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Mechanics of Materials

Supporting material for combined study

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Contents

Chapter 1

Informations about subject

Abilities and knowledge

Ability to understand and solve basic tasks of non-linear mechanics (plasticity, stability). Ability to identify type of non-linear problem and to select proper solution approach.

Abstract

In this subject there are given information about non-linear behavior of building structures and building materials. There are discussed problems of structural non-linearity, of constitutive modelling which includes time-dependent problems and basics of fracture mechanics. There are also discussed problems of geometrical non-linearity. Methods of solution of these problems are also introduced. The practical part of the subject is based on solution of typical problems with use on analytical and numerical methods.

Literature

- JIRASEK, Milan. a Z. P. BAZANT. Inelastic analysis of structures. New York, NY: Wiley, 2002. ISBN 978-0-471-98716-1.

- BORESI A. P., SCHMIDT, R. J.: Advanced Mechanics of Materials,John Wiley and Sons, Chichester, USA 2003

- BELYTSCHKO, Ted, W. K. LIU a B. MORAN. Nonlinear finite elements for continua and structures. New York: Wiley, c2000. ISBN 0471987743.

- BAZANT, Z. P., F.-J. ULM, Hamlin. JENNINGS a Roland. PELLENQ. Mechanics and physics of creep, shrinkage, and durability of concrete: a tribute to Zdenk P. Baant : proceedings of the Ninth International Conference on Creep, Shrinkage, and Durability Mechanics (CONCREEP-9), September 22-25, 2013 Cambridge, Massachusetts. Reston, Virginia: American Society of Civil Engineers, 2013.

Contents

Chapter 2

Introduction

2.1 Types of non-linearities

Non-linearities are usually categorised by their type. Obviously, in many cases the solved problem incorporates several non-linearities.

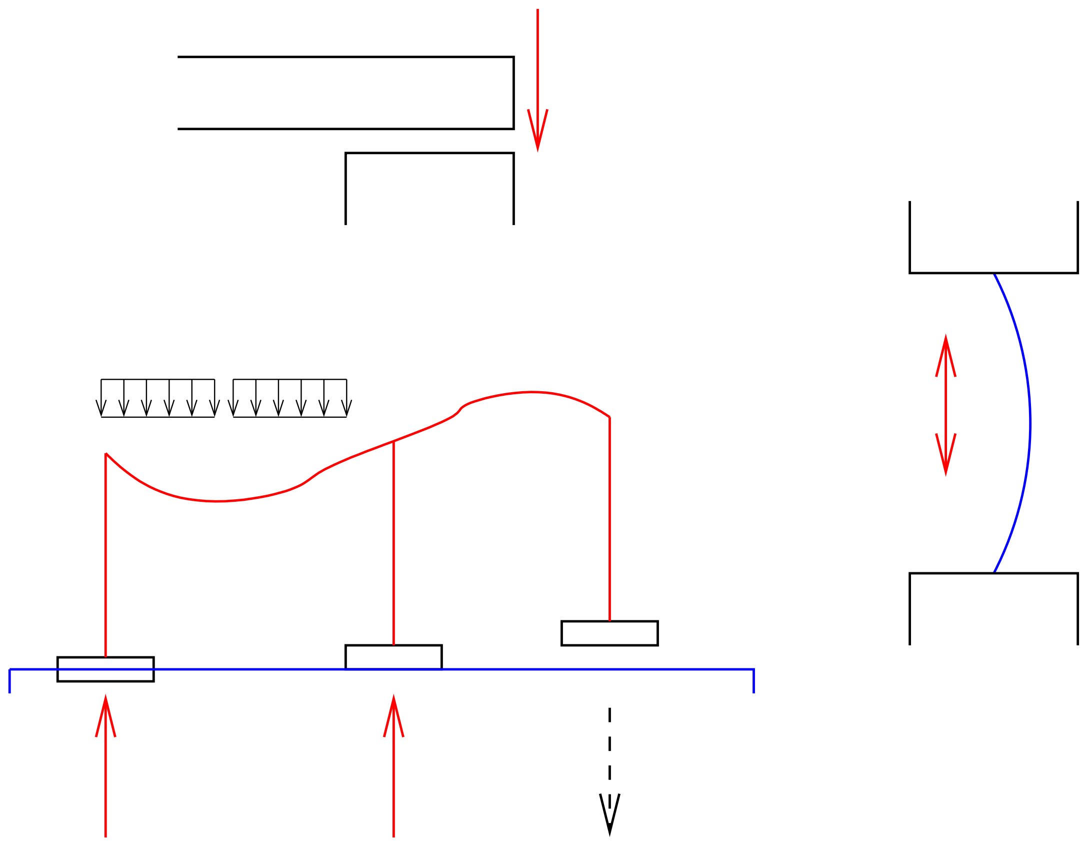
The basic categorisation is:

- structural (model) non-linearities – supports and structural elements that are working only in certaion conditions (compression-only etc.),
- physical (constitutive) non-linearities – material behaviour is not linear (differs from Hooke law) (non-linear elasticity, plasticity, fracture mechanics,. . .),
- geometric non-linearity – large deformations (displacements, rotations,. . .).

Chapter 3

Model non-linearity

3.1 Typical cases



On the picture above there are shown the common cases of model non-linearity – supports or member that are actin only in some load cases (tension-only mebers or compression-only supports).

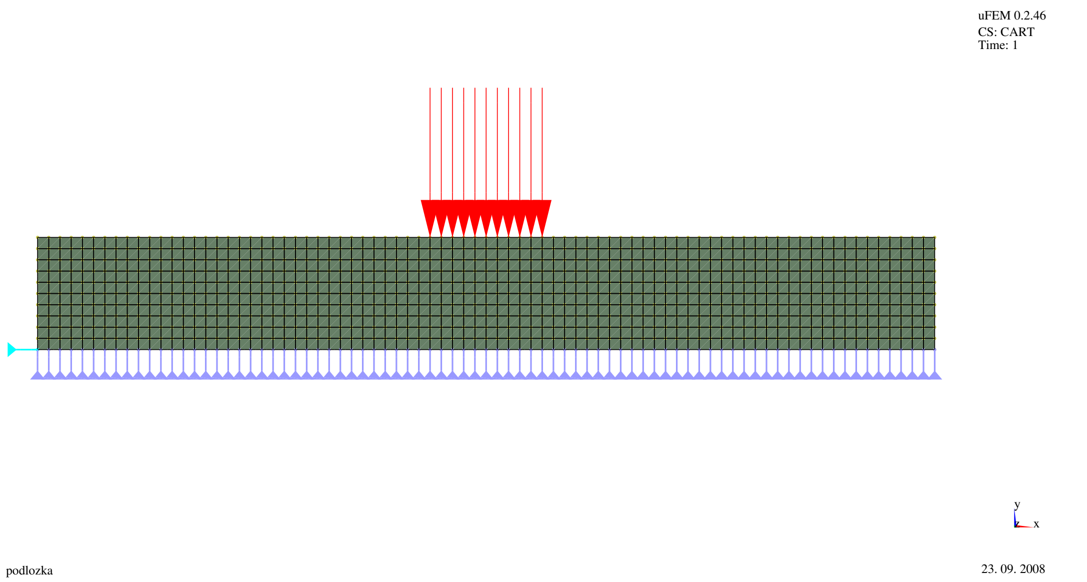
These types of model non-linearity usually require iterative solution.

3.2 Supports working only for certain load

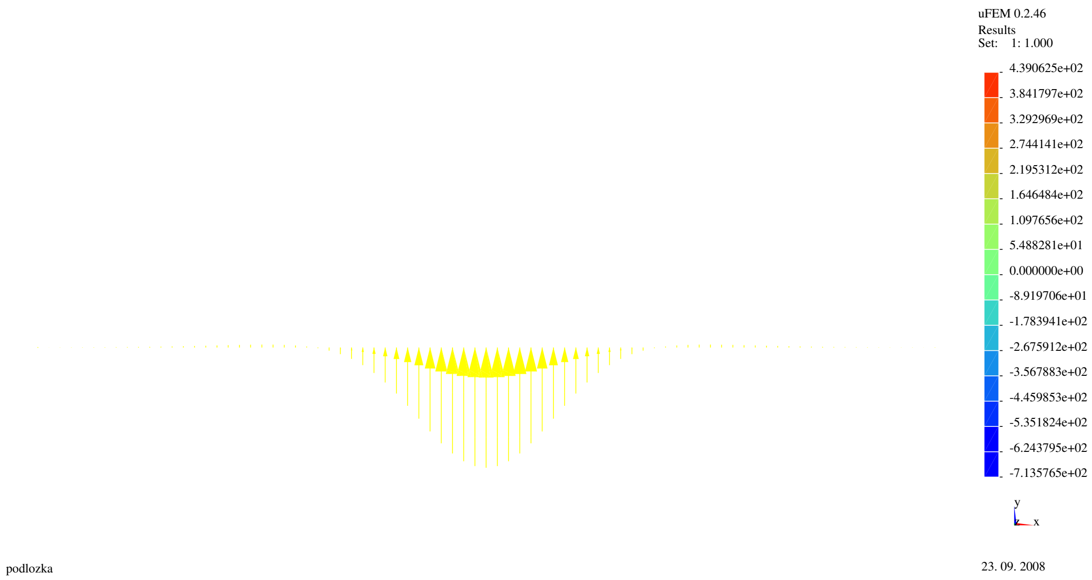
On the pictures below it is illustrated a case of structural non-linearity: a beam supported by compression-only supports.

3.3 Example

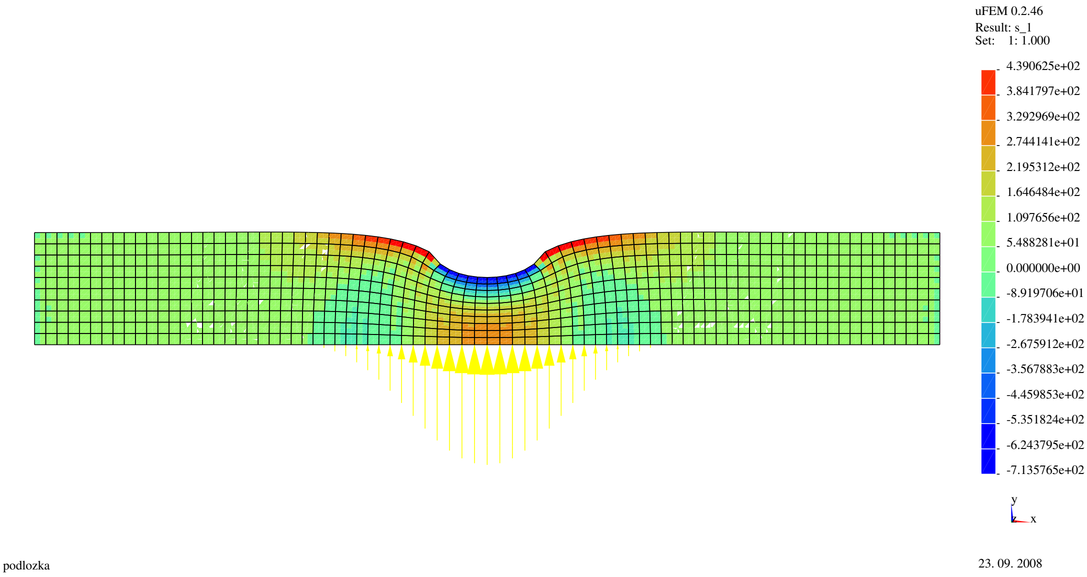
The model:



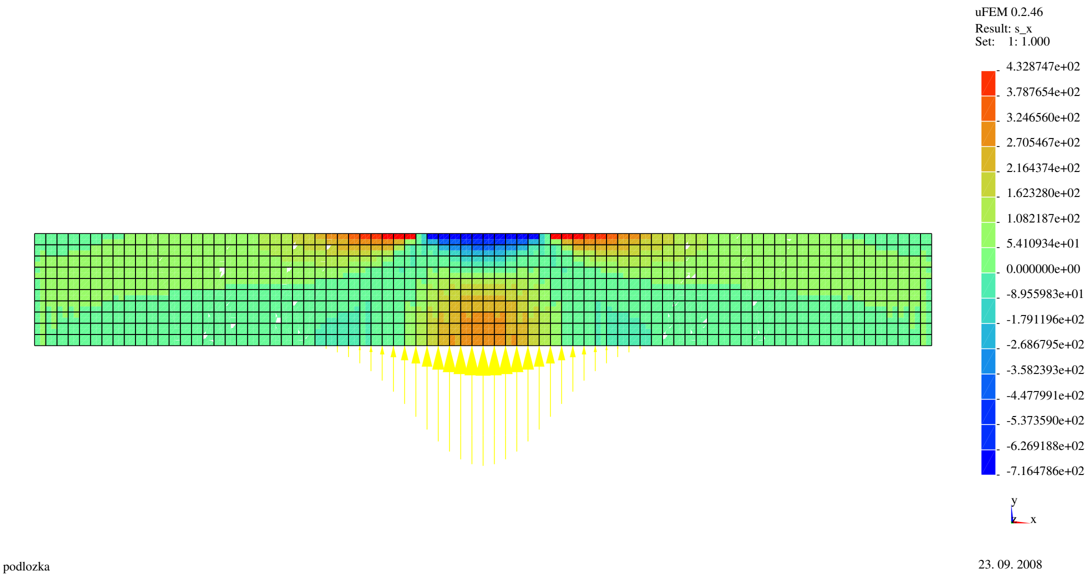
Reactions (usual supports):



