimum return on total investment should be 15%. Formulate as LPP.

7. A company manufactures two products P_1 and P_2. Profit per unit for P_1 is Rs. 200 and for P_2 is Rs. 300. Three raw materials M_1, M_2 and M_3 are required. One unit of P_1 needs 5 units of M_1 and 10 units of M_2. One unit of P_2 needs 18 units of M_2 and 10 units of M_3. Availability is 50 units of M_1, 90 units of M_2 and 50 units of M_3. Formulate as LPP.

8. A firm produces two products X and Y. Minimum 50 units of X should be produced. There is no limit for producing Y. Profit per unit is Rs. 100 for X and Rs. 150 for Y.

Product	Resource Requirement	Resource Availability
X	20 Machine Hours	Machine Hours = 2500
	10 Labour Hours	Labour Hours = 3000
Y	10 Machine Hours	
	15 Labour Hours	

Formulate as LPP.

9. A patient has been recommended two nutrients N_1 and N_2 everyday. Minimum intake is 10g for N_1 and 15g for N_2 everyday.

These nutrients are available in two products P_1 and P_2. One unit of P_1 contains 2g of N_1 and 3g of N_2. One unit of P_2 contains 1g of N_1 and 2g of N_2. Cost per unit is Rs. 200 for P_1 and Rs. 150 for P_2.

Formulate as LPP such that nutrient requirement can be fulfilled at the lowest cost.

10. Two vitamins A and B are to be given as health supplements on daily basis to students. There are two products Alpha & Beta which contain vitamins A and B. One unit of Alpha contains 2g of A and 1g of B. One unit of Beta contains 1g of A and 2g of B. Daily requirements for A and B are atleast 10g each. Cost per unit of Alpha is Rs. 20 and of Beta is Rs. 30. Formulate as LPP to satisfy the requirements at minimum cost.

------xxx------xxx-----

UNIT 3
LINEAR PROGRAMMING SOLUTION - GRAPHICAL METHOD

INTRODUCTION

There are two methods available to find optimal solution to a Linear Programming Problem. One is graphical method and the other is simplex method.

Graphical method can be used only for a two variables problem i.e. a problem which involves two decision variables. The two axes of the graph (X & Y axis) represent the two decision variables X_1 & X_2.

METHODOLOGY OF GRAPHICAL METHOD

Step 1: Formulation of LPP (Linear Programming Problem)

Use the given data to formulate the LPP.

Maximisation

Example 1

A company manufactures two products A and B. Both products are processed on two machines M_1 & M_2.

	M_1	M_2
A	6 Hrs/Unit	2 Hrs/Unit
B	4 Hrs/Unit	4 Hrs/Unit
Availability	7200 Hrs/month	4000 Hrs/month

Profit per unit for A is Rs. 100 and for B is Rs. 80. Find out the monthly production of A and B to maximise profit by graphical method.

Formulation of LPP

X_1 = No. of units of A/Month

X_2 = No. of units of B/Month

Max $Z = 100 X_1 + 80 X_2$

Subject to constraints:

$6 X_1 + 4 X_2 \leq 7200$

$2 X_1 + 4 X_2 \leq 4000$

$X_1, X_2 \geq 0$

Step 2: Determination of each axis

Horizontal (X) axis: Product A (X_1)

Vertical (Y) axis: Product B (X_2)

Step 3: Finding co-ordinates of constraint lines to represent constraint lines on the graph.

The constraints are presently in the form of inequality ($\leq$). We should convert them into equality to obtain co-ordinates.

Constraint No. 1: $6 X_1 + 4 X_2 \leq 7200$

Converting into equality:

$6 X_1 + 4 X_2 \leq 7200$

X_1 is the intercept on X axis and X_2 is the intercept on Y axis.

To find X_1, let $X_2 = 0$

$6 X_1 = 7200$

$6 X_1 = 7200$ $\therefore X_1 = 1200; \ X_2 = 0 \ (1200, 0)$

To find X_2, let $X_1 = 0$

$4 X_2 = 7200$ $X_2 = 1800; \ X_1 = 0 \ (0, 1800)$

Hence the two points which make the constraint line are:

(1200, 0) and (0, 1800)

Note: When we write co-ordinates of any point, we always write (X_1, X_2). The value of X_1 is written first and then value of X_2. Hence, if for a point X_1 is 1200 and X_2 is zero, then its co-ordinates will be (1200, 0).

Similarly, for second point, X_1 is 0 and X_2 is 1800. Hence, its co-ordinates are (0, 1800).

Constraint No. 2:

$2 X_1 + 4 X_2 \leq 4000$

To find X_1, let $X_2 = 0$

$2 X_1 = 4000$ $\therefore X_1 = 2000; \ X_2 = 0 \ (2000, 0)$

To find X_2, let $X_1 = 0$

$4 X_2 = 4000$ $\therefore X_2 = 1000; \ X_1 = 0 \ (0, 1000)$

Each constraint will be represented by a single straight line on the graph. There are two constraints, hence there will be two straight lines.

The co-ordinates of points are:

1. Constraint No. 1: (1200, 0) and (0, 1800)
2. Constraint No. 2: (2000, 0) and (0, 1000)

Step 4: Representing constraint lines on graph

To mark the points on the graph, we need to select appropriate scale. Which scale to take will depend on maximum value of X_1 & X_2 from co-ordinates.

For X_1, we have 2 values $\longrightarrow$ 1200 and 2000

$\therefore$ Max. value for $X_2 = 2000$

For X_2, we have 2 values $\longrightarrow$ 1800 and 1000

$\therefore$ Max. value for $X_2 = 1800$

Assuming that we have a graph paper 20 X 30 cm. We need to accommodate our lines such that for X-axis, maximum value of 2000 contains in 20 cm.

$\therefore$ Scale 1 cm = 200 units

$\therefore$ 2000 units = 10 cm (X-axis)
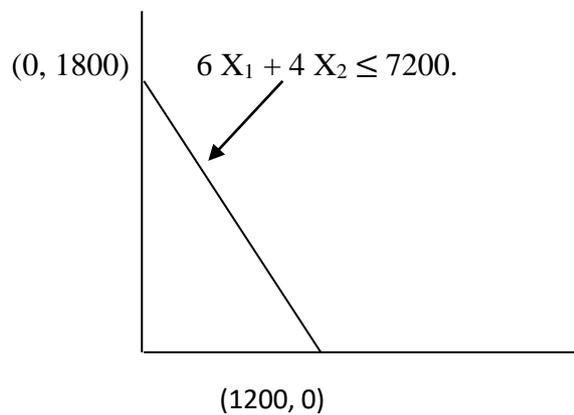
1800 units = 9 cm (Y-axis)

The scale should be such that the diagram should not be too small.

Constraint No. 1:

The line joining the two points (1200, 0) and (0, 1800) represents the constraint

$6 X_1 + 4 X_2 \leq 7200$.

Fig 1.

