6. THE NORMAL DISTRIBUTION AND THE LIMIT THEOREMS

Study time: 60 minutes

Aim - you will be able to

- characterize the normal and the standard normal distribution
- formulate and use the limit theorems
- describe special distributions

Explication

6.1. Normal Distribution

Astronomers were responsible for one of the earliest attempts to formally model the random variation inherent in the measurement process. The probability density function which was adopted at that time has been alternatively called the error function, the Gaussian curve, and today most commonly, the normal distribution. The normal distribution is the most widely used model of random variation. Its popularity is partly based on its intuitive appeal as a simple mathematical model of our instinctual notions of what constitutes random variation. However, there is also sound theoretical support for the belief that the normal distribution frequently occurs in practice.

The form of the normal density model is a simple symmetric bell-shaped curve with a single mode. This shape is achieved by the use of a negative exponential function whose argument is the square of the distance from the mode. Since squared distance makes values near zero smaller, the normal curve has a smooth rounded shape in the region of its mode. The normal density has two parameters, μ, its mode and the point about which the density is symmetric, and σ, a scale or dispersion parameter which determines the concentration of the density about the mode and the rate of decrease of the density towards the tails of the distribution. The family of normally distributed random variables is denoted by

$$X \rightarrow N(\mu, \sigma^2)$$

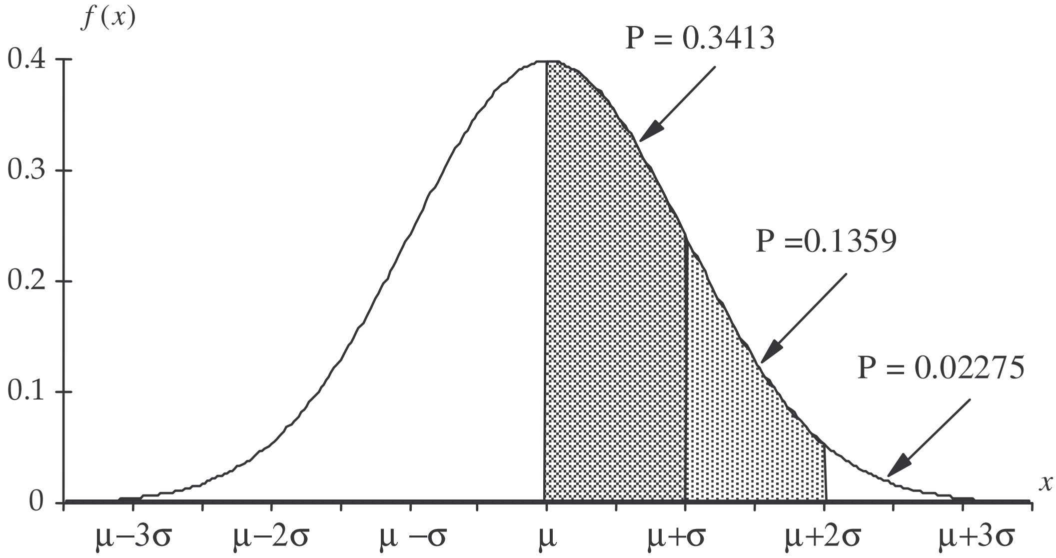
The probability density of the random variable X with the normal distribution: $X \rightarrow N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; -\infty < x < +\infty$$

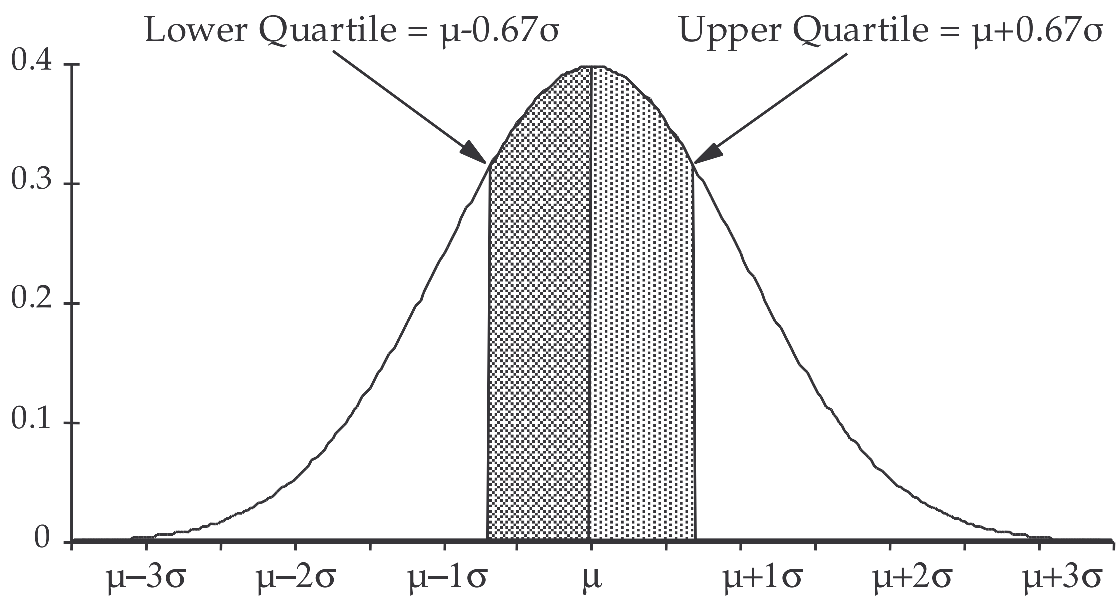
$$\text{pro} - \infty < \mu < \infty , \quad \sigma^2 > 0$$