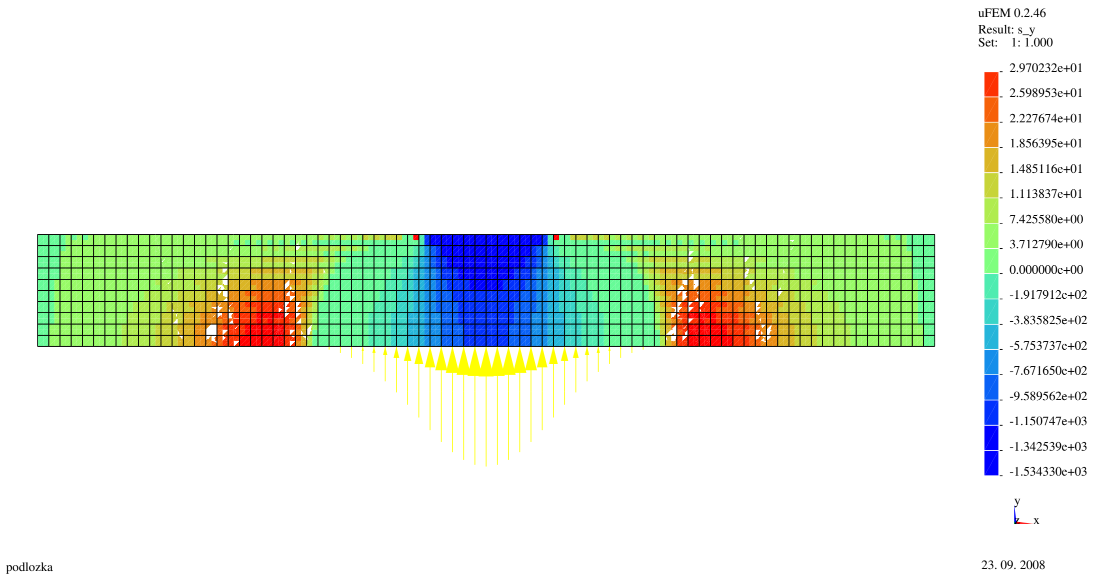
Deformed shape (usual supports):



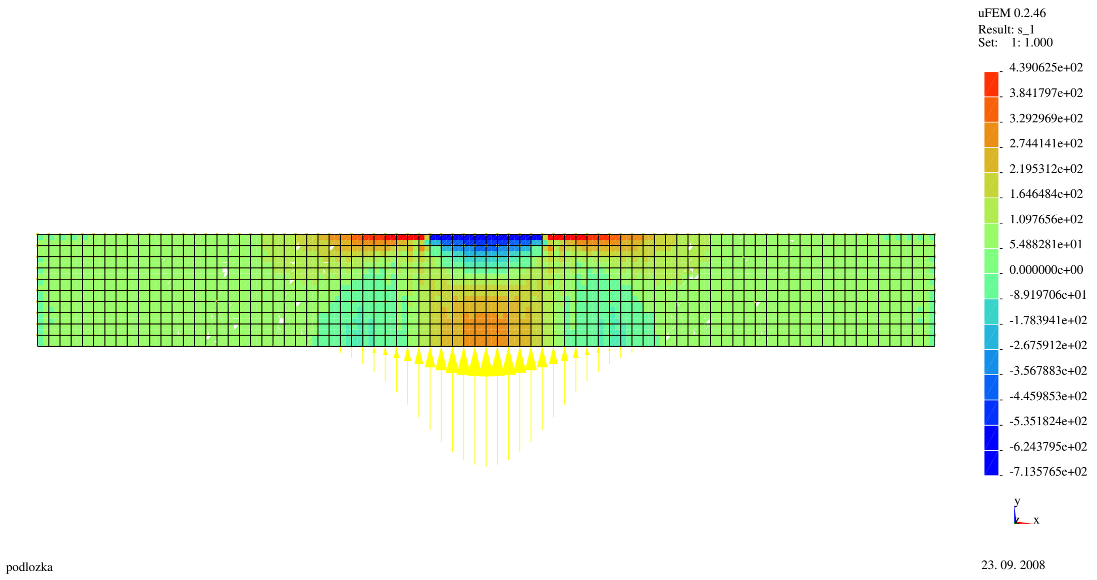
Normal stress σ_x (usual supports):



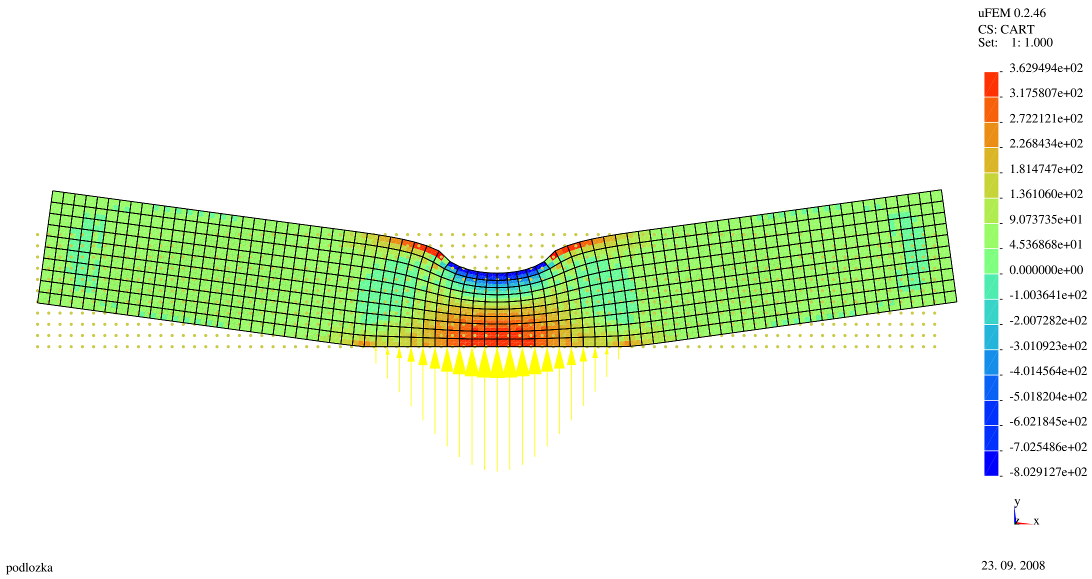
Normal stress σ_y (usual supports):



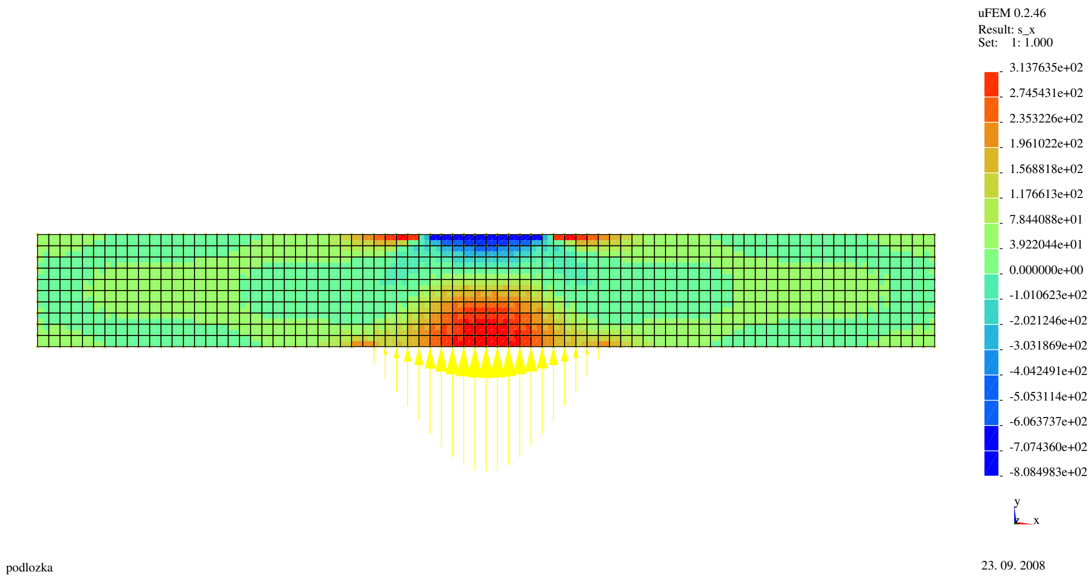
Normal stress σ_1 (usual supports):



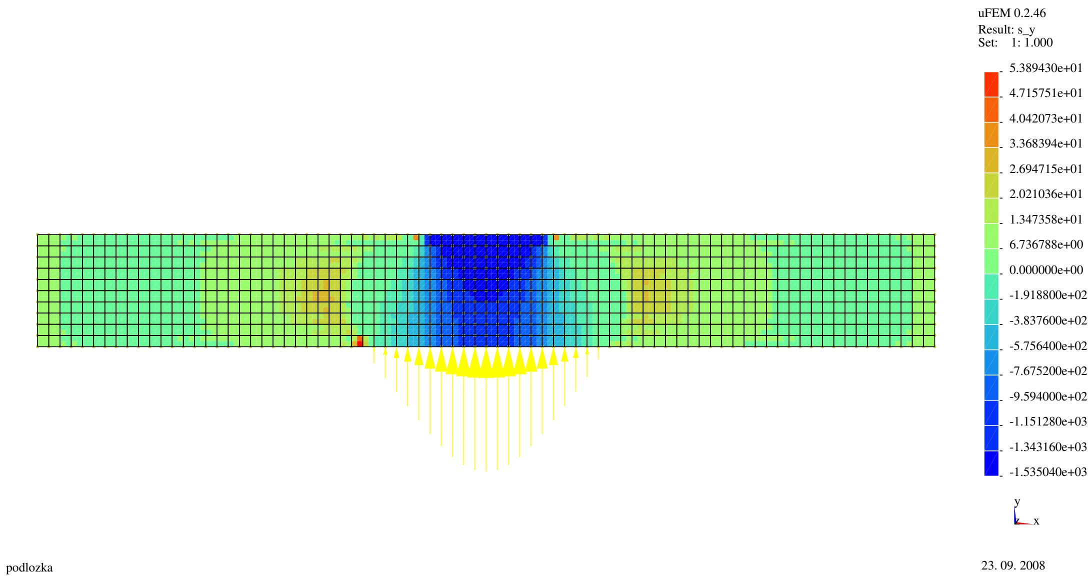
Deformed shape (compression-only supports):



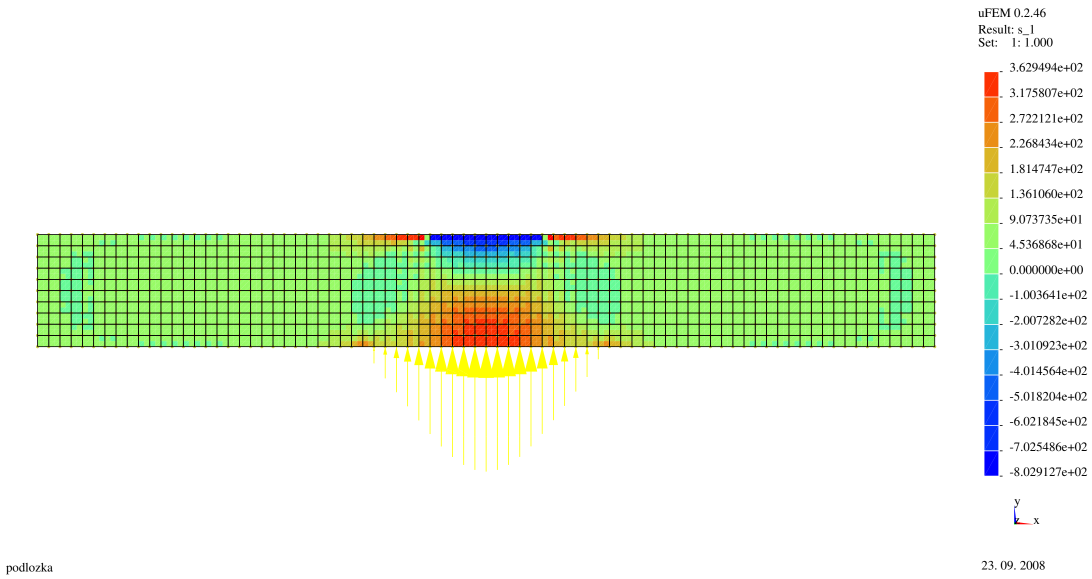
Normal stress σ_x (compression-only supports):



Normal stress σ_y (compression-only supports):



Normal stress σ_1 (compression-only supports):



3.4 Problem for individual work

- In software of your choice prepare 3D model of hall with tension-only stiffeners (recommended: SCIA Engineer). Compare results of linear elastic solution and non-linear one. How large is the difference in maximal internal forces?

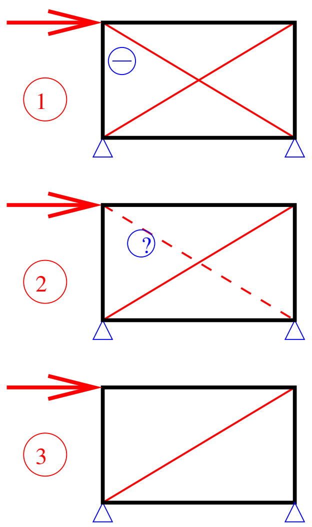
Chapter 4

Methods of solution on non-linear problems

- Iterational solution.
- Steps-based solution (Euler method).
- Combination of steps and iterations (Newton-Raphson method, Arc-lenght method).

4.1 Iterational solution

It can be used for problems of structural (construction) non-linearity (see previous chapter).



4.1.1 Algorithm of iterational solution

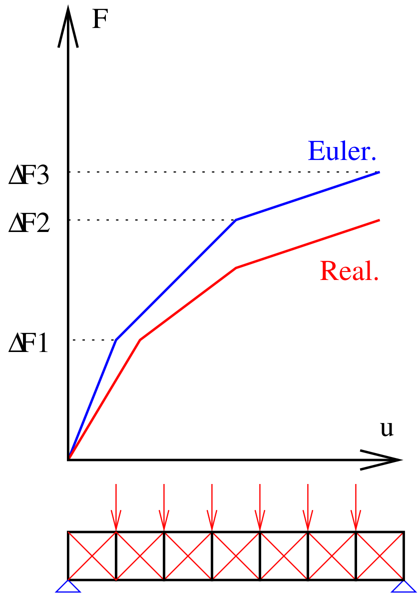
1. Linear solution.
2. Changes in structure related to computed stresses and strains (de-activation of compression-oly supports or changes of member stifnesses, for example).