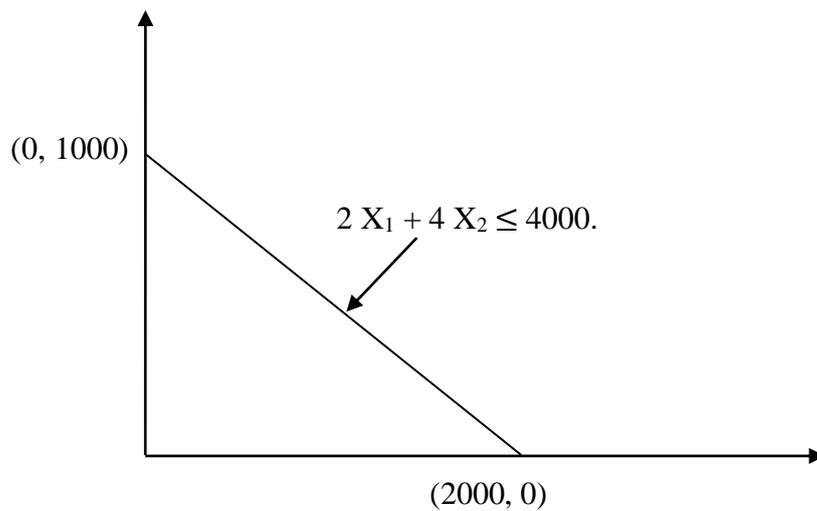(0, 1800) $6 X_1 + 4 X_2 \leq 7200.$

(1200, 0)

Every point on the line will satisfy the equation (equality) $6 X_1 + 4 X_2 \leq 7200$.
Every point below the line will satisfy the inequality (less than) $6 X_1 + 4 X_2 \leq 7200$.
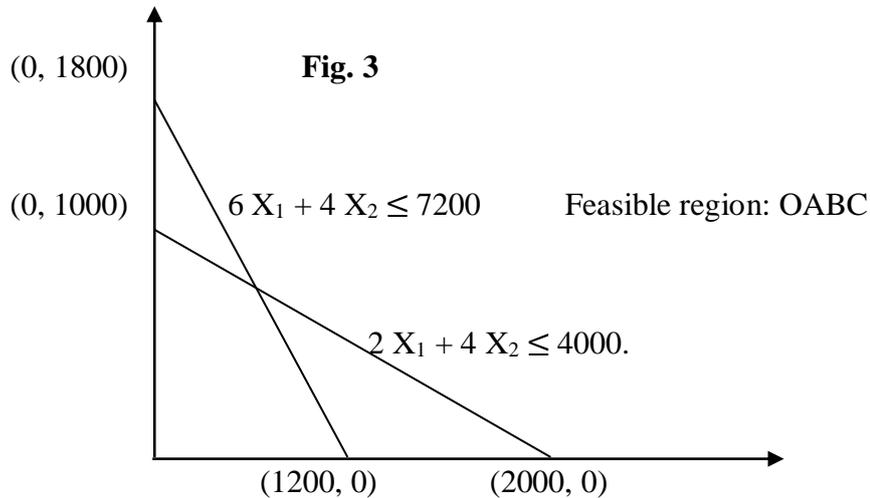
Constraint No. 2:

The line joining the two points (2000, 0) and (0, 1000) represents the constraint
$2 X_1 + 4 X_2 \leq 4000$
Every point on the line will satisfy the equation (equality) $2 X_1 + 4 X_2 \leq 4000$.
Every point below the line will satisfy the inequality (less than) $2 X_1 + 4 X_2 \leq 4000$.

Fig 2

(0, 1000) $2 X_1 + 4 X_2 \leq 4000.$

(2000, 0)

Now the final graph will look like this:

Fig. 3

(0, 1800)

(0, 1000) $6 X_1 + 4 X_2 \leq 7200$ Feasible region: OABC

$2 X_1 + 4 X_2 \leq 4000.$
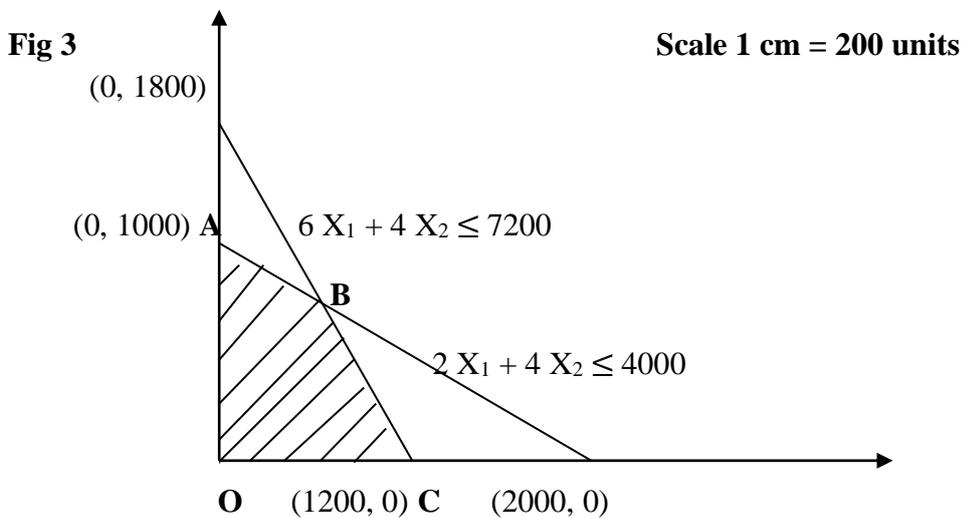
(1200, 0)   (2000, 0)

Step 5: Identification of Feasible Region

The feasible region is the region bounded by constraint lines. All points inside the feasible region or on the boundary of the feasible region or at the corner of the feasible region satisfy all constraints.

Both the constraints are 'less than or equal to' ($\leq$) type. Hence, the feasible region should be inside both constraint lines.

Hence, the feasible region is the polygon OABC. 'O' is the origin whose coordinates are (0, 0). O, A, B and C are called vertices of the feasible region.

Fig 3   **Scale 1 cm = 200 units**

(0, 1800)

(0, 1000) **A** $6 X_1 + 4 X_2 \leq 7200$

B

$2 X_1 + 4 X_2 \leq 4000$

O   (1200, 0) **C**   (2000, 0)

Step 6: Finding the optimal Solution

The optimal solution always lies at one of the vertices or corners of the feasible region.

To find optimal solution:

We use corner point method. We find coordinates (X_1, X_2 Values) for each vertex or corner point. From this we fine 'Z' value for each corner point.

Vertex	Co-ordinates	$Z = 100 X_1 + 80 X_2$
O	$X_1 = 0$, $X_2 = 0$ From Graph	$Z = 0$
A	$X_1 = 0$, $X_2 = 1000$ From Graph	$Z =$ Rs. 80, 000

B	$X_1 = 800, X_2 = 600$ From Simultaneous equations	Z = Rs. 1, 28, 000
C	$X_1 = 1200, X_2 = 0$ From Graph	Z = Rs. 1, 20, 000

Max. Z = Rs. 1, 28, 000 (At point B)

For B $\longrightarrow$ B is at the intersection of two constraint lines $6 X_1 + 4 X_2 \leq 7200$ and $2 X_1 + 4 X_2 \leq 4000$. Hence, values of X_1 and X_2 at B must satisfy both the equations.

We have two equations and two unknowns, X_1 and X_2. Solving simultaneously.

$6 X_1 + 4 X_2 \leq 7200$ (1)

$2 X_1 + 4 X_2 \leq 4000$ (2)

        $4 X_1 = 3200$ Subtracting (2) from (1)

         $X_1 = 800$

Substituting value of X_1 in equation (1), we get

        $4 X_2 = 2400$ $\therefore X_2 = 600$

Solution

Optimal Profit = Max Z = Rs. 1, 28, 000

Product Mix:

                X_1 = No. of units of A / Month = 800

                X_2 = No. of units of A / Month = 600

ISO Profit line:

ISO profit line is the line which passes through the points of optimal solution (Maximum Profit). The slope of the iso-profit line depends on the objective function.

In the above example, the objective function is:

Max. $Z = 100 X_1 + 80 X_2$

How to find slope of iso-profit line:

Equation of a straight line: $y = mx + c$

where,   m = slope of the straight line

In our case, y means ' X_2' and x means ' X_1'.

c means 'Z'.