This density is symmetric about μ and therefore the mean, median, and mode are all equal to μ. Also due to the symmetric bell shape of this density, the interquartile range equals the Shorth which is twice the MAD.

The following charts illustrate the distribution of probabilities for a normal random variable. The first chart shows that the probability of being between 0 and 1 standard deviation (σ) above the mean (μ) is 0.3413 or approximately one-third. Since the distribution is symmetric, the probability of being between 0 and 1 standard deviation (σ) below the mean (μ) is also approximately one third. Therefore the probability of being more than one standard deviation from the mean in either direction is again one third.



Conversely, the upper and lower quartiles are two thirds of a standard deviation above and below the mean. Thus μ+/- 0.67 σ divides the probability of the distribution into four equal parts of 25%.



The mean and variance of a normal random variable are equal to its location parameter μ, and the square of its scale parameter σ^2, respectively.

$$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \mu$$

$$V(X) = E[(X - E(X))^2] = \int_{-\infty}^{+\infty} \frac{(x-\mu)^2}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sigma^2$$

The distribution function of the normal distribution:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}dt} dt$$

Standard normal distribution

A normal random variable with location parameter 0 and scale parameter 1 is called a standard normal random variable. Because of the form of the normal density, it is possible to determine probabilities for any normal random variable from the distribution function of the standard normal variable. Consequently, the standard normal random variable has been given the special symbolic destination, Z, from which the z-score derives. The standard normal distribution function is given the special symbol, Φ.

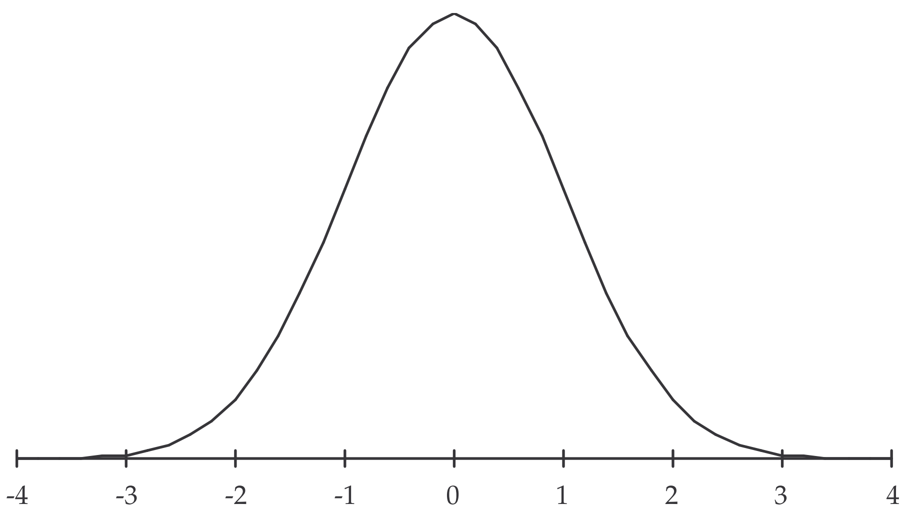
Then,

$$\Phi(z) = P(Z < z) = \int_{-\infty}^{z} \varphi(u)du = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

where φ(z) is the standard normal density

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} ;-\infty < z < +\infty$$

Standard Normal Distribution



Standardization - relation between normal and standard normal distribution

Therefore if X is any normal random variable $N(\mu,\ \sigma^2)$ we can define a related standard normal random variable $Z = \dfrac{X-\mu}{\sigma}$ and it has the standard normal distribution.

$$X \ldots N(\mu,\ \sigma^2) \ \Rightarrow\ Z = \frac{X-\mu}{\sigma} \quad , Z \ldots \Phi(\ 0,\ 1\ )$$