3. Linear solution of chnaged structure.

4. If changes of results are minimal then solution is done, otherwise solution should continue in the step 2.

Note: In some case the iterational solution can be slow. For example. it can be case of compression-only supports if number of support is large.

4.2 Euler method



Loads F are applied step-by-step (with step size ΔF, for example). There is no iteration.

4.2.1 Algorithm ot the Euler method

1. Solution for the first ΔF_1

2. Changes in structure related to computed stresses and strains.

3. Solution for the next ΔF_2.

4. Sum of results.

5. Vyhodnocen zmn v konstrukci (vylouen prut,...)

6. Changes in structure related to computed stresses and strains.

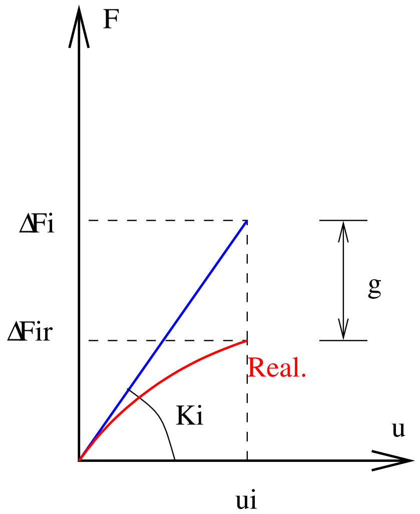
7. Solution for the next $\Delta F_2 \ldots$

8. Solution is done after the total load reaches $F = \sum_{i=1}^{n} \Delta F_i$.

Note: This method highly depends on step size and is subject to roudning errors. It is not used for practical computation. IT is shown here like step between iterative solutions and the Newton-Raphson method.

4.3 Newton-Raphson method

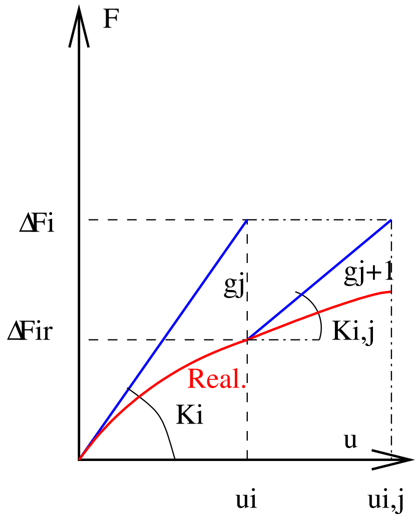
There are many methods that combine loads applied in steps with iterational procedures but the Newton-Raphson method seems to be the most common.



The Newton-Raphson method can be viewed as an extension of the Euler method:

- Load is applied in steps ΔF.
- There is an iteration in every load step.
- Iteration shoudl minimise the **unballanced forces** g.
- The g represents difference between stress state elated to initial stiffness. E_i and actual stress state related to updated stiffness. It can be obtained from equilibrium conditions in finite element nodes, for example.

4.3.1 Algorithm of one step Newton-Raphson method



1. Computation for the ΔF_1:

$$\mathbf{K_i}(\mathbf{u}) \times \Delta\mathbf{u_i} = \Delta\mathbf{F_i}. \tag{4.1}$$

2. Changes in structure related to actual stress and strain state.

3. Computation of unballanced forces$\mathbf{g_j}$:

$$\mathbf{K_{i,j}}(\mathbf{u}) \times \Delta\mathbf{u_{i,j}} = \mathbf{g_j} \tag{4.2}$$

4. Changes in structure related to actual stress and strain state....

5. Repeating until $\mathbf{g_{j+x}}$ is not small enough.

The solution is repeated for every addition of load ΔF_i.

4.3.2 Convergence criteria

The iteration is finisted when the unballanced forces are small enough. It can be detected by use of vector norms:

- Size of unballanced forces:

$$\frac{||\mathbf{g}||}{||\Delta\mathbf{F_i}||} < \varepsilon \tag{4.3}$$

- Size of displacements increase between iterations:

$$\frac{||\Delta\mathbf{u_{i,j}}||}{||\Delta\mathbf{u_i}||} < \varepsilon \tag{4.4}$$

Where ε is required precision (for example $\varepsilon = 0,00001$).

As a **vector norm** it is often used the Euclide norm:

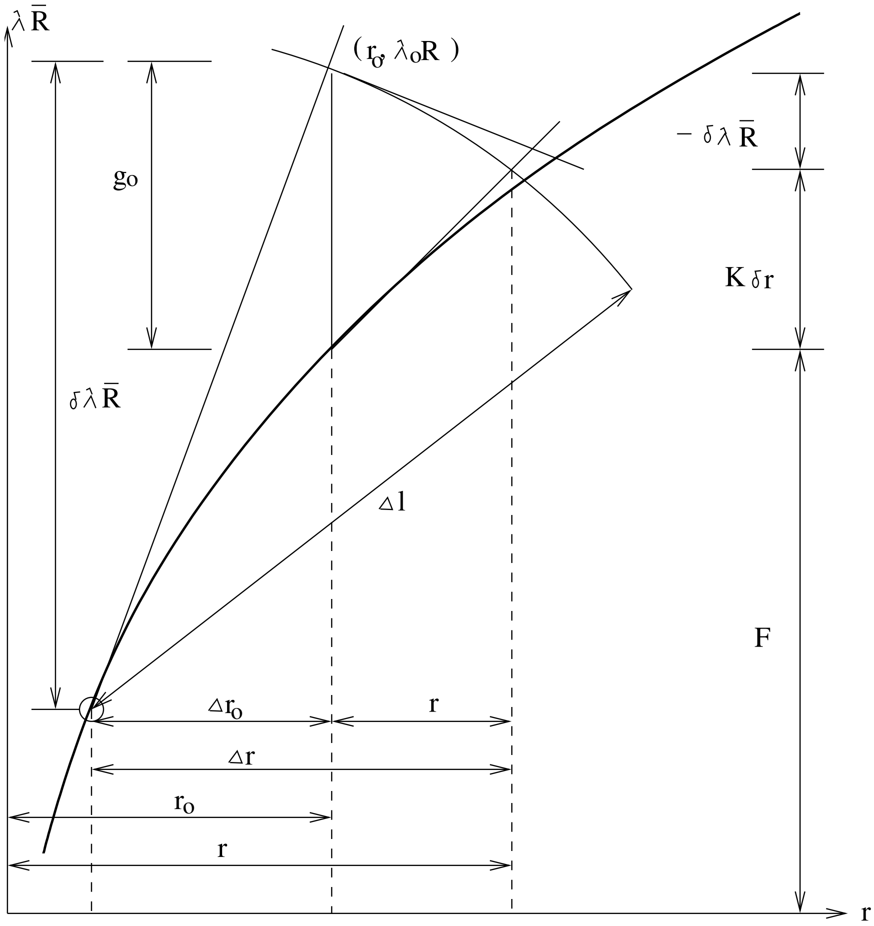
$$||\mathbf{u}|| = \sqrt{\sum_{i=1}^{n} u_i^2}. \tag{4.5}$$

4.4 Arc-lenght method

The Newton-Raphson method is not suitable for computation of post-peak behaviour of structures. Thus it is not the best one for problems like large deformation of shell structures of for analysis of progresivelly cracking concrete strucures.

The Arc-lenght method is an extension of the Newton-Raphson method which uses variable ΔF_i step size which is based on a relation of vector norms of the load vector $\mathbf{\Delta F}(\delta\mathbf{R})$ and the vector of deformations $\delta\mathbf{u}(\delta r)$ (thus the ΔF can be even negative).

The load vector addition ΔF (marked as δR on the next picture) is computed from a condition that in every step the lenght on a load–displacement path s should be constant.



4.4.1 Linearised Arc-lenght method

The Arc-lenght method can be derived ins several ways. The different derivations are known to work well for some types of problems but not for all non-linear problems.

Many of computational codes (for example ANSYS, uFEM) use so-called **linearised Arc-lenght method** which compute the load multiplied $\delta\lambda$ in this way:

$$\delta\lambda = -\frac{\frac{a_o}{2} + \Delta\mathbf{r_o}^T\delta\overline{\mathbf{r}}}{\Delta\mathbf{r_o}\delta\mathbf{r_t} + \Delta\lambda_o\psi^2\overline{\mathbf{R}}^T\overline{\mathbf{R}}}. \tag{4.6}$$

Note: the ψ parameter to adjust effects of load vector to solution speed. For example in computations with large deformations it can be usefull to set $\psi = 0$.

4.5 Problem for individual work

- Why it the Newton-Raphson method unsuitable for post-peak analysis of structures? Draw the load-displacement diagram with a peak and try to map load increments ΔF to the diagram.

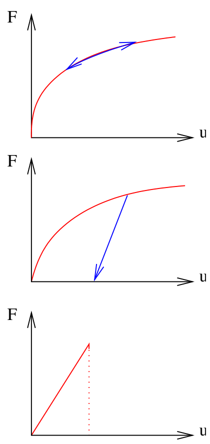
Chapter 5

Constitutive (material) non-linearity

There are these most common types on non-linear material behaviour:

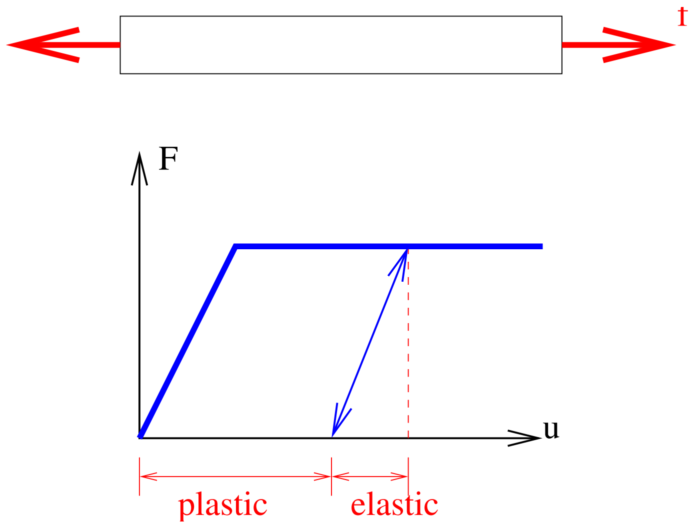
- Non-linear elasticity:
  - Hooke law is not respected,
  - there are no non-reversible deformations.
- Elastic-plastic behaviour:
  - there are non-reversible (plastic) deformations.
- Viskoelasticity, viskoplasticity,. . . :
  - there are time-dependent elastic (or plastic) deformations.
- Fragile materials:
  - these materials have fragile (brittle) behaviour,
  - they are studied in detail by *Fracture mechanics*.

The picture illustrates (from the top): non-linear elasticity, plasticity, frageile behaviour:



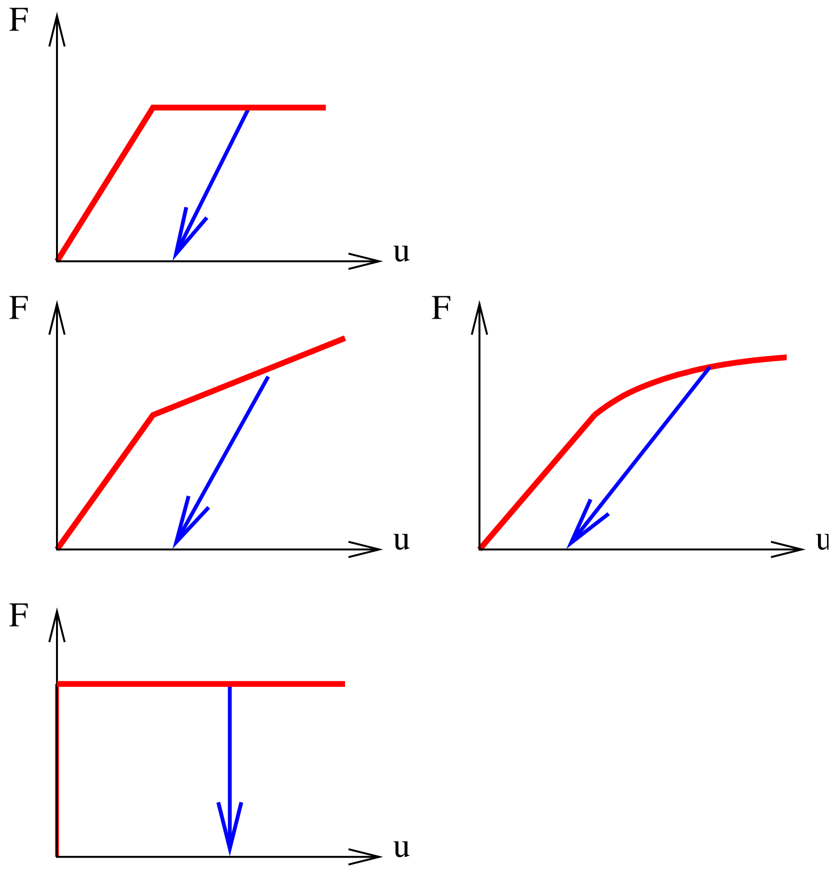
5.1 Elastic-plastic behaviour

Load-displacement diagral for axially loaded member:



One can see the main parts of load-displacement diagram: the initial elastic part, the plastic part and typical behaviour during unloading. The deformation called ”plastic” remains even after the load is fully removed.

There can be shown three principal variants of elastic-plastic material as shown in the picture below.