$\therefore X_2 = m . X_1 + Z$

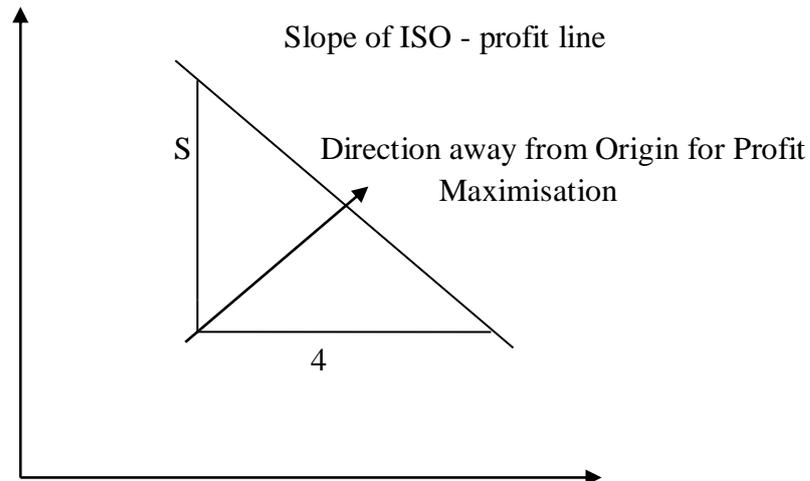Converting original objective function in this format:

Max. $Z = 100 X_1 + 80 X_2$

$\therefore 80 X_2 = Z - 100 X_1 = - 100 X_1 + Z$

$\therefore X_2 = \frac{-100}{80}X_1 + \frac{Z}{80}$

$\therefore X_2 = \frac{-5}{4}X_1 + \frac{Z}{80}$

$\therefore$ Slope of ISO profit line $= \frac{-5}{4}$

Negative sign indicates that the line will slope from left to right downwards. And slope will be 5/4. Every 4 units on X-axis for 5 units on Y-axis.

Slope of ISO - profit line

Direction away from Origin for Profit Maximisation

As we start from the origin and go away from origin to maximise profit, 'B' is the last point on the feasible region that is intersected by the iso-profit line. Hence, B is the optimal solution.

MINIMISATION

Example 2

A firm is engaged in animal breeding. The animals are to be given nutrition supplements everyday. There are two products A and B which contain the three required nutrients.

Nutrients	Quantity/unit		Minimum Requirement
	A	**B**	
1	72	12	216
2	6	24	72
3	40	20	200

Product cost per unit are: A: rs. 40; B: Rs. 80. Find out quantity of product A & B to be given to provide minimum nutritional requirement.

Step 1: Formulation as LPP

X_1 - Number of units of A

X_2 - Number of units of B

Z - Total Cost

Min. $Z = 40 X_1 + 80 X_2$

Subject to constraints:

$72 X_1 + 12 X_2 \geq 216$

$6 X_1 + 24 X_2 \geq 72$

$40 X_1 + 20 X_2 \geq 200$

$X_1, X_2 \geq 0$.

Step 2: Determination of each axis

Horizontal (X) axis:　　Product A (X_1)

Vertical (Y) axis:　　　Product B (X_2)

Step 3: Finding co-ordinates of constraint lines to represent the graph

All constraints are 'greater than or equal to' type. We should convert them into equality:

1. Constraint No. 1: $72 X_1 + 12 X_2 \geq 216$

Converting into equality

$72 X_1 + 12 X_2 = 216$

To find X_1, let $X_2 = 0$

$72 X_1 = 216$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\therefore X_1 = 3, X_2 = 0$ $\quad\quad$ (3, 0)

To find X_2, let $X_1 = 0$

$12 X_2 = 216$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\therefore X_1 = 0, X_2 = 18$ $\quad\quad$ (0, 18)

2. Constraint No. 2:

$6 X_1 + 24 X_2 \geq 72$

To find X_1, let $X_2 = 0$

$6 X_1 = 72$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\therefore X_1 = 12, X_2 = 0$ $\quad\quad$ (12, 0)

To find X_2, let $X_1 = 0$

$24 X_2 = 72$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\therefore X_1 = 0, X_2 = 3$ $\quad\quad$ (0, 3)

3. Constraint No. 3:

$40 X_1 + 20 X_2 \geq 200$

To find X_1, let $X_2 = 0$

$40 X_1 = 200$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\therefore X_1 = 5, X_2 = 0$ $\quad\quad$ (5, 0)

To find X_2, let $X_1 = 0$

$20 X_2 = 200$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\therefore X_1 = 0, X_2 = 10$ $\quad\quad$ (0, 10)

The co-ordinates of points are:

1. Constraint No. 1: (3, 0) & (0, 18)
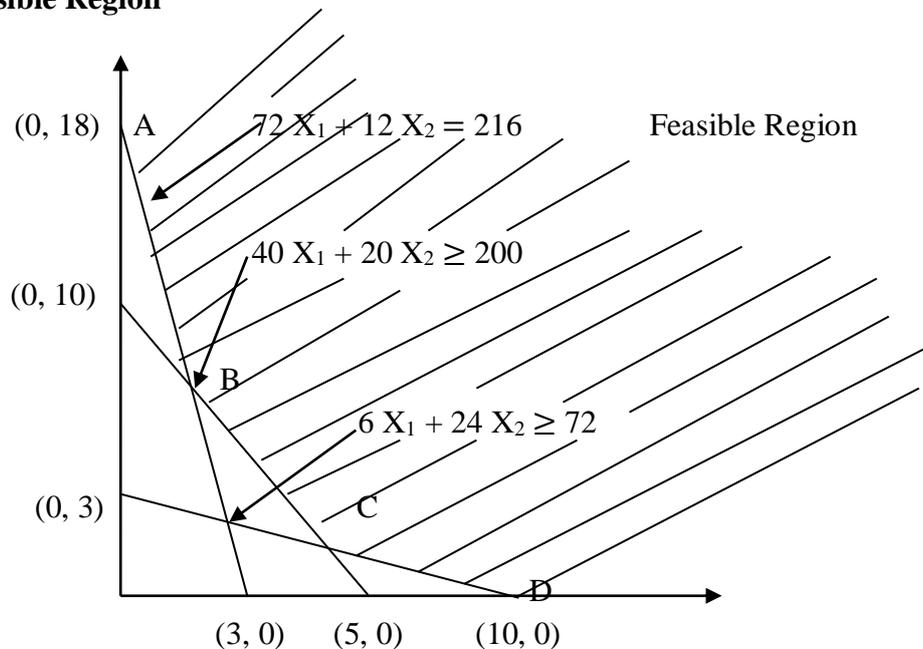
2. Constraint No. 2: (12, 0) & (0, 3)

3. Constraint No. 3: (5, 0) & (0, 5)

Every point on the line will satisfy the equation (equality) $72 X_1 + 12 X_2 = 216$.

Every point above the line will satisfy the inequality (greater than) $72 X_1 + 12 X_2 = 216$.

Similarly, we can draw lines for other two constraints.

Step 5: Feasible Region

All constraints are greater than or equal to ($\geq$) type. Hence, feasible region should be above (to the right of) all constraints.

The vertices of the feasible region are A, B, C & D.

Step 6: Finding the optimal solution

Corner Point Method

Vertex	Co-ordinates	$Z = 40 X_1 + 80 X_2$
A	$X_1 = 0$, $X_2 = 18$ From Graph	$\therefore Z = 1, 440$
B	$X_1 = 2$, $X_2 = 6$ From Simultaneous Equations	$\therefore Z = 560$
C	$X_1 = 4$, $X_2 = 2$ From Simultaneous Equations	$\therefore Z = 320$
D	$X_1 = 12$, $X_2 = 0$ From graph	$\therefore Z = 480$

$\therefore$ Min. Z = Rs. 320 (At point 'C')

For B - Point B is at intersection of constraint lines '$72 X_1 + 12 X_2 \geq 216$' and '$40 X_1 + 20 X_2 \geq 200$'. Hence, point B should satisfy both the equations.

$72 X_1 + 12 X_2 = 216$ (1)

$40 X_1 + 20 X_2 = 200$ (2)

$\therefore 360 X_1 + 60 X_2 = 1080$ (1) x 5

$120 X_1 + 60 X_2 = 600$ (2) x 3

$\therefore 240 X_1 = 480$

       $X_1 = 2$

Substituting value of X_1 in equation (1), we get:

$12 X_2 = 216 - 144 = 72$

$X_2 = 6$

For C - Point C is at intersection of constraint lines '$6 X_1 + 24 X_2 = 72$' and '$40 X_1 + 20 X_2 = 200$'. Hence, point C should satisfy both the equations.