The distribution function of X can therefore be computed from the derived random variable Z which has a standard normal distribution:

$$F(x) = P(\ X\ < x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}dt} dt = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = \Phi(\ \frac{x-\mu}{\sigma}\ )$$

Solved example

X ... N(2, 25), determine P(2 <X <8)

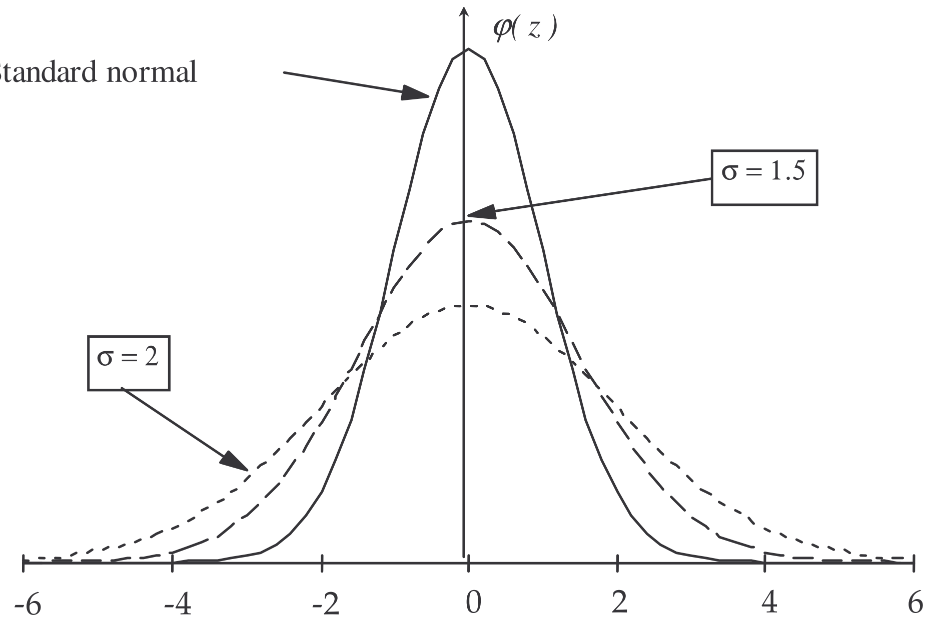
Solution:

$$P(\ 2 < X < 8\ ) = F(\ 8\ ) - F(\ 2\ ) = \Phi(\ \frac{8-2}{\sqrt{25}}\ ) - \Phi(\ \frac{2-2}{\sqrt{25}}\ ) = \Phi(\ 1.2\ ) - \Phi(\ 0\ ) = 0.885 - 0.5 = 0.385$$

In the tables or by suitable software we can find: $\Phi(1.2) = 0.885$, $\Phi(0) = 0.5$

Example:

The following chart illustrates the normal density with zero mean for selected values of σ. It is clear that the mean, median, and mode of a normal random variable are all equal, and the two parameters of the normal distribution are the embodiment of our intuitive notions about the general distributional characteristics of location and scale.



6.2. Limit Theorems

Definitions of the basic notions

Convergence in probability:

$\lim_{n\to\infty} P(|X_n - X| < \varepsilon) = 1 \quad \Rightarrow X_n \xrightarrow{p} X$; *i.e. a sequence of random variables* $\{X_n\}$ *converges in probability to random variable X*

Convergence in distribution:

$\{F_n(x)\}$ ... is a sequence of distribution functions corresponding to random variables $\{X_n\}$

The sequence $\{X_n\}$ *converges towards X in distribution,* if:

$$\lim_{n\to\infty} F_n(x) = F(x),$$

for every real number x at which F is continuous.

Consequence:

The sequence $\{X_n\}$ converges in distribution to distribution $N(\mu, \sigma^2)$, i.e. the random variable X_n has *asymptotical normal distribution* if $\lim_{n\to\infty} F_n(x) = \Phi(\frac{x-\mu}{\sigma})$.

Limit theorems

Chebyshev's inequality:

X ... is an arbitrary random variable with mean EX and variance DX.
Then

$$P(\,|X - EX| \geq \varepsilon) \leq \frac{DX}{\varepsilon^2}, \qquad \varepsilon > 0$$

This relation results from the variance definition:

$$D(X) = \int_{-\infty}^{+\infty} (x\text{-}E(X))^2 f(x)dx = \left\{ \int_{|x-E(X)|<\varepsilon} (x\text{-}E(X))^2 f(x)dx + \int_{|x-E(X)|\geq\varepsilon} (x\text{-}E(X))^2 f(x)dx \right\}$$
$$\geq \int_{|x-E(X)|\geq\varepsilon} (x\text{-}E(X))^2 f(x)dx \geq \varepsilon^2 P(|X - E(X)| \geq \varepsilon)$$