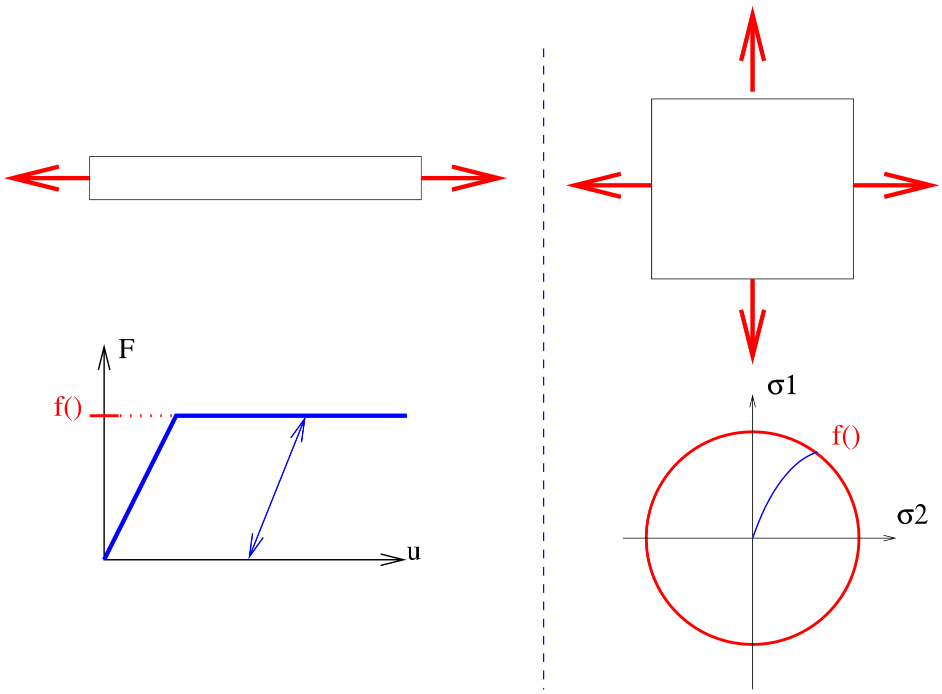


The types of elastic-plastic behaviour:

1. Ideally elastic–plastic: when material reaches some point then the behaviour became plastic (deformation continues for constant force or stress)
2. Elastic–plastic with hardening: stress should rise in plastic stage.
3. Stiff–plastic: there is no elastic part of behaviour. This model is theoretical but it was commonly used for plastic analysis of frames because of its simplicity.

5.2 Plasticity condition

In the 1D case the *plasticity condition* is a given stress (or force, named $f()$ below) value. When such point is aproached then material change it's behaviour from elastic to plastic.



In 2D case the plasticity condition is a closed, convex, curve. Here it is shown in the plane of principal stresses $\sigma_1 - -\sigma_2$ (red curve labelled $f()$).

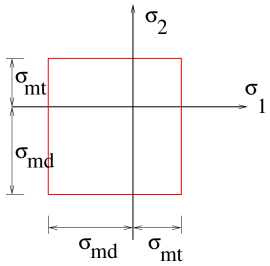
5.3 Plasticity conditions common in civil engineering

5.3.1 Maximal normal stresses theory (Rankine)

$$-\sigma_{md} \leq \sigma_1 \leq \sigma_{mt} \quad (5.1)$$

$$\sigma_1 - \sigma_{mt} = 0 \quad (5.2)$$

$$\sigma_2 - \sigma_{md} = 0 \quad (5.3)$$



This condition was originaly developed in simpler form as a failure condition. It is sometimes used as a plasticity condition for material which cannot be easily described by other plasticity conditions.

5.3.2 Maximal shear stresses theory (Tresca)

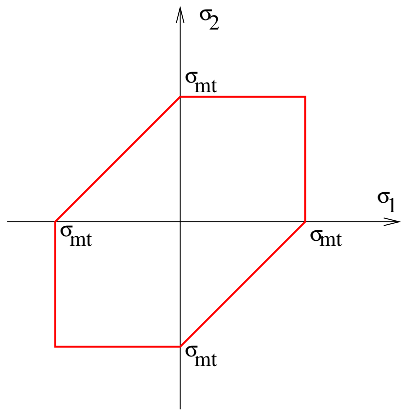
This condition uses shear stresses. It can be used for metal materials.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} - \tau_m = 0 \quad (5.4)$$

$$\sigma_1 - \sigma_3 - \sigma_{mt} = 0 \quad (5.5)$$

$$(\tau_m = \frac{\sigma_{mt}}{2}) \quad (5.6)$$

$$\sigma_{md} = \sigma_{mt} \quad (5.7)$$

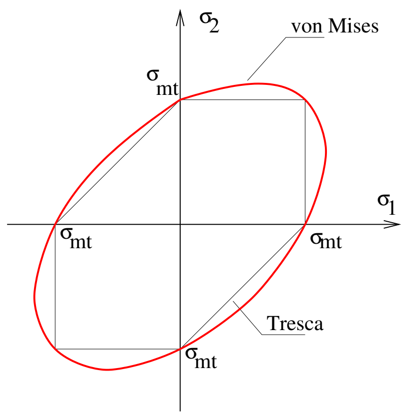


5.3.3 Energy shape condition change (von Mises or von Mises, Huber, Hencky)

Derivation of such condition is more complex. The reaulting equation is:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_3 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_{mt}^2 \quad (5.8)$$

$$\sigma_{md} = \sigma_{mt}$$



Note: one can find so-called „von Mises stress“ which is defined as:

$$\sigma_{vmis} = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_3-\sigma_2)^2+(\sigma_1-\sigma_3)^2}{2}} \leq \sigma_{mt} \tag{5.9}$$

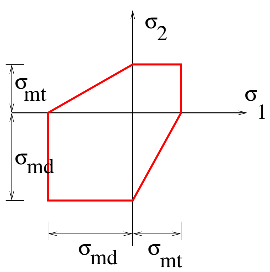
Thus σ_{vmis} represents state of material. If $\sigma_{mt} = f_y$ is the yield stress value then σ_{vmis} can be used as an indication if studied material is elastic or not.

The von Mises criteria is better suited for computatioonal mechanics than the Tresca one because it is represented by a smooth curve without edges. It is often used in FEA software for representation of elastic-plastic behaviour of metals.

5.3.4 Mohr–Coulomb condition

This condition can be viewed as a modified Tresca with $\sigma_{md} \neq \sigma_{mt}$.

$$\begin{aligned} \sigma_1 - \frac{\sigma_{mt}}{\sigma_{md}}\sigma_3 - \sigma_{mt} &= 0 \\ \sigma_{md} &\neq \sigma_{mt} \end{aligned} \tag{5.10}$$



This condition is suitable for geotechnical materials (soils). It is available in FEA packages like the PLAXIS. The Drucker–Prager condition (which is represented by smooth curve over the edges of the Mohr–Coulomb) is also often used.

5.3.5 Chen–Chen condition

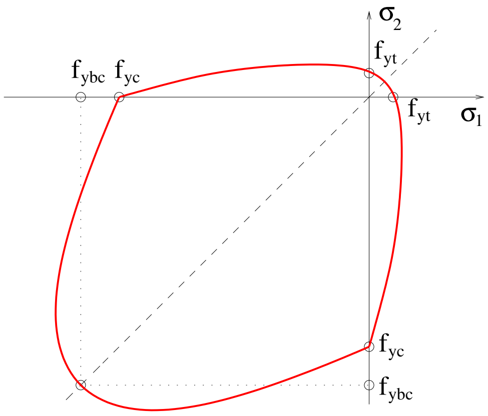
This condition was derived for concrete. Its shape is was created to represent data obtained from expewrimental researches (Kupfer et al and others).

For compression–compression area ($\sigma_1 < 0$ a $\sigma_2 < 0$, $\sigma_3 < 0$):

$$J_2 + \frac{A_{yc}}{3} I_1 - \tau_{yc}^2 = 0 \tag{5.11}$$

For other areas:

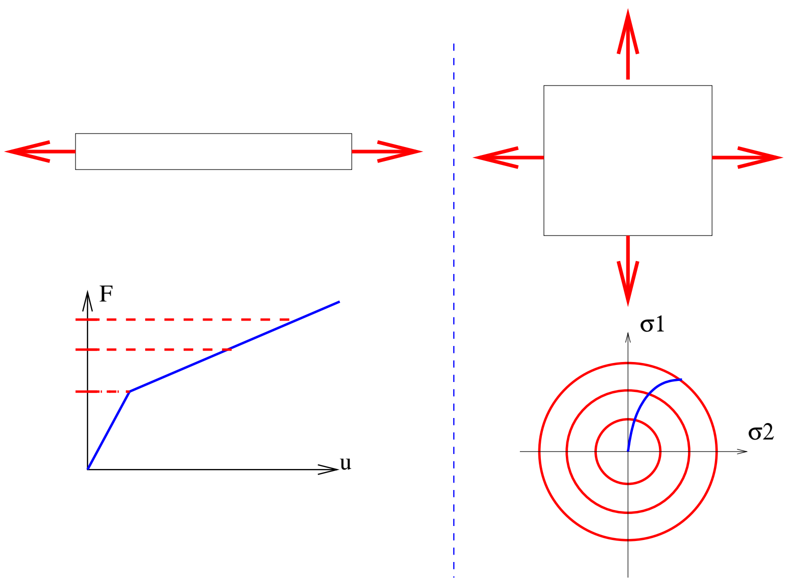
$$J_2 - \frac{1}{6} I_1^2 + \frac{A_{yt}}{3} I_1 - \tau_{yt}^2 = 0 \tag{5.12}$$



This condition is discussed in greater details in the further text.

5.4 Hardening

Hardening stage of elastic-plastic behaviour begins after plasticity condition is reached. The next picture illustrates hardening in 1D and 2D.



There are several main types of hardening behaviour:

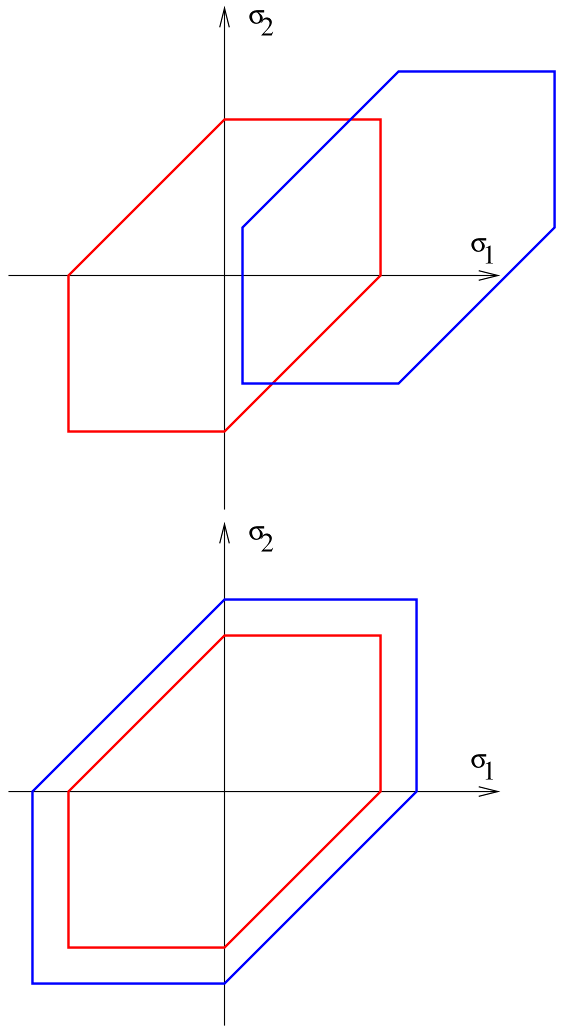
- Kinematic
  - subsequent plasticity conditions are moving,

- no change of shape and size.

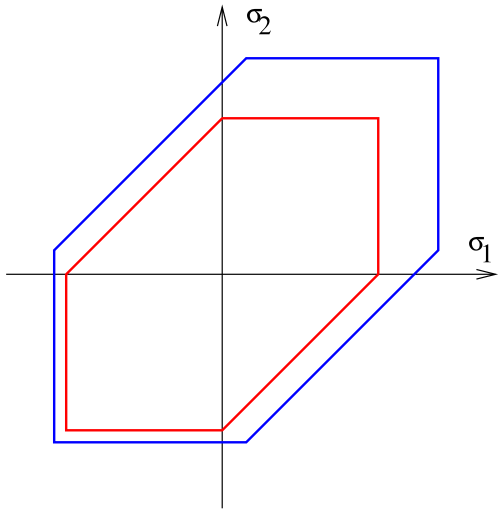
- Izotropic
  - subsequent conditions are changin size proportionally,
  - no moves.

- Combined
  - combination of kinematic and isotropic hardenings.

The next picture illustrates the kinematic hardening (top) and the isotropic one (botton).



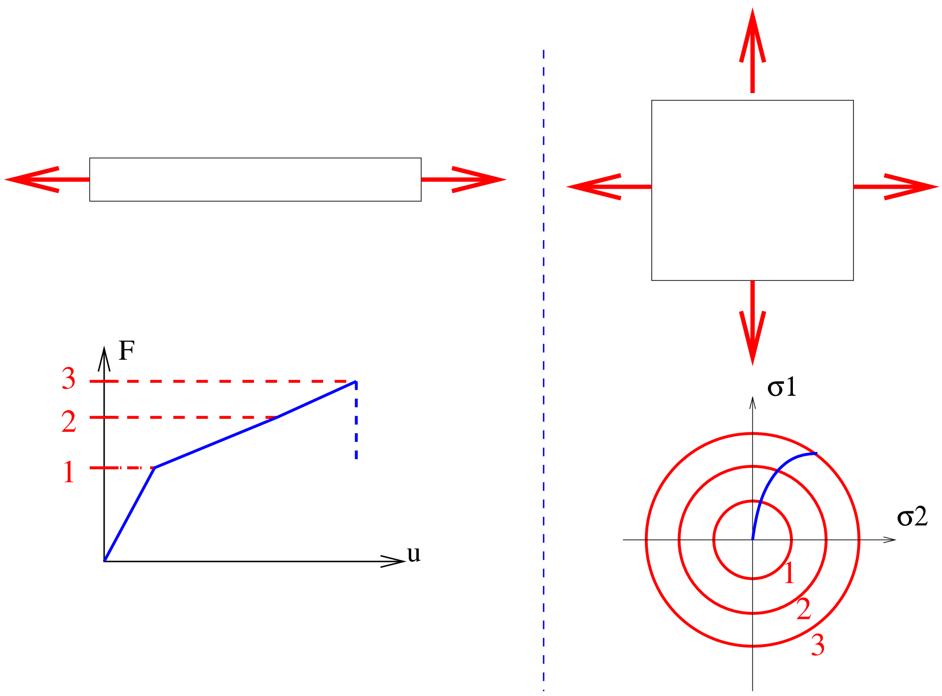
In many cases, the **combined hardening** is more close to real material behaviour. It is illustrates on the next picture.