$6 X_1 + 24 X_2 = 72$ (1)

$40 X_1 + 20 X_2 = 200$ (2)

$30 X_1 + 120 X_2 = 360$ (1) x 5

$240 X_1 + 120 X_2 = 1200$ (2) x 6

$210 X_1 = 840$

$X_1 = 4$

Substituting value of X_1 in equation (1), we get

$24 X_2 = 72 - 24 = 48$

$X_2 = 2$

Solution

Optimal Cost = Z min = Rs. 320

Optimal Product Mix:

               X_1 = No. of units of product A = 4

               X_2 = No. of units of product B = 2

ISO Cost Line

ISO Cost line passes through the point of optimal solution (Minimum cost)

Objective function: $Z = 40 X_1 + 80 X_2$

Equation of straight line: $y = mx + c$
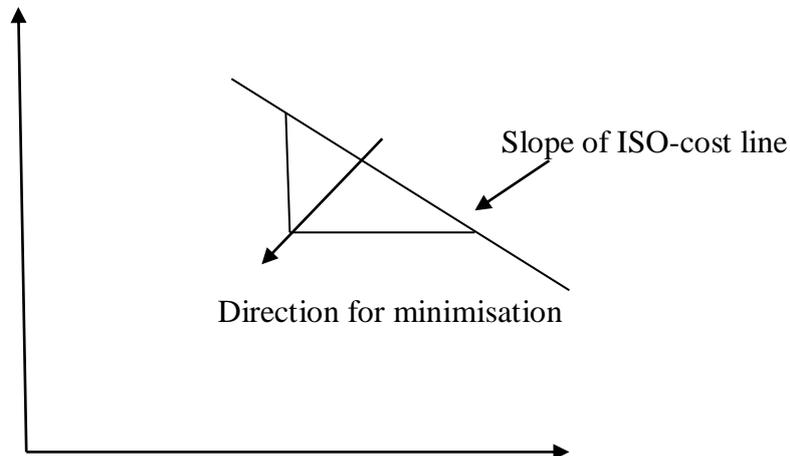
where m = slope

In this case, y is X_2 & x is X_1

$Z = 40 X_1 + 80 X_2$

$\therefore 80 X_2 = - 40 X_1 + Z$

$\therefore X_2 = -1/2 x_1 + Z/80$

$\therefore$ Slope of ISO-cost line $= -1/2$

Sloping from left to right downwards

Slope of ISO-cost line

Direction for minimisation

$\therefore$ Point C is the nearest to the origin.

MAXIMISATION-MIXED CONSTRAINTS

Example 1

A firm makes two products P_1 & P_2 and has production capacity of 18 tonnes per day. P_1 & P_2 require same production capacity. The firm must supply at least 4 t of P_1 & 6 t of P_2 per day. Each tonne of P_1 & P_2 requires 60 hours of machine work each. Maximum machine hours available are 720. Profit per tonne for P_1 is Rs. 160 & P_2 is Rs. 240. Find optimal solution by graphical method.

LPP Formulation

X_1 = Tonnes of P_1 / Day

X_2 = Tonnes of P_2 / Day

Max. $Z = 160 X_1 + 240 X_2$

Subject to constraints

$X_1 \geq 4$

$X_2 \geq 6$

$X_1 + X_2 \leq 18$

$60 X_1 + 60 X_2 \leq 720$

$X_1, X_2 \geq 0$

Coordinates for constraint lines:

1. $X_1 \geq 4$ (4, 0) .... No value for X_2, $\therefore X_2 = 0$

2. $X_2 \geq 6$ (0, 6) .... No value for X_1, $\therefore X_1 = 0$

3. $X_1 + X_2 \leq 18$ (18, 0) (0, 18)

4. $60 X_1 + 60 X_2 \leq 720$ (12, 0) (0, 12)

If $X_1 = 0$, $60 X_2 = 720$ $\therefore X_2 = 12$ (0, 12)
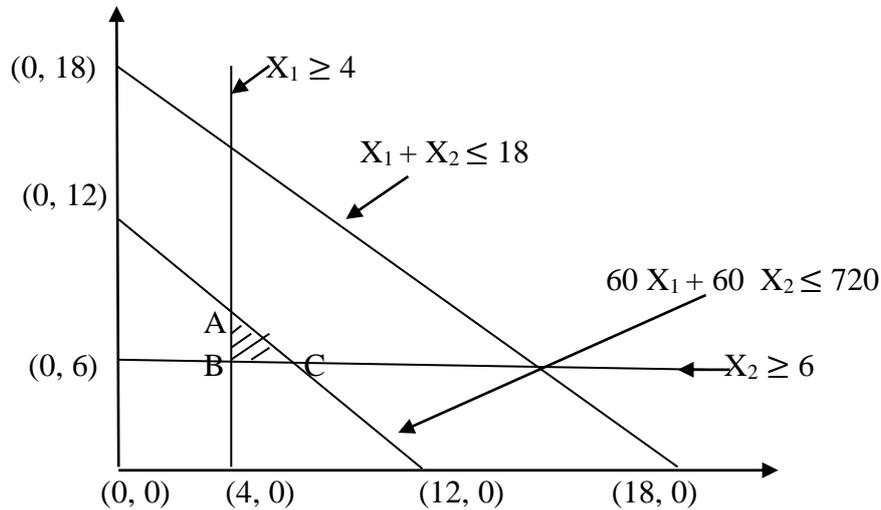
If $X_2 = 0$, $60\ X_1 = 720$ $\therefore X_1 = 12$ (12, 0)

Graph: X_1: X Axis

   X_2: Y Axis

Scale:

Maximum value for $X_1 = 18$; Maximum value for $X_2 = 18$; $\therefore$ Scale: 1 cm = 2 Tonnes.

Two constraints are 'greater than or equal to' type. Hence, feasible region will be above or to the right of these constraint lines. Two constraints are 'less than or equal to' type. Hence, feasible region will be below or to the left of these constraint lines. Hence, feasible region is ABC.

Optimal Solution

Corner Point Method

Vertex	Coordinates	$Z = 160\ X_1 + 240\ X_2$
A	$X_1 = 4$, $X_2 = 8$ Simultaneous Equation	$\therefore Z$ = Rs. 2, 560
B	$X_1 = 4$, $X_2 = 6$ From Graph	$\therefore Z$ = Rs. 2, 080
C	$X_1 = 6$, $X_2 = 6$ Simultaneous Equations	$\therefore Z$ = Rs. 2, 400

For A $\longrightarrow$ $X_1 = 4$ from graph

A is on the line $60\ X_1 + 60\ X_2 = 720$

$60\ X_2 = 720 - 60\ (4) = 480$ $\therefore X_2 = 8$

For C $\longrightarrow$ $X_2 = 6$ from graph

A is on the line $60\ X_1 + 60\ X_2 = 720$

$60\ X_1 = 720 - 60\ (6) = 360$ $\therefore X_1 = 6$

$\therefore Z$ = Rs. 2, 560 (At point 'A')

Solution

Optimal Profit Z. Max = Rs. 2, 560

X_1 = Tonnes of P_1 = 4 tonnes

X_2 = Tonnes of P_2 = 8 tonnes.

MINIMISATION MIXED CONSTRAINTS

Example 1:

A firm produces two products P and Q. Daily production upper limit is 600 units for total production. But at least 300 total units must be produced every day. Machine hours consumption per unit is 6 for P and 2 for Q. At least 1200 machine hours must be used daily. Manufacturing costs per unit are Rs. 50 for P and Rs. 20 for Q. Find optimal solution for the LPP graphically.

LPP formulation

X_1 = No. of Units of P / Day

X_2 = No. of Units of Q / Day

Min. $Z = 50 X_1 + 20 X_2$

Subject to constraints

$X_1 + X_2 \leq 600$

$X_1 + X_2 \geq 300$

$6 X_1 + 2 X_2 \geq 1200$

$X_1, X_2 \geq 0$

Coordinates for Constraint lines

1. $X_1 + X_2 = 600$

If $X_1 = 0$, $X_2 = 600$		$\therefore$ (0, 600)
If $X_2 = 0$, 60 $X_1 = 600$		$\therefore$ (600, 0)

2. $X_1 + X_2 = 300$

If $X_1 = 0$, $X_2 = 300$		$\therefore$ (0, 300)
If $X_2 = 0$, 60 $X_1 = 300$		$\therefore$ (300, 0)

3. $6 X_1 + 2 X_2 \geq 1200$

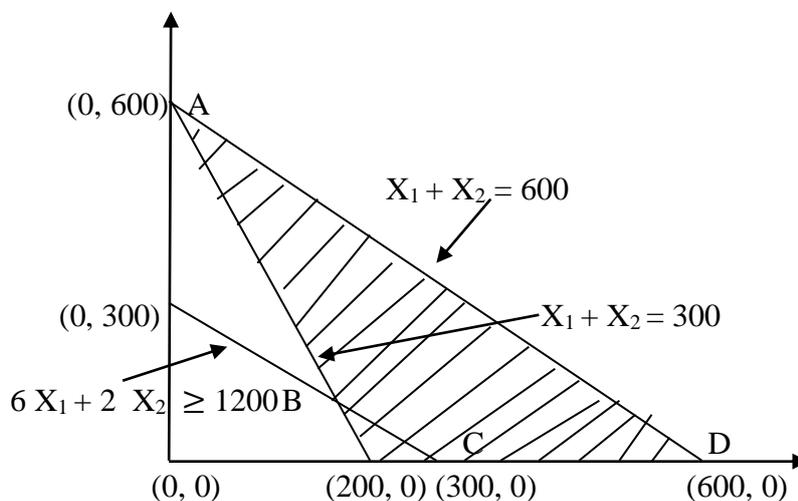|		
If $X_1 = 0$, $2X_2 = 1200$	$\therefore X_2 = 600$	(0, 600)
If $X_2 = 0$, 6 $X_1 = 1200$	$\therefore X_1 = 200$	(200, 0)

Graph: X_1: X Axis

X_2: Y Axis

Scale:

Maximum value for $X_1 = 600$; Maximum value for $X_2 = 600$; $\therefore$ Scale: 1 cm = 50 units.

Feasible region is ABCD.

Two constraints are 'greater than or equal to' type. Hence, feasible region will be above or to the right of these constraint lines. Two constraints are 'less than or equal to' type. Hence, feasible region will be below or to the left of these constraint lines. Hence, feasible region is ABCD.

Optimal Solution

Corner Point Method

Vertex	Coordinates	$Z = 160 X_1 + 240 X_2$
A	$X_1 = 0$, $X_2 = 600$ From Graph	$\therefore Z = $ Rs. 12, 000
B	$X_1 = 150$, $X_2 = 150$ Simultaneous Equations	$\therefore Z = $ Rs. 10, 500
C	$X_1 = 300$, $X_2 = 0$ From Graph	$\therefore Z = $ Rs. 15, 000
D	$X_1 = 600$, $X_2 = 0$ From Graph	$\therefore Z = $ Rs. 30, 000

Min. Z = Rs. 10, 500

For B - B is at intersection of two constraint lines '$6 X_1 + 2 X_2 \geq 1200$' and '$X_1 + X_2 = 300$'.

$6 X_1 + 2 X_2 \geq 1200$ $\qquad$ (1)

$X_1 + X_2 = 300$ $\qquad$ (2)

$2X_1 + 2X_2 = 600$ $\qquad$ (2) X 2

$2X_1 = 600$

$X_1 = 150$

Substituting value in Equation (2), $X_2 = 150$.

Solution

Optimal Cost = Rs. 10, 500/-

$X_1 = $ No. of Units of P = 150

$X_2 = $ No. of Units of P = 150.

EXERCISES

1. What is meant by feasible region in graphical method.

2. What is meant by 'iso-profit' and 'iso-cost line' in graphical solution.

3. Mr. A. P. Ravi wants to invest Rs. 1, 00, 000 in two companies 'A' and 'B' so as not to exceed Rs. 75, 000 in either of the company. The company 'A' assures average return of 10% in whereas the average return for company 'B' is 20%. The risk factor rating of company 'A' is 4 on 0 to 10 scale whereas the risk factor rating for 'B' is 9 on similar scale. As Mr. Ravi wants to maximise his returns, he will not accept an average rate of return below 12% risk or a risk factor above 6.

Formulate this as LPP and solve it graphically.

4. Solve the following LPP graphically and interpret the result.

Max. $Z = 8X_1 + 16 X_2$

Subject to:

$X_1 + X_2 \leq 200$

$X_2 \leq 125$

$3X_1 + 6X_2 \leq 900$

$X_1, \; X_2 \geq 0$

5. A furniture manufacturer makes two products - tables and chairs.

Processing of these products is done on two types of machines A and B. A chair requires 2 hours on machine type A and 6 hours on machine type B. A table requires 5 hours on machine type A and no time on Machine type B. There are 16 hours/day available on machine type A and 30 hours/day on machine type B. Profits gained by the manufacturer from a chair and a table are Rs. 2 and Rs. 10 respectively. What should be the daily production of each of the two products? Use graphical method of LPP to find the solution.

CHAPTER 5

LINEAR PROGRAMMING

SYLLABUS

Linear Programming - Special Properties, Interpretation Of Final Tableau, Concept Of Shadow Price, Concept Of Primal Dual Analysis

Special Cases in Linear Programming

a. Infeasible Solution (Infeasibility)

Infeasible means not possible. Infeasible solution happens when the constraints have contradictory nature. It is not possible to find a solution which can satisfy all constraints.

In graphical method, infeasibility happens when we cannot find Feasible region.
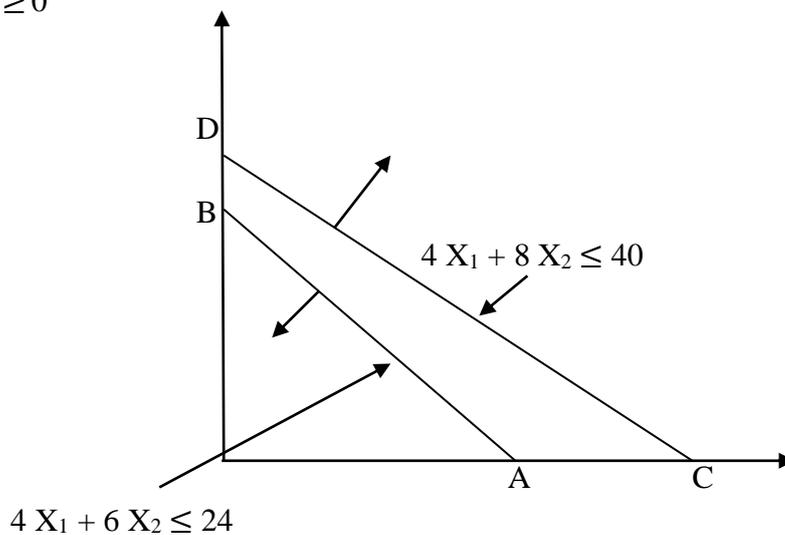
Example 1:

Max. $Z = 5 X_1 + 8 X_2$

Subject to constraints

$4 X_1 + 6 X_2 \leq 24$

$4 X_1 + 8 X_2 \leq 40$

$X_1, X_2 \geq 0$
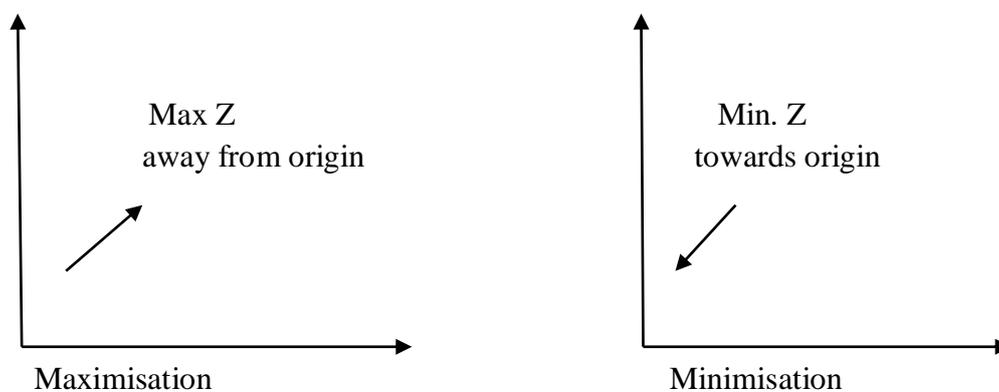

$4 X_1 + 6 X_2 \leq 24$

There is no common feasible region for line AB and CD.