5.5 Plasticity and failure conditions

The picture below demonstrates all interesting parts of elastic–plastic behaviour in 1D (left) and 2D:

1. Initial plasticity condition.
2. Subsequent plasticity condition (during hardening).
3. Failure condition (theoretical failure of material).



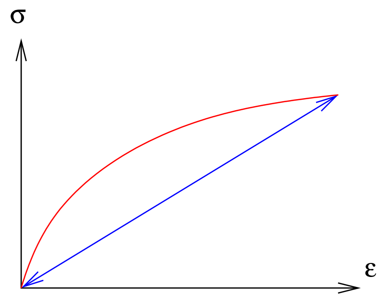
5.6 Variants of theory of plasticity

5.6.1 Theory of plastic deformations

- It uses relations between total deformations and stresses:

$$\boldsymbol{\sigma} = \mathrm{D}^{\mathrm{EP}} \ \boldsymbol{\varepsilon}$$

- The solution **does not** depend on loading path.



The theory of plastic deformations assumes that it is possible to find a direct relation between stresses and strains for any point on loading path (illustrated by blue line with arrows on the picture). Obviously, it is possible only for selected and relatively simple problems.

5.6.2 Plastic flow theory

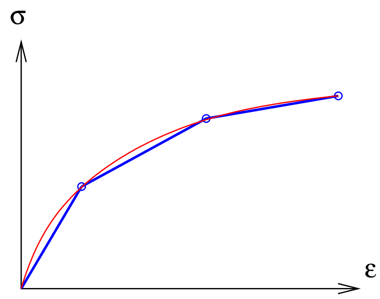
This theory assumes that:

- relation between changes ("speeds") of deformations and stresses is:

$$\dot{\boldsymbol{\sigma}} = \mathrm{D}^{\mathrm{ep}} \ \dot{\boldsymbol{\varepsilon}} \tag{5.13}$$

,

- solution **depends** on loading path,
- solution can be divided to a set of linearised steps.



This approach is much more robust and it is used in subsequent text.

5.7 Plastic flow theory

5.7.1 Unknowns – changes ("speeds"):

- Stresses: $\dot{\boldsymbol{\sigma}} = \{\dot{\sigma}_x, \dot{\sigma}_y, \dot{\sigma}_z, \dot{\tau}_{yz}, \dot{\tau}_{yz}, \dot{\tau}_{xy}\}^T$
- Relative deformations (strains): $\dot{\boldsymbol{\varepsilon}} = \{\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_z, \dot{\gamma}_{yz}, \dot{\gamma}_{yz}, \dot{\gamma}_{xy}\}^T$
- Displacements (and rotations): $\dot{\boldsymbol{u}} = \{\dot{u}, \dot{v}, \dot{w}\}$

5.7.2 Assumptions:

- Initial stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ state must be known.
- Solution have to respect boundary conditions.

5.7.3 Elastic–plastic material matrix

- Constitutive equations:

$$\dot{\boldsymbol{\sigma}} = \mathbf{D^{ep}} \;\; \dot{\boldsymbol{\varepsilon}}$$

  $\mathbf{D^{ep}}$ ... elastic–plastic material matrix (have to be found).

- Division of change of deformations to elastic and plastic part:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{e}} + \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{p}}$$

- Plastic condition (it is used for description of change from elastic to plastic state):

$$f(\boldsymbol{\sigma}, \boldsymbol{k}) = 0$$

- Consistence condition of plastic material:

$$df = \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \{d\boldsymbol{\sigma}\} + \left\{\frac{\partial f}{\partial \mathbf{k}}\right\}^T \{d\mathbf{k}\} = 0$$

- Speed of plastic deformation (plastic deformation law):

$$\dot{\boldsymbol{\varepsilon}}_p = d\lambda \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}$$

- Stress changes:

$$\dot{\boldsymbol{\sigma}} = d\boldsymbol{\sigma} = \mathbf{D_e}\left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}_p\right) = \mathbf{D_e}\left(\dot{\boldsymbol{\varepsilon}} - d\lambda \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}\right)$$

- Equivalent plastic deformation:

$$d\varepsilon_p = \sqrt{\dot{\boldsymbol{\varepsilon}_{\boldsymbol{p}}}^T \; \dot{\boldsymbol{\varepsilon}_{\boldsymbol{p}}}} = d\lambda \sqrt{\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}}$$

- From consistence condition:

$$\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \mathbf{D_e} d\boldsymbol{\varepsilon} - d\lambda \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \mathbf{D_e} \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\} + d\lambda \frac{\partial f}{\partial \varepsilon_p} \sqrt{\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}} = 0 \tag{5.14}$$

Computation of the $d\lambda$ parameter:

$$d\lambda = \frac{\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \mathbf{D_e} \dot{\boldsymbol{\varepsilon}}}{\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \mathbf{D_e} \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\} + \frac{\partial f}{\partial \varepsilon_p} \sqrt{\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}}} \tag{5.15}$$

If $d\lambda$ is substituted into $\dot{\boldsymbol{\sigma}} = \mathbf{D_e}\left(\dot{\boldsymbol{\varepsilon}} - d\lambda \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}\right)$ then:

$$\dot{\boldsymbol{\sigma}} = \mathbf{D_e} \left(\dot{\boldsymbol{\varepsilon}} - \frac{\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \mathbf{D_e} \dot{\boldsymbol{\varepsilon}}}{\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \mathbf{D_e} \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\} + \frac{\partial f}{\partial \varepsilon_p} \sqrt{\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}^T \left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\}}} \left\{\frac{df}{d\boldsymbol{\sigma}}\right\} \right) \tag{5.16}$$

The equation for $\dot{\boldsymbol{\sigma}}$ can be simplified:

$$\dot{\boldsymbol{\sigma}} = \mathbf{D_{ep}} \;\; \dot{\boldsymbol{\varepsilon}}_{ep}, \tag{5.17}$$

where elastic–plastic material matrix $\mathbf{D_{ep}}$ is:

$$\mathbf{D_{ep}} = \mathbf{D_e} - \frac{\mathbf{D_e} \left\{\frac{\partial \mathbf{f}}{\partial \boldsymbol{\sigma}}\right\} \left\{\frac{\partial \mathbf{f}}{\partial \boldsymbol{\sigma}}\right\}^{\mathbf{T}} \mathbf{D_e}}{\left\{\frac{\partial \mathbf{f}}{\partial \boldsymbol{\sigma}}\right\}^{\mathbf{T}} \mathbf{D_e} \left\{\frac{\partial \mathbf{f}}{\partial \boldsymbol{\sigma}}\right\} - \frac{\partial \mathbf{f}}{\partial \boldsymbol{\varepsilon}_\mathbf{p}} \sqrt{\left\{\frac{\partial \mathbf{f}}{\partial \boldsymbol{\sigma}}\right\}^{\mathbf{T}} \left\{\frac{\partial \mathbf{f}}{\partial \boldsymbol{\sigma}}\right\}}} \tag{5.18}$$

This formulation is used in many FEA packages in conjunction of substep-based metods for solution on non-linear problems (the Newton-Raphson–type methods).

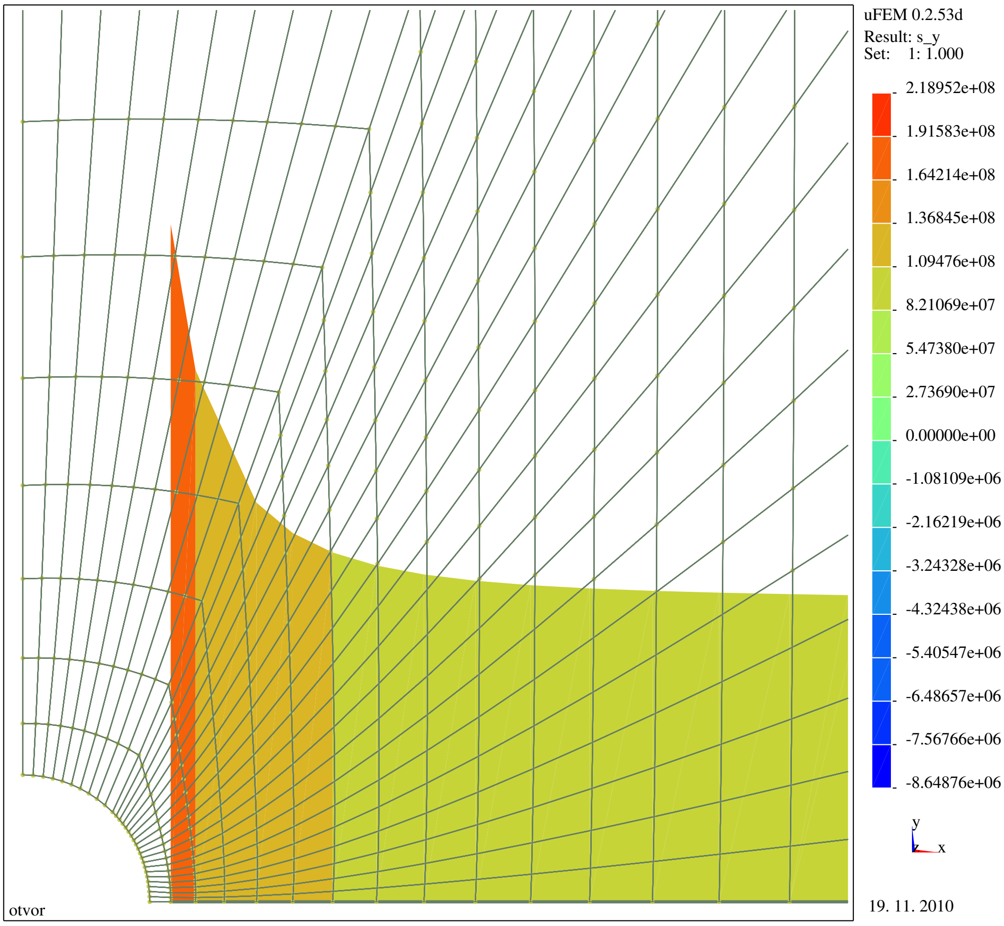
5.8 Problem for individual work

- Use the software of your choice to analyse 2D problem (a perpendicular wall loaded on its upper edge and fixed on sides). Use Newton-Raphson method and von Mises model. Prepare parametric study with different values of hardening parameters and compare the results. Then do the same with the Drucker-Prager or the Mohr-Coulomb models (set f_{yc} to the same value as in the von Mises case and $f_{yt} = \frac{f_{yc}}{10}$).

Chapter 6

Introduction to Fracture Mechanics

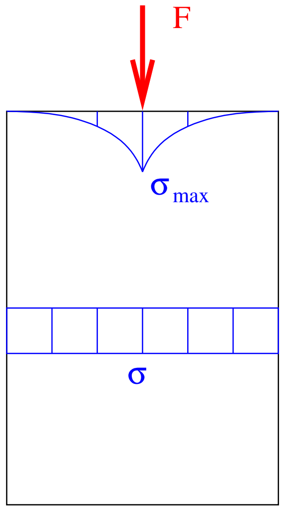
6.1 Stress concentrations



There are many cases when stress concentrations can be found:

- Near holes and cracks.
- Angles, especially sharp ones.
- Concentrated loads.

6.2 Saint–Venant Principle



Saint–Venant Principle: *The stress in the material is independent on the form of the load if the distance from the load is sufficient.*

The problem is that problems (stress concentrations, cracks, fractures,...) often occur in the areas close to load or geometry changes.

Common criteria for bearing capacity (plasticity conditions or failure conditions) assume the Saint-Venant Principle and thus:

- They are usefull if stress gradients are small.
- They don't work well for large stress gradient (where Saint-Venant cannot be used).

6.3 Introduction to fracture mechanics

Fracture mechanics was developed to address these problems:

- Faults are developing in places of stress concentrations.
- Real structures **always** include such concentrations.
- Real structures always broke earlier than it can be assumed from classic plasticity/failure conditions.

Typical problems which should be studied with use of fracture mechanics are:

- Large steel structures (bridges, towers, ships,...).
- Welded structures .
- Structures exposed to big temperature changes.
- Massive reinforced concrete structures (pilllars, dams).

For metals there are two types of fracture :

- *Brittle failure*: effects of normal stress on layers of atoms
- *Ductile failure*: effects of shear stress on layers of atoms

Thus Fracture Mechanics studies failure caused by cracks. There are several tasts to bear:

- Crack identification.
- Determination of conditions when the crack is stable or unstable:
  - acceptable crack size,
  - critical size of crack.

6.4 Linear Fracture Mechanics

Linear fracture mechanics assume that studied material is linear elastic. This is valid only for material like glass and high-quality steels.

We will briefly review these theories of the like fracture mechanics:

- Griffith theory
  - fragile materials – glass.
- Irwin–Orowan theory
  - extension of Griffith theory to other materials (metals, some plastics).