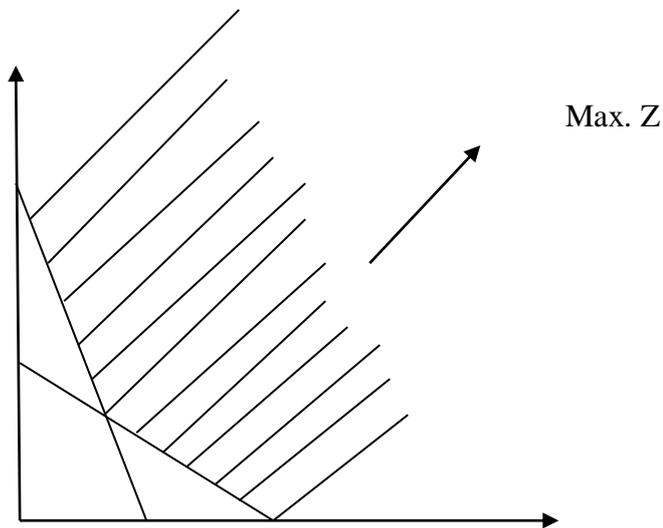Hence, solution is infeasible.

b. Unbounded Solution (Unboundedness)

Unbounded mean infinite solution. A solution which has infinity answer is called unbounded solution.

In graphical solution, the direction with respect to origin is as follows:

Now, in a maximisation problem, if we have following feasible region:

There is no upper limit (away from origin), hence the answer will be infinity. This is called unbounded solution.

c. Redundant Constraint (Redundancy)

A constraint is called redundant when it does not affect the solution. The feasible region does not depend on that constraint.

Even if we remove the constraint from the solution, the optimal answer is not affected.

Example

Max. $Z = 5 X_1 + 8 X_2$
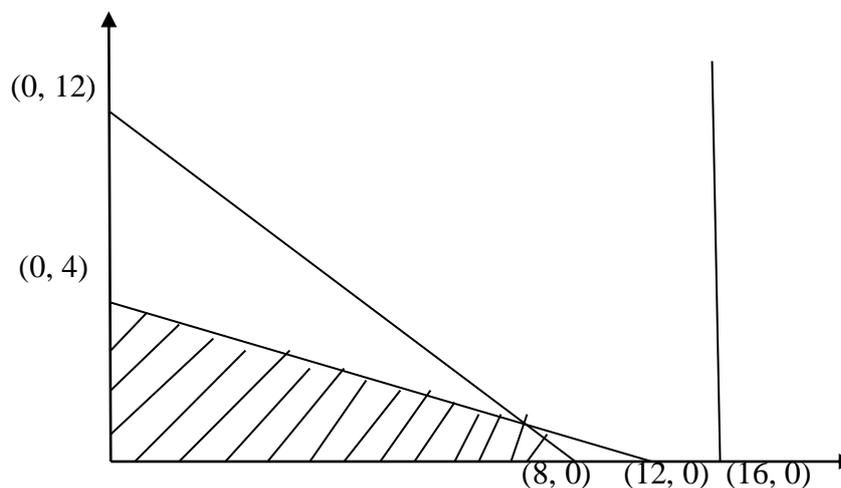
Subject to Constraints

$3 X_1 + 2 X_2 \leq 24$

$X_1 + 3 X_2 \leq 12$

$X_1 \leq 16$

$X_1, X_2 \geq 0$

The feasible region for the above problem is OABC. The 3rd constraint does not affect the feasible region.

Hence, the constraint $X_1 \leq 16$ is redundant constraint.

d. Alternate Optimal Solution: (Multiple Optimal Solution)

Alternate or multiple optimal solution means a problem has more than one solution which gives the optimal answer.

There are two or more sets of solution values which give maximum profit or minimum cost. In graphical method, we come to know that there is optimal solution which is alternative when:

The iso-profit or iso-cost line is parallel to one of the boundaries of feasible region (they have the same slope value).

SPECIAL CASES IN SIMPLEX

1. Unbounded Solution

Max $Z = 60 X_1 + 20X_2\Delta$

Subject to

$2 X_1 + 4X_2 \geq 120$

$8 X_1 + 6X_2 \geq 240$

$X_1, X_2 \geq 0$

Solution

Max $Z = 60 X_1 + 20X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to

$2 X_1 + 4X_2 - S_1 + A_1 = 120$

$8 X_1 + 6X_2 - S_2 + A_1 = 240$

$X_1, X_2, S_1, S_2, A_1, A_2, \geq 0$

When we solve this LPP by simplex method, we will get the following values in 4th Simplex Table.

C_j			60	20	0	0	- M	- M	R.R
C	X	B	X_1	X_2	S_1	S_2	A_1	A_2	
0	S_2	240	0	10	- 4	1			- 60
60	X_1	60	1	2	-1/2	0			- 120
$Z_j \longrightarrow$			60	120	- 30	0			
$\Delta = C_j - Z_j \longrightarrow$			0	- 100	30	0			
					$\uparrow$				

Max Positive $C_j - Z_j = 30$

Key Column $= S_1$

But there is no positive Replacement Ration R means there is an Entering variable, but there is no outgoing variable.

Hence, the solution is unbounded or infinity.

The value of Z (Profit) keeps on increasing infinitely.

2. Infeasible Region

Max $Z = 3 X_1 + 2 X_2$

Subject to:

$X_1 + X_2 \leq 4$

$2 X_1 + X_2 \geq 10$

$X_1, X_2 \geq 0$

Solution

Standard Form

Max $Z = 3 X_1 + 2 X_2 + 0S_1 + 0S_2 - MA_1$

Subject to

$X_1 + X_2 + S_1 = 4$

$2X_1 + X_2 - S_2 + A_1 = 10$

$X_1, X_2, S_1, S_2, A_1 \geq 0$

When we solve this LPP by simplex method, we will get the following values in 2nd Simplex Table.

C_j			60	20	0	0	- M	R.R
C	X	B	X_1	X_2	S_1	S_2	A_1	
3	X_1	4	1	1	1	0	0	
- M	A_1	2	- 2	- 1	-1/2	- 1	1	
Z_j ⟶			3	3 + M	3 + 2M	M	- M	
$\Delta = C_j - Z_j$ ⟶			0	- 1 - M	-3 -2M	- M	0	

No positive Δ value.

All $C_j - Z_j$ values are either zero or negative. Hence, test of optimality is satisfied. So, the solution appears to be optimal. But an artificial variable (A_1) is present in the basis, which has objective function coefficient of - M (infinity).

Hence, the solution is infeasible (Not feasible).

Infeasibility occurs when there is no solution which satisfies all the constraints of the LPP.

Concept of Shadow Price

Shadow price of resource means value of one extra unit of resource. It is the maximum price the company should pay for procuring extra resources from market. It also indicates profitability or profit contribution of each resource (per unit).

Shadow price = 'Z_i' value of slack variables.

S_1 - slack variable of resource 1 and S_2 - Slack Variable of resource 2. A slack variable represents unutilised capacity of a resource. Slack Variable is represented by 'S'.

Concept of Duality

Every linear programming problem has a mirror image associated with it. If the original problem is maximisation, the mirror image is minimisation and vice versa.

The original problem is called 'primal' and the mirror image is called 'dual'.

The format of simplex method is such that when we obtain optimal solution of any one out of primal or dual, we automatically get optimal solution of the other.

For example, if we solve dual by simplex method, we also get optimal solution of primal.

Characteristics of Dual Problem

1. Dual of the dual is primal.

2. If either the primal or dual has a solution, then the other also has a solution. The optimal value of both the solutions is equal.

3. If any of the primal or dual is infeasible then the other has an unbounded solution.

Advantages of Duality

1. If primal problem contains a large number of rows and a smaller number of columns we can reduce the computational procedure by converting into dual.

2. Solution of the dual helps in checking computational accuracy of the primal.

3. Economic interpretation of the dual helps the management in decision making.

For example,

Minimisation LPP can be solved by two methods:

1. Simplex of Dual Method and

2. Artificial Variable Method

Method:1 Simplex of Dual Method

The original problem is called 'Primal'. We convert the problem in its 'Dual'.

Primal	Dual
1. Minimisation Problem Min. Z	Maximisation Problem Max. Z^*
2. Constraints are of ' $\geq$ ' type	Constraints are of '$\leq$' type.
3. Decision variables are X_1, X_2 etc.	Decision variables are Y_1, Y_2 etc.

4. Objective function coefficients of primal (4, 3) become RHS of constraints in Dual.

5. RHS of Constraints of Primal (4000, 50, 1400) become objective function coefficients of Dual.

6. In the LHS (left side) of constraints, all vertical values are written horizontally in Dual.

7. No. of Decision Variables in Primal = No. of constraints in Dual

8. No. of Constraints in Primal = No. of Decision Variables in Dual

For example,

Primal	Dual
Min Z = 4 X_1 + 3 X_2 Subject to: 200 X_1 + 100 $X_2 \geq$ 4000 1 X_1 + 2 $X_2 \geq$ 50 40 X_1 + 40 $X_2 \geq$ 1400	Max Z = 4000 Y_1 + 50 Y_2 + 1400 Y_3 Subject to: 200 y_1 + 1y_2 + 40$y_3 \leq$ 4 100 y_1 + 2y_2 + 40$y_3 \leq$ 3

The numerical is calculated as shown in Simplex Method.

EXERCISES

1. Why an optimal solution to an unbounded maximisation LPP cannot be found in Simplex Method?

2. What is meant by shadow price of a resource?

UNIT 6
SENSITIVITY ANALYSIS

The solution to a LPP is determined by simplex method is a static solution, It means that solution corresponds to:

1. Value of profit coefficients in the objective function, and
2. Availability of resources (i.e R.H.S of constraints)

But in reality, profit coefficients of variables may increase or decrease. Similarly, availability of resources may also increase or decrease. In that case, the optimal profit and optimal quantity (b values) of variables calculated as per Simplex solution will change.

The objective of sensitivity analysis is to determine the new values of solution. If possible, from the given simplex solution. This will be possible only if the changes (increase/decrease) in the objective function or constraint capacities is in certain limits. These will be two limits, lower limit (max. possible decrease) and upper limit (max. possible increase). These two limits provide the range within which the present simplex table remains optimal.

Hence, if the change in profit or change in capacity constraints is in the range, we can find new values of the solution. If it is not in the range, we cannot find the new values of the solution. Because the present simplex table will not remain optimal any more.

EXAMPLE 1: An electronics firm manufactures three products - transistors, resistors and capacitors which give profit of Rs. 100, 60 and 40 per unit respectively.
The firm uses three resources - Engineering, Direct Labour and Admin. capacities are 100, 600 and 300 hrs. respectively.
Following simplex solution is obtained.

C_j			100	60	40	0	0	0
C	X	B	X_1	X_2	X_3	S_1	S_2	S_3
60	X_2	200/3	0	1	5/6	1	- 1/6	0
100	X_1	100/3	1	0	1/6	0	1/6	0
0	S_3	100	0	0	4		0	1
Z_j ⟶			100	60	200/3	0	20/3	0
$\Delta = C_j - Z_j$ ⟶			0	0	-80/3	0	- 20/3	0

Note: The solution is optimal as there is no positive $C_j - Z_j$.

Optimal Product Mix is:
$X_1 = 100/3 =$ No. of units of transistors.
$X_2 = 200/3 =$ No. of units of resistors
X_3 (capacitors) is not produced.

Optimal Profit is:
Max. Z = [60 X 200/3] + [100 X 100/3] = [4000 + 10, 000/3]
Max. Z = Rs. 22, 000/3

A. Range for profit coefficients of products:

Formula $= \dfrac{\Delta j}{Xn}$

B. Range for profit coefficients of X_1:

Formula $= \dfrac{\Delta j}{Xi}$

We take ration of 'Δ' (C_j - Z_j) row and ' X_1' row:

The ratios will be:

$\dfrac{-80/3}{1/6} = \dfrac{-80}{3} \text{ x } \dfrac{6}{1} = -160$

$\dfrac{-\frac{100}{3}}{-2/3} = \dfrac{-100}{3} \text{ x } -\dfrac{3}{2} = 50$

$\dfrac{-20/3}{1/6} = \dfrac{-20}{3} \text{ x } \dfrac{6}{1} = -40$

'-' sign indicates decrease in profit & '+' sign indicates increase in profit. It means possible decrease is 160 or 40. Hence, we can decrease profit by only 40. Possible increase is 50.

$\therefore$ Range for profit coefficient of X_1 is:

Original Profit $\pm$ (increase or decrease)

$100 + 50 = 150$

$100 - 40 = 60$

$\therefore$ Rs. 60 to Rs. 150

It means profit of X_1 (transistors) can fluctuate within the range of Rs. 60 to Rs. 150. The simplex solution will remain optimal in this range.

2. Range for profit coefficient of X_2

Formula $= \dfrac{\Delta 1}{X2}$

We take ration of 'Δ' (C_j - Z_j) row and 'X_2' row:

$\dfrac{-80/3}{5/6} = \dfrac{-80}{3} \text{ x } \dfrac{6}{5} = -32$

$\dfrac{-\frac{100}{3}}{5/3} = \dfrac{-100}{3} \text{ x } -\dfrac{3}{5} = -20$

$\dfrac{-20/3}{1/6} = \dfrac{-20}{3} \text{ x } \dfrac{6}{1} = 40$

Hence, possible decrease is Rs. 20 and possible increase is Rs. 40. Range for profit coefficient of X_2 is

$60 - 20 = 40$

$60 + 40 = 100$

Rs. 40 to Rs. 100.

It means the present simplex solution will remain optimal even if profit of X_2 (resistors)fluctuates in the range of Rs. 40 to Rs. 100.

3. Range for profit coefficient of X_3

X_3 capacitors is not produced.

Δ of X_3 = -80/3

It means if we produce X_3, we will incur a loss of Rs. 80/3 per unit of X_3. So, X_3 will be produced only if present profit of X_3 is increased by Rs. 80/3.

$\therefore$ X_3 will be produced if its profit becomes [40 + 80/3] = Rs. 200/3 or more than that.

Hence, the present simplex solution remains optimal till the profit value of 200/3 for X_3.

∴ Range for profit coefficient of X_3 is

Rs. zero to 200/3.

B. Range for capacity of resources

OR Range for validity of shadow prices of Resources

Formula = - $[\frac{b}{Sn}]$

1. Range for capacity or availability of Engineering hours

Engineering hours is represented in simplex table by slack variable s_1:

Formula = - $[\frac{b}{S1}]$

We will take ratio of 'b' column and 'S$_1$' column.

The ratios will be

- $[\frac{200/3}{5/6}]$ = - $[\frac{200}{3}$ x $\frac{3}{5}$] = - 40

- $[\frac{100/3}{-2/3}]$ = - $[\frac{100}{3}$ x $\frac{-3}{2}$]= 50

- $[\frac{100/3}{-2/3}]$ = - [- 50] = 50

Hence, possible decrease in capacity is 40 hrs and possible increase in capacity is 50 hrs.