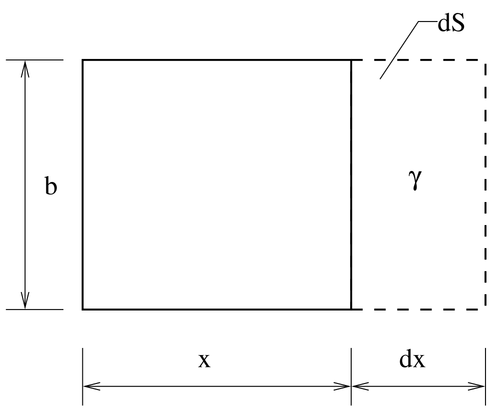
6.5 Griffith Theory

Assumes that there exist *surface stress* γ
(for example, $\gamma = 0,55\frac{N}{m}$ for glass at $15^o\ C$).

Change of surface stress if the area is extended for $dS = dx \times b$:

$$d\Gamma = \gamma \times b \times dx \tag{6.1}$$

$$\gamma = \frac{d\Gamma}{dS} \tag{6.2}$$

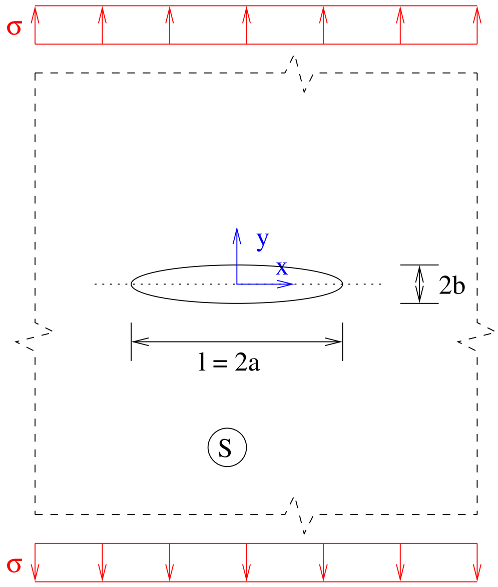


Infinite wall of unit thickness with crack
Westergaard solution of elliptic hole – crack (linear elasticity):

$$b = \frac{2a}{E}\sigma, \tag{6.3}$$

where E is Young modullus.

Infinite wall of unit thickness with crack



Energy of wall with crack:

$$U = U_1 - U_2, \tag{6.4}$$

where:
U_1 ... energy of the wall.
U_2 ... deformational energy necessary to close the crack.

Energy of wall with crack:

$$U_1 = \frac{1}{2}\int_V \boldsymbol{\sigma}\boldsymbol{\varepsilon} dV \tag{6.5}$$

$$U_1 = \frac{1}{2}\int_S \sigma\frac{\sigma}{E}1dS \tag{6.6}$$

$$U_1 = \frac{\sigma^2}{2\,E}S \tag{6.7}$$

Energy of crack:

$$dU_2 = \int_0^y \sigma^* dxdy = ... = \frac{1}{2}\sigma y dx \tag{6.8}$$

$$U_2 = \sigma \int_{S_T} dU_2, \tag{6.9}$$

where $S_T = \pi\ a\ b \dots$ is crack area.

From Westergaard solution ($b = \frac{2\ a}{E}\sigma$)

$$U_2 = \frac{\pi^2\ \sigma^2\ l^2}{4\ E}, \tag{6.10}$$

thus:

$$U = \frac{\sigma^2}{2\ E}S - \frac{\pi^2\ \sigma^2\ l^2}{4\ E}. \tag{6.11}$$

Extension δl of existing crack:

$$-\delta U = -\frac{\partial U}{\partial l}\delta l = \frac{\partial U_2}{\partial l}\delta l \tag{6.12}$$

$$\frac{\partial U_2}{\partial l} = \frac{\partial\left(\frac{\pi\sigma^2 l^2}{4E}\right)}{\partial l} \tag{6.13}$$

$$\frac{\partial U_2}{\partial l} = \frac{\pi\sigma^2 l}{2E} \tag{6.14}$$

Crack surface extension $2\ \delta l$.

Energy needed for creation of new surface must be equal to released energy of internal forces:

$$-\delta U = 2\delta l\ \gamma \tag{6.15}$$

Enlargement of existing crack for δl: From previous equations (6.13)–(6.15):

$$\frac{\pi\ \sigma^2\ l}{2\ E} = 2\gamma \tag{6.16}$$

It defines **critical state of crack**. One can define critical stress for given crack size or critical crack lenght for given stress.

Critical stress for given crack lenght l:

$$\sigma_{crit} = \sqrt{\frac{4\ \gamma\ E}{\pi\ l}} \tag{6.17}$$

Critical crack lenght for given stress σ:

$$l_{crit} = \frac{4\ \gamma\ E}{\pi\ \sigma^2} \tag{6.18}$$

6.6 Irwin theory

This is an application of the Griffith theory to other materials. It can be used for metals and some plastics and thus it assumes **plastic zone on the crack tip**. Energy necessary for plastic deformation is about 1000 times higher than surface stress thus surface stress can be neglected.

There is a new entity – the work necessary for creation of plastic zone G. Deformation necessary for enlergement of crack for δl:

$$dV = G\,dl \tag{6.19}$$

$$\frac{\pi\,\sigma^2}{2\,E}\,l dl = G dl \tag{6.20}$$

After modification:

$$\sigma\sqrt{\pi\,a} = \sqrt{G\,E} \tag{6.21}$$

For critical state of crack:

$$\sigma\sqrt{\pi\,a} = \sqrt{G\,E} \tag{6.22}$$

6.6.1 Parameters for description of crask state:

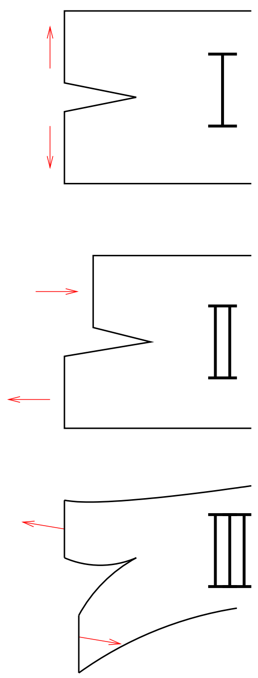
- **Stress Intensity Factor** (it depends on stress):

$$K_I = \sigma\sqrt{\pi\,a} \tag{6.23}$$

- **Fracture Toughness** (in depends on material properties):

$$K_{IC} = \sqrt{G\,E} \tag{6.24}$$

6.7 Modes of fracture



The picture illustrates three modes of fracture. More complex modes can be created a combination of them.

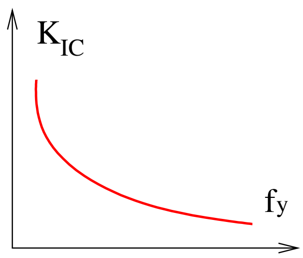
- Mode I – **opening**.

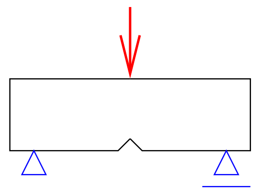
- Mode II – **sliding**.
- Mode III – **tearing**.

In the previous text the Mode I was always used, thus the parameters have had to be indexed by the mode number: K_I, K_{IC}.

6.8 Fracture Toughness

Fracture toughness is usually obtained from experimental tests.





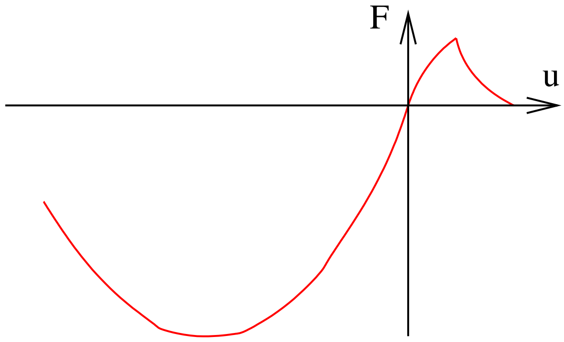
The oblatined value depends on many factors and thus its use is more complicated than use of material parameters available in linear mechanics. It may depend on:

- size of structure,
- environmental effects,
- temperature,
- stress state,
- loading speed,
- initial stresses casued by production of material or structural member.

6.9 Material models for concrete – quasi-brittle materials

6.9.1 Mechanical properties of concrete

Concrete **is not** elastic-plastic material. In some situations it can be approaximately modelled as such. However, for more precise modelling it is necessary to use other models.



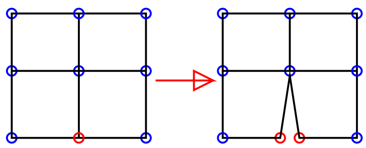
Concrete has these properties:

- There is no linear elastic behaviour.
- There is non-reversible non-linear behaviour (non-reversible deformations, cracking,. . .)
- It has different behaviour for different types of loading (tension vs compression).

6.10 Constitutive models for concrete

There are several basic groups of constitutive models:

- **Discrete models:** individual cracks are modelled (usually by finite element mesh changes – see picture).



- **Continuum models:** it is assumed that model remains continuosus but with changing material properties:
  - Models based on non-linear fracture mechanics (smaered cracks, non-local continuum, microplane models)
  - Elastic–plastic models.
  - Combined models (for example: elastic–plastic behaviour for compression and fracture–based model for tension).

6.11 Chen elastic–plastic model

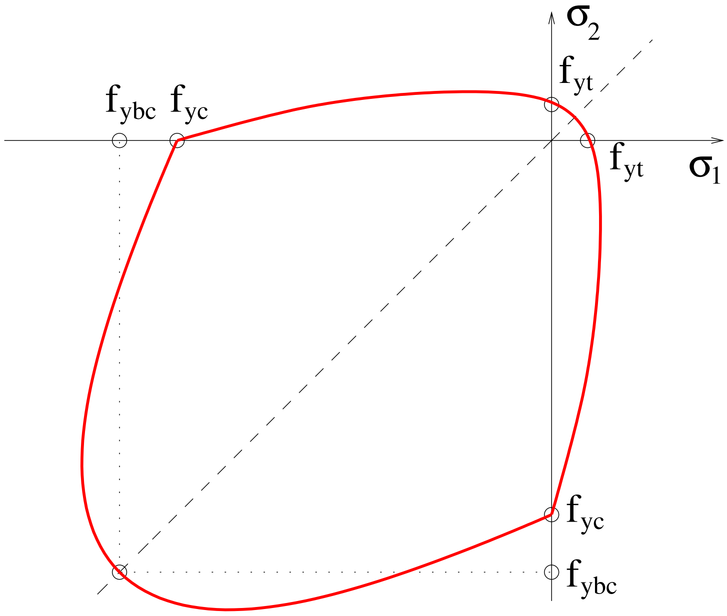
6.11.1 Chen model and plasticity condition

Usual plasticity conditions don't satisfy concrete behaviour (von Mises, Tresca). Experiment research by plane stress concrete samples (prof. Kupfer, Germany) were used by several authors to construct failure conditions, plasticity conditions and complete constitutive models for concrete (Kupfer; Chen and Chen; Willam and Warnke;...).

Chen (Chen and Chen) condition:

- Uses aproximation of Kupfer data by polynomic functions.
- The condition can be used both for plasticity and for failure.

6.11.2 Parameters of Chen plasticity condition



For compression–compression zone ($\sigma_1 < 0$ a $\sigma_2 < 0$, $\sigma_3 < 0$):

$$J_2 + \frac{A_{yc}}{3} I_1 - \tau_{yc}^2 = 0 \tag{6.25}$$

For all ther zones:

$$J_2 - \frac{1}{6} I_1^2 + \frac{A_{yt}}{3} I_1 - \tau_{yt}^2 = 0 \tag{6.26}$$

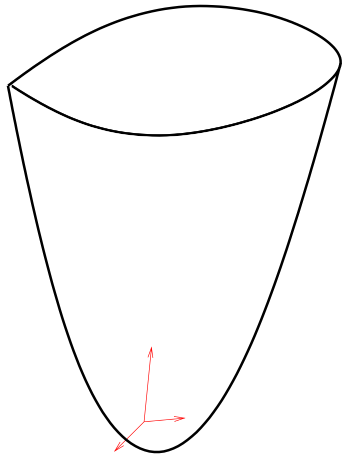
where:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \tag{6.27}$$

and

$$J_2 = \frac{1}{2} \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) \tag{6.28}$$

These function define open, convex, shape in 3D:



Constants A_{yx}, τ_{yx} are based on material data:

$$\begin{aligned} A_{yc} &= \frac{f_{ybc}^2 - f_{yc}^2}{2f_{ybc} - f_{yc}} \\ \tau_{yc}^2 &= \frac{f_{ybc} f_{yc}(2f_{yc} - f_{ybc})}{3(2f_{ybc} - f_{yc})} \\ A_{yt} &= \frac{f_{yc} - f_{yt}}{2} \\ \tau_{ut}^2 &= \frac{f_{yc} f_{yt}}{6}, \end{aligned} \quad (6.29)$$

where:
f_{yc} ... yield stress in uniaxial compression,
f_{ybc} ... yield stress in biaxial compression,
f_{yt} ... yield stress in uniaxial tension.

6.11.3 Chen failure condition

The failure condition can be defined in the same manner but for ultimate stresses (f_{uc}, f_{ubc}, f_{ut}):

$$J_2 + \frac{A_{uc}}{3} I_1 - \tau_{uc}^2 = 0 \quad (6.30)$$

and

$$J_2 - \frac{1}{6} I_1^2 + \frac{A_{ut}}{3} I_1 - \tau_{ut}^2 = 0 \quad (6.31)$$

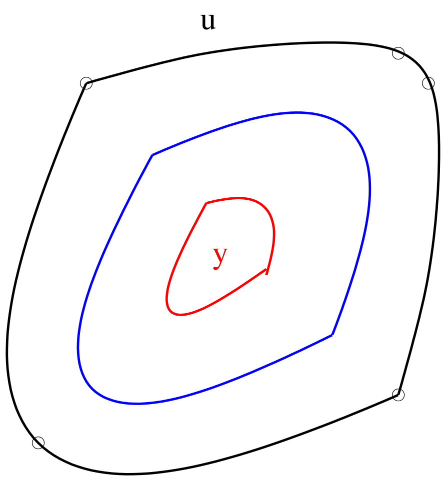
The constant have analogical meanings:

$$
\begin{aligned}
A_{uc} &= \frac{f_{ubc}^2 - f_{uc}^2}{2f_{ubc} - f_{uc}} \\
\tau_{yc}^2 &= \frac{f_{ubc} f_{yc}(2f_{uc} - f_{ubc})}{3(2f_{ubc} - f_{uc})} \\
A_{ut} &= \frac{f_{uc} - f_{ut}}{2} \\
\tau_{ut}^2 &= \frac{f_{uc} f_{ut}}{6}
\end{aligned}
\tag{6.32}
$$

For material states between plasticity and failure conditions it is necessary to define relations:

$$
\begin{aligned}
A_c &= \alpha_c \tau_c^2 + \beta_c, \\
A_t &= \alpha_t \tau_t^2 + \beta_t.
\end{aligned}
\tag{6.33}
$$

These conditions are illustrated on the picture below (red: plasticity, black: failure, blue: intermediate state).



Where the parameters α, β are defined as:

$$
\begin{aligned}
\alpha_c &= \frac{A_{uc} - A_{yc}}{\tau_{uc}^2 - \tau_{yc}^2} \\
\beta_c &= \frac{A_{yc}\tau_{uc}^2 - A_{yc}\tau_{yc}^2}{\tau_{uc}^2 - \tau_{yc}^2} \\
\alpha_t &= \frac{A_{ut} - A_{yt}}{\tau_{ut}^2 - \tau_{yt}^2} \\
\beta_t &= \frac{A_{yt}\tau_{ut}^2 - A_{yt}\tau_{yt}^2}{\tau_{ut}^2 - \tau_{yt}^2}
\end{aligned}
\tag{6.34}
$$

Note: such model shown good agreement with experimental results for reinforced concrete.

6.11.4 Related conditions

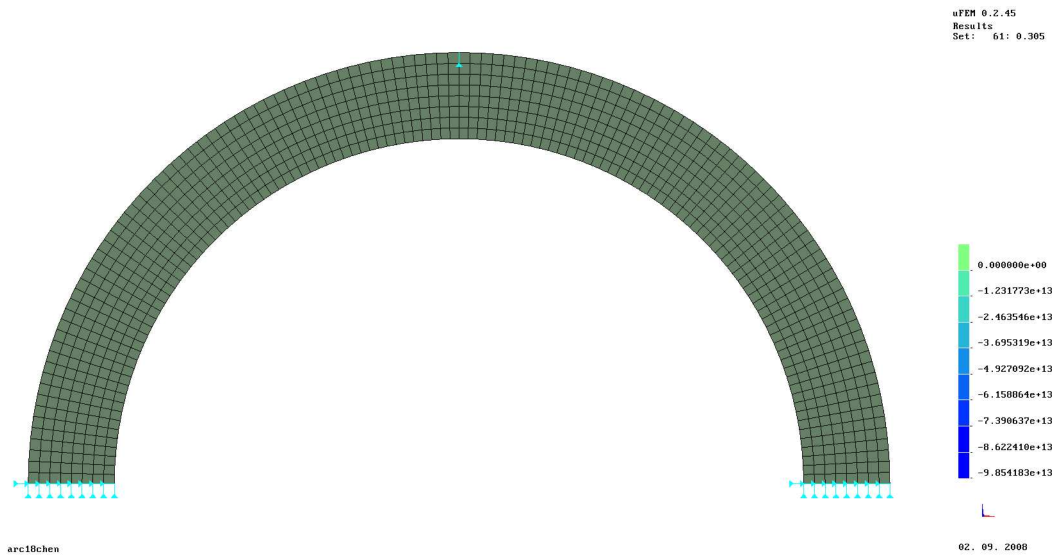
There are other models and conditions which can give similar results to the Chen model, for example:

- **Kupfer** failure condition:
  - it is defined for 2D stress state (only),
  - it uses data from standard tests (cyllindric strength of concrete).
- **Willam–Warnke** condition:
  - it is defined for 3D stress state,
  - very similar to Chen one in term of input data and shape but uses different formulation:

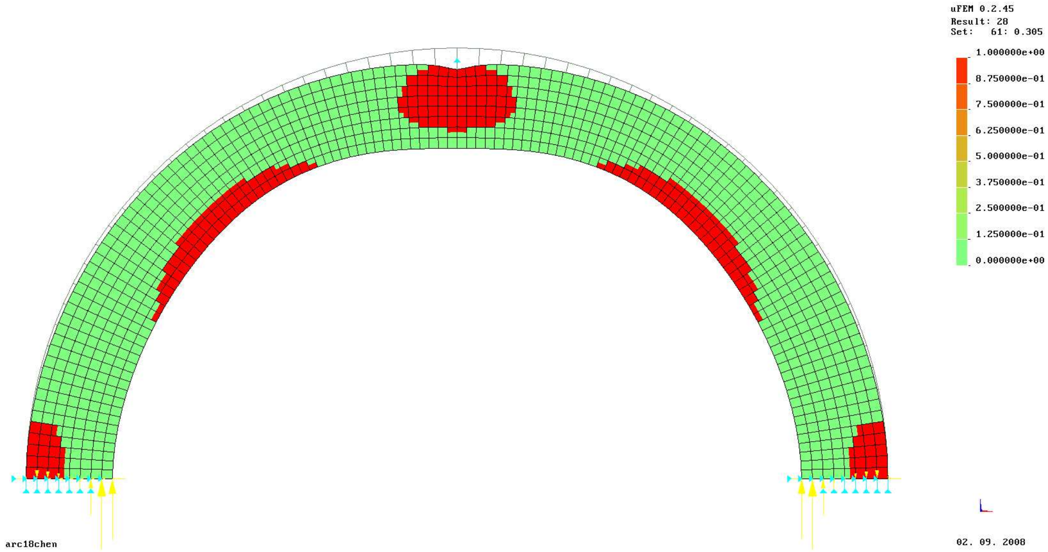
$$f = \frac{1}{3z}\frac{I_1}{\sigma_c} + \sqrt{\frac{2}{5}}\frac{1}{r(\theta)}\frac{J_2}{\sigma_c} - 1 = 0, \tag{6.35}$$

  , where r and z are constant based on material properties. They are deffined in similar manner as the A and τ properties of Chen model.

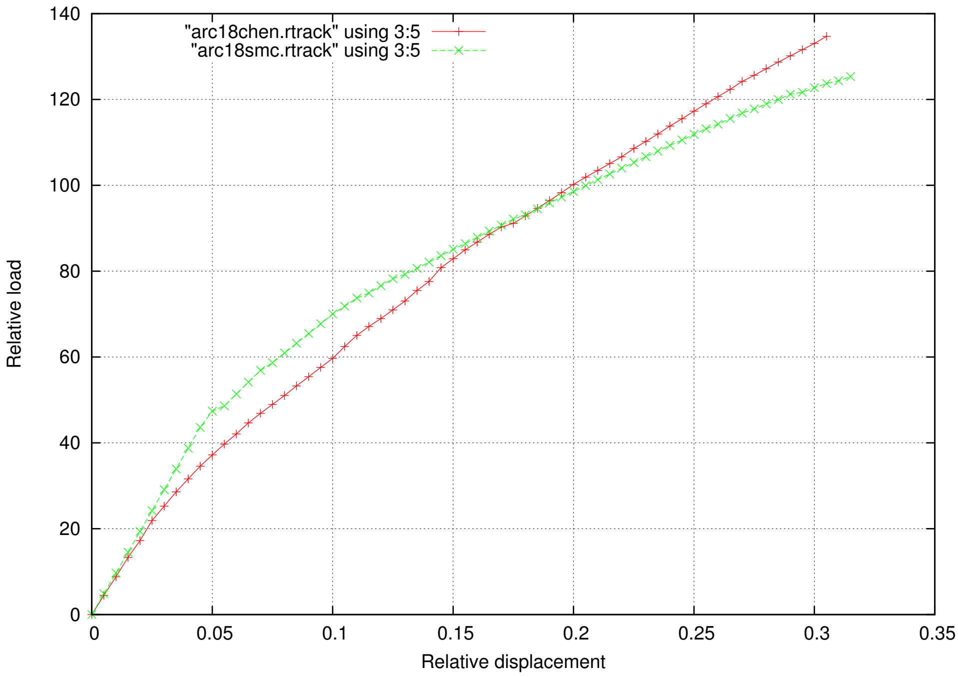
6.11.5 Example – finite element model of concrete arc



The picture above shows 2D model of reinforced concrete arc modelled with use of the abovementioned Chen model. The next picture show location of plasticised areas:



On the next picture there is also shown obtained load–displacement relation for the model (red curve; the green curve was obtained from solution based on smeared crack model whic his discussed below).



6.12 Smeared crack model

6.12.1 Basic principles

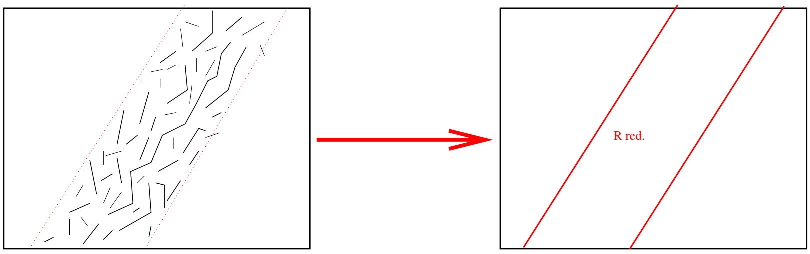
Elastic–plastic models for concrete can be usefull if concrete is properly reinforced and when cracking is not progressive. If these assumptions are not valid the different models have to be used.

In many case the principles of fracture mechanics are used through the smeared crack model.

The smeared crack model is based on these principles:

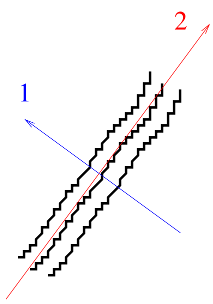
- Modelling of damaged area (cracks,...) is done by reduction of material properties (E, ν)
- The model is continuous, area damaged by cracking has reduced material properties.
- Cracks are expected to be small (large discrete cracks should require different approaches).

Idealization of real structure with cracks to mathematical model is illustrated in the next picture,



The damaged area can be modelled as an **orthotropic material** with axes of orthotropy oriented in dependence of crack oreintation:

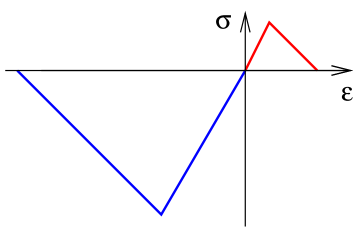
$$\mathbf{D} = \frac{R_2}{R_2 - \mu^2\ R_1} \begin{bmatrix} R_1 & \mu R_1 & 0 \\ \mu R_1 & R_2 & 0 \\ 0 & 0 & \frac{\beta\ G}{R_2/(R_2 - \mu^2\ R_1)} \end{bmatrix},$$



The orientation is usually computed in relation to direction of principal stresses.

To define parameters of the orthotropic material, there are many approaches. One of the simplest is use of two main parts:

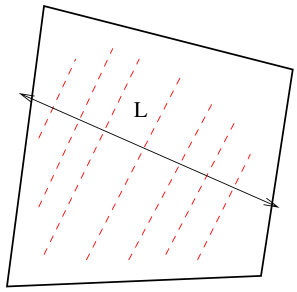
- One-dimensional equivalent stress–strain law.



- 2D condition for adjusting of one-dimensional law. The 2D condition may be the Kupfer condition or the Chen condition, for example.

6.12.2 Bažant's crack band model

The abovementioned approach has one disadvantage: in practical use the actual results depend on size of finite elements. To counter this, many approaches have been developed. The simplest one is the crack band model proposed by Bažant. It assumes that energy spent for full opening of a crack (fracture energy A_G) can be used as a material property and can be used to adjust the model.



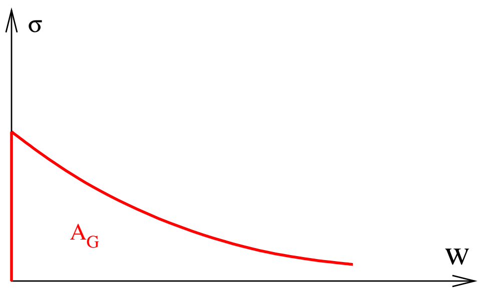
Fracture energy :

$$G_F = A_G\, L = const., \tag{6.36}$$

where L is width of finite element.