Range for resource capacity of Engineering is:

Original Capacity ± (increase or decrease)

100 + 950) = 150

100 - 40 = 60

∴ Range is 60 hrs to 150 hrs.

It means the present simplex solution will remain optimal even if availability of Engineering resource fluctuates between 60 hrs to 150 hrs.

Note: In other words the shadow price of Engineering resource [which is Rs. 100/3] will remain valid even if the resource availability fluctuates between 60 hrs. to 150 hrs.

2. Range for availability of Direct Labour

Formula = - $[\frac{b}{S2}]$

The ratios will be

- $[\frac{200/3}{-1/6}]$ = - $[\frac{200}{3}$ x$\frac{-6}{1}$] = 400

- $[\frac{100/3}{-1/6}]$ = - $[\frac{100}{3}$ x $\frac{6}{1}$]= - 200

- $[\frac{100}{0}]$ = Infinity.

∴ Range for resource capacity of Direct Labour is:

600 + 400 = 1000

600 - 200 = 400

∴ Range is 400 hrs to 1000 hrs.

It means the present simplex solution will remain optimal even if availability of Direct Labour resource fluctuates between 400 hrs to 1000 hrs.

Note: In other words, the shadow price of Direct Labour resource (which is Rs. 20/3) will remain valid even if the resource availability fluctuates between 400 hrs to 1000 hrs.

3. Range for availability of Admin:

Formula = - $[\frac{b}{S3}]$

The ratios will be

$-\left[\dfrac{200/3}{0}\right] = -[\text{Infinity}] = \text{Infinity}$

$-\left[\dfrac{100/3}{0}\right] = -[\text{Infinity}] = \text{Infinity}$

$-\left[\dfrac{100}{1}\right] = -100.$

It means the capacity can increase upto infinity.

Possible decrease = 100 hrs.

Range for resource capacity of Admin is

300 + Infinity = Infinity

300 - 100 = 200

∴ Range is 200 hrs to Infinity.

It means the present simplex solution will remain optimal even if availability of Admin resource fluctuates between 200 hrs to Infinity.

Note: In other words, the shadow price of Admin resource (which is Rs. zero) will remain valid even if the resource availability fluctuates between 200 hrs to infinity.

c. Effect on the solution due to increase or decrease in the availability of resources

1. What will be the effect on solution if capacity of Engineering is increased by 30%?

Answer:

Original capacity = 100 hrs.

Increase = 30%

New capacity = 130 hrs.

Note:

To find the effect on solution we need to find the range of resource capacity. We can find the effect on solution only if the new capacity is in the range.

If the new capacity goes out of the range, then we cannot find effect on solution.

Because in that case, the present simplex solution does not remain optimal any more.

From earlier calculation, we know that, Range for resource capacity of Engineering is 60 hrs to 150 hrs.

∴ New capacity is in the range.

Change in capacity = 130 - 100 = + 30 hrs.

Hence, we multiply column S_1 by + 30.

From that we will get change in 'b' column.

	S_1 X (+30) = Change in 'b' column
X_2 ⟶	5/3 X + 30 = + 50
X_1 ⟶	-2/3 X + 30 = - 20

Now we can find new basis values.

New Basis:

c	X	New b
60	X_2	200/3 + 50 = 350/3
100	X_1	100/3 - 20 = 40/3

New Z = (60 x 350/3) + (100 X 40/3)

New Z = 25, 000/3 Rs.

Increase in optimal profit = 25, 000 - 22, 000/3

= Rs. 1, 000

New Optimal Product Mix:

$X_1 = 40/3$ units

$X_2 = 350/3$ units.

2. Can you find out effect on optimal solution if excess capacity of abundant resource is transferred to Direct Labour?

Answer:

In the optional solution, S_3 is present in the basis.

$S_3 = 100$

S_3 represents slack value of Admin.

Hence, Admin is abundant resource and its excess (unused) capacity is 100 hrs.

Capacity of Direct Labour = 600 hrs.

If 100 hrs are transferred,

New capacity of Direct labour = 700 hrs.

We, know that range of Direct Labour capacity is 400 hrs to 1000 hrs.

New capacity is in the range.

	S_1 X (+30) = Change in 'b' column
$X_2 \longrightarrow$	-1/6 X + 100 = - 100/6 - -50/3
$X_1 \longrightarrow$	1/6 X + 100 = 100/6 = 50/3

New Basis:

c	X	New b
60	X_2	200/3 - 50/3 = 150/3 = 50
100	X_1	100/3 + 50/3 = 150/3 = 50

New Z = (60 x 50) + (100 + 50) = Rs. 8, 000

Increase in optimal profit = 8, 000 - 22, 000/3

= Rs. 2, 000/3

New Optimal Product Mix:

$X_1 = 50$ units

$X_2 = 50$ units.

3. What will be the effect on optimal solution if capacity of Admin is reduced to 175 hrs?

Answer: Range for capacity of Admin = 200 hrs to Infinity.

Since, 175 hrs is out of the range, if Admin capacity is reduced to 175 hr. solution will not remain optimal.

EXERCISES

1. An engineering company BMS Ltd. produces three products A, B and C using three machines M_1, M_2 and M_3. The resource constraints on M_1, M_2 and M_3 are 96, 40 and 60 hours respectively. The profits earned by the products A, B and C are Rs. 2, Rs.5 and Rs. 8 per unit respectively. A simplex optimal solution to maximize the profit is given below where X_1, X_2 and X_3 are quantities of products A, B and C produced by the company s_1, s_2 and s_3 represent the slack in the resources M_1, M_2 and M_3. Study the solution given below and answer the following questions:

C	X Variables in the basis	X_1	X_2	X_3	S_1	S_2	S_3	B Solution Values
5	X_2	1/3	1	0	1/6	- 1/3	0	8/3
8	X_3	5/6	0	1	- 1/12	2/3	0	56/3
0	S_3	7/3	0	0	- 1/13	- 1/3	1	44/3
	$\Delta = C - Z$	- 19/3	0	0	- 1/6	- 11/3	0	

1. Indicate the shadow price of each resource. Which of the resources are abundant and which are scarce?

2. What profit margin for product A do you expect the marketing department to secure if it is to be produced, and justify your advice?

3. Within what range, the profit of product B can change for the above solution to remain optimal?

4. How would an increase of 10 hours in the resource M_2 affect the optimality?

5. If the company BMS Ltd. wishes to raise production which of the three resources should be given priority for enhancement?

2. A business problem is formulated and expressed below as an LPP. (Profit is in Rs. and Resources are in units).

Objective function

Maximise $Z = 80 X_1 + 100 X_2$

Subject to resource constraints,

$X_1 + 2X_2 \leq 720$	(Resource 1)
$5X_1 + 4X_2 \leq 1800$	(Resource 2)
$3X_1 + X_2 \leq 900$	(Resource 3)

X_1, $X_2 \geq 0$

Simplex algorithm of LPP, applied to the above problem yielded following solution:

Basis							B_1
C_b	X_b	X_1	X_2	S_1	S_2	S_3	
100	X_2	0	1	5/6	- 1/6	0	300
80	X_2	1	0	- 2/3	1/3	0	120
0	S_3	0	0	7/6	- 5/6	1	240
	C_j	80	100	0	0	0	
	$\Delta = C_j - Z_j$	0	0	- 30	- 10	0	

1. Answer the following questions with justification:

a. Is the solution optimal and unique?

b. Is the above solution infeasible?

c. What is the maximum profit as per optimal solution?

d. Which resources are abundant and which are scarce as per optimal solution?

2. Find out the range of coefficient of X_1 in the objective function for which the above solution remains optimal.

3. Can you obtain the solution values of basic variables form the optimal solution when resource constraint (a) Changes to 750 units? If yes, find the new values of the basic variables.

.