Then:

$$G_F = \int_0^\infty \sigma_n(w)\, dw, \tag{6.37}$$

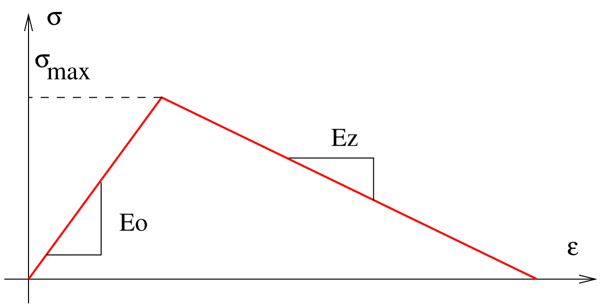


Total cracks width on a finite element width can be computed:

$$w = \varepsilon\, L \tag{6.38}$$

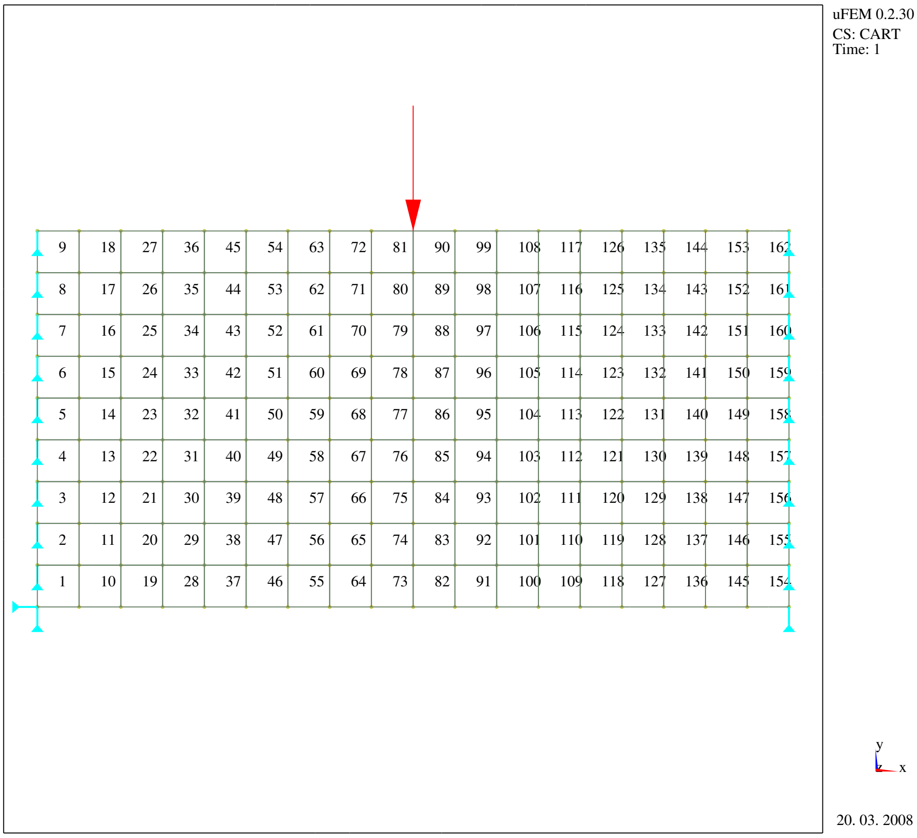
Descending modulus for one-dimensional law (see next picture) can be then defined:

$$E_z = \frac{E_o}{1 - \frac{2 G_F E_o}{L\, \sigma_{max}^2}}. \tag{6.39}$$

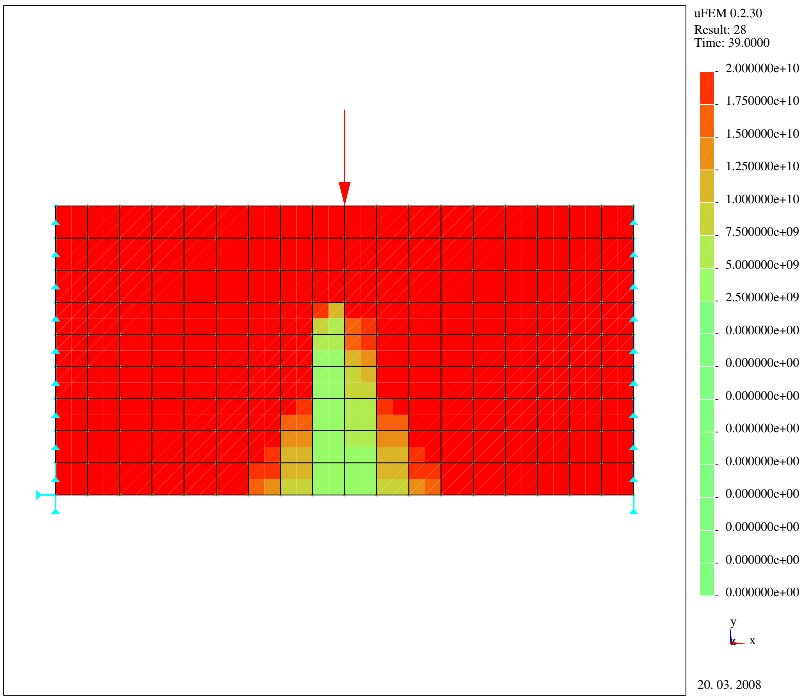


6.12.3 Example of smeared crack model

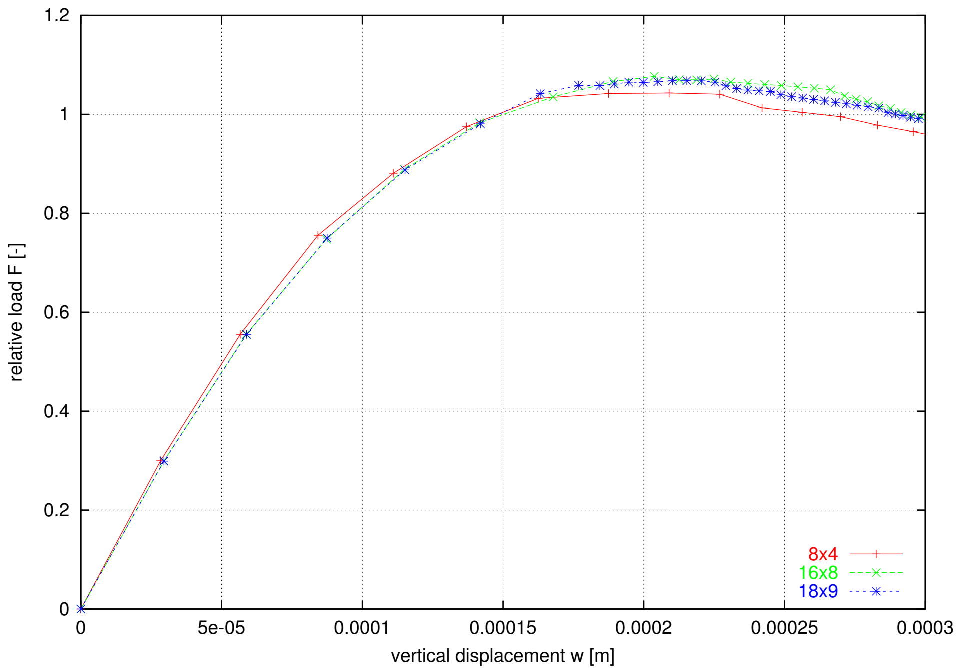
The abovementioned approach was implemented into the uFEM software and a simple example is presented below.



The residual stiffness (reduced stiffness of orthotropic material in direction perpendicular to cracks) is shown in the next picture.



Load–displacement curves for several mesh sizes (to demonstrate effects of the crack band model) are shown on the last picture.



6.13 Problem for individual work

- Find at least 3 software packages which can use at least one of the above-mentioned models for concrete.

Chapter 7

Geometric non–linearity

7.1 Non-linear behaviour related to geometry

Constitutive non-linearities are often coupled with large deformations or at least in states when deformation of structure has important effect to structural behaviour.

Even in the case of small deformations it can be usefull to compute equilibrium equations on deformed structure ("2^{nd} order theory"). The most common use is for **linear stability** problems but it can be usefull in many cases:

- Pre-stressed structures (pre-stressed concrete, rope structures).
- Structures with progressive damage.
- Thin structures (including slender beams).

We will demonstrate the 2^{nd} order theory on the well-known Euler problem.

7.2 Euler solution – 2^{nd} order theory

This is the problem known from basic elasticity courses:

- Loss of stability of axially loaded beam stability.
- Linear theory gives incorrect solution (too optimistic).
- Equlilibrium equation must be used on deformed beam ⇒ 2nd order theory.

Bending moment in point x:

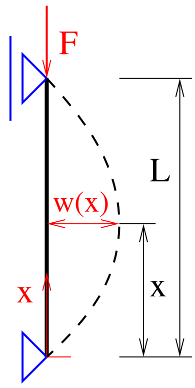
$$M = F\,w \tag{7.1}$$

Displacement function:

$$w^{''} = -\frac{M}{EI} = -\frac{F\,w}{EI} \tag{7.2}$$

If we will use $\alpha^2 = \frac{F}{EI}$:

$$w =^{''} + \alpha^2 w = 0 \tag{7.3}$$



Equation (7.3) has a solution:

$$w = C_1 \sin \alpha x + C_2 \cos \alpha x \tag{7.4}$$

Boundary conditions:

- for $x = 0$ it is $w(x = 0) = 0$:

$$0 = C_1 \sin \alpha\, 0 + C_2 \cos \alpha\, 0 \Rightarrow C_2 = 0 \tag{7.5}$$

,

- for $x = L$ it is $w(x = L) = 0$:

$$0 = C_1 \sin \alpha\, L + 0 \Rightarrow 0 = C_1 \sin \alpha\, L \tag{7.6}$$

For $C_1 \neq 0$ it must be $\sin \alpha L = 0$:

$$\alpha L = k\, \pi \ldots k = 1, 2, 3, ... \tag{7.7}$$

After use of boundary conditions:

$$w = C_1 \sin \frac{k\pi\, x}{L} \tag{7.8}$$

One can use α^2:

$$\alpha^2 = \frac{F}{E\, I} \Rightarrow F = \alpha^2 EI \;\; ... \;\; \alpha\, L = 1\, \pi \tag{7.9}$$

After moditication and after use of $F_{cr} = F$

$$F_{cr} = \pi^2 \frac{EI}{L^2}. \tag{7.10}$$

The F_{cr} is the well-known **Euler's limit force**.

7.3 Ritz method for Euler problem

It is possible to use the Ritz method to analyse the same problem. The advantage of this method is, of course, it's ability to be used for more complex structures.

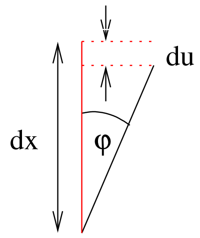
Approximation of deformation:

$$w = a_1 \sin \frac{\pi x}{L} \tag{7.11}$$

Potential energy:

1. $\Pi_N = -F\, u_a = -F\frac{FL}{EA}$, $u_a \ldots$ shortening of beam due to classical linear theory (does not depend on w)

2. $\Pi_M = -F u_b$, $u_b \ldots$ shortening due to rotation of beam



Shortening of beam due to rotation:

$$du = dx - dx\cos\varphi \tag{7.12}$$

We can write (7.12) as a Taylor sequence:

$$du \approx dx - dx(1 - \frac{1}{2}\varphi^2) = \frac{1}{2}\varphi^2 dx \approx \frac{1}{2}(w^{'})^2 dx. \tag{7.13}$$

In a short form: $du \approx \frac{1}{2}(w^{'})^2 dx$

For the whole beam:

$$u_b = \frac{1}{2}\int_0^L (w^{'})^2 dx \tag{7.14}$$

Aproximation of w:

$$w = a_1 \sin \frac{\pi x}{L} \tag{7.15}$$

Derivation:

$$w^{'} = a_1 \frac{\pi}{L}\cos\frac{\pi x}{L}, \quad w^{''} = -a_1\frac{\pi^2}{L^2}\sin \pi x L \tag{7.16}$$

Use of $u_b = \frac{1}{2}\int_0^L (w^{'})^2 dx$:

$$u_b = \frac{\pi^2}{2L^2}\, a_1^2 \int_0^L \cos^2\frac{\pi x}{l} dx = \frac{\pi^2}{4L} a_1^2 \tag{7.17}$$

Potential energy:

$$\Pi_e = -F\, u_b = -\frac{\pi^2}{4L} F a_1^2 \tag{7.18}$$

$$\Pi_i = \frac{1}{2}\int_0^L EI(w'')^2 dx = \frac{1}{2}EIA_1^2\frac{\pi^4}{L^4}\int_0^L \sin^2\frac{\pi x}{L}dx = \frac{\pi^4}{4}\frac{EI}{L^3}a_1^2 \tag{7.19}$$

Total potential energy of the system:

$$\Pi = \Pi_e + \Pi_i = \left(-\frac{\pi^2}{4L}F + \frac{\pi^4}{4}\frac{EI}{L^3}\right)a_1^2 \quad (+\Pi_N) \tag{7.20}$$

We can find of extremal value of potential energy with use of $\frac{\Pi}{a_1} = 0$:

$$\frac{\Pi}{a_1} = \left(-\frac{\pi^2}{4L}F + \frac{\pi^4 EI}{4L^3}\right)2a_1 = 0 \tag{7.21}$$

If we assume that $a_1 \neq 0$:

$$-\frac{\pi^2}{4L}F + \frac{\pi^4 EI}{4L^3} = 0 \tag{7.22}$$

Then the **result** is (it is identical to Euler's solution):

$$F = F_{cr} = \frac{\pi^2 EI}{L^2}. \tag{7.23}$$

7.4 Geometric non-linearity and FEM

7.4.1 Strains – more precisely derived

By ommiting of simplications done in basic elasticity courses during derivation of geometry equation one can obtain:

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
\varepsilon_z &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \\
\gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}
\end{aligned}$$

In many cases the equations 7.24 are simplified in order to obtain 2^{nd} order theory solution.

7.4.2 Geometric non-linearity and FEM

Non-linear members of 7.24 can be included in the K_G matrix:

$$(\mathbf{K} + \mathbf{K_G})\Delta\mathbf{r} = \Delta\mathbf{F} \tag{7.24}$$

The $\mathbf{K_G}$ is usually called ”geometric matrix” or ”initial stiffness matrix”, and it depends on current stress state of material

7.4.3 Linear stability and FEM

- An approach similar to the Euler theory can be used.
- It is based on the equation $(\mathbf{K} + \mathbf{K_G})\mathbf{r} = \mathbf{F}$.
- The critical load (just before loss of stability) is have to be found:

$$(\mathbf{K} + \lambda\mathbf{K_G})\mathbf{r} = \mathbf{0} \tag{7.25}$$

  It can be viewed as an analogy of „$M = F\,u$“ in Euler problem.

- The problem $(\mathbf{K} + \lambda\mathbf{K_G})\mathbf{r} = \mathbf{0}$ can be defiined as a searching of eigenvalues, where $\lambda\ldots$ (eigenvalues) are multipliers of critical load. The matrix $\mathbf{K_G}$ depends on internal stress state and thus on loads, too.

7.5 Problem for individual work

- Use software of your choice to compute critical load on an Euler beam (divide the model to at least 10 finite elements). Try also different boundary conditions ot the beam. Compare analytical and numerical results.