Using of the Chebyshev's inequality (for calculation probabilities):

$$P(|X - E(X)| < k\sigma) > 1 - \frac{1}{k^2}$$

e.g. we can apply it to $X = \overline{X}$ with respect to following limit theorems:

$$P(|\overline{X} - \mu| < k\frac{\sigma}{\sqrt{n}}) > 1 - \frac{1}{k^2}$$
$$P(\frac{|\overline{X} - \mu|}{\sigma} < \frac{k}{\sqrt{n}}) > 1 - \frac{1}{k^2}$$

Law of large numbers:

$\{X_n\}$ ... is a sequence of independent random variables, each having a mean $EX_n = \mu$ and a variance $DX_n = \sigma^2$.

Define a new variable

$$\overline{X_n} = \frac{1}{n} . \sum_{j=1}^{n} X_j\,, \qquad n \in \mathrm{N}$$

The sequence { $\overline{X_n}$ } converges in probability to μ: $\overline{X_n} \xrightarrow{p} \mu$.

Notion: This affirmation results from the Chebyshev's inequality.

Bernoulli theorem: $\{X_n\}$ ... is a sequence of the binomial independent random variables with parameters $n=1$ a $p \in (0,1)$ (so-called alternative random variable, let $X_n = 1$ in case the event

will be at one trial and $X_n = 0$ in case the event won't be; $P(X_n = 1) = p$ a $P(X_n = 0) = 1\text{-}p$).
Then we know that

$$\overline{X_n} = \frac{1}{n}.\sum_{j=1}^{n} X_j \xrightarrow{p} p$$

The expression on the left side represents a relative frequency of the event occurrence in the sequence n trials. That is why we can estimate a probability ingoing any occurrence by relative frequency of this event occurrence in the sequence n trials when we have a great number of the trials.

Central limit theorem

Lindeberg's theorem:
Let $X_1, X_2, \ldots, X_n \ldots$ be a sequence of independent random variables, $n \to \infty$.
X_i ... have the same probability distribution, $EX_i = \mu$, $DX_i = \sigma^2$.
Then

$$Y_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}$$ has an asymptotic normal distribution $N(0,1) \Rightarrow \lim_{n\to\infty} P(Y_n < u) = \Phi(u)$

for $-\infty < u < \infty$, it's mean that Y_n converges in distribution to distribution $N(0,1)$.

For sufficiently large numbers n holds:

1. $X_n = \sum_{i=1}^{n} X_i \Rightarrow EX = n\mu,\ DX = n\sigma^2$, we can approximate the distribution X_n by the distribution $N(n\mu, n\sigma^2)$, i.e. X_n *has the asymptotic normal distribution*,

   $X_n = \sum_{i=1}^{n} X_i \to N(n\mu, n\sigma^2)$.

2. Analogous for $\overline{X}$:

   $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ *has the asymptotic normal distribution with parameters* $E\overline{X} = \mu$,

$D\overline{X} = \frac{\sigma^2}{n}$,

$$\overline{X} \to N(\mu, \frac{\sigma^2}{n})$$

Special case of the above theorem is Moivre-Laplace theorem:

Let S_n ... Bi(n, p); E$S_n = np$; D$S_n = np(1\text{-}p)$

[$S_n = \sum_{i=1}^{n} X_i$, X_i ... has the alternative distribution thus binomial Bi(1, p)]

then for large n hold that $U_n = \frac{S_n - np}{\sqrt{np(1-p)}} \to N(0,1)$.

Applications of the Central Limit Theorem – Normal Approximations to the binomial and Poisson distributions

Taking n observations of a Bernoulli distribution and computing the sample average $\hat{p}$ is equivalent to defining the sample proportion random variable

$$\hat{p} = \frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} X_i}{n} = \frac{k}{n} \quad \text{... proportion of “successes“ in } n \text{ Bernoulli trials}$$

The sample proportion will have a Binomial distribution will the values re-scaled from 0 to 1. That is, if X has a binomial distribution with parameters n and p.

$$\mathrm{P}\left[\hat{p} = \frac{k}{n}\right] = \mathrm{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

Furthermore:

$$E(\hat{p}) = p\,; \qquad D(\hat{p}) = \frac{p(1-p)}{n}$$

Therefore by the Central Limit Theorem, we can approximate the binomial distribution by the normal distribution for large n.

$$\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \to N(0,1)$$

$$\frac{X-\mathrm{n}p}{\sqrt{np(1-p)}} \to N(0,1)$$

Probabilities concerning ranges of value for these variables can then be calculated as

$$P(p_1 < \hat{p} < p_2) = \Phi\left(\frac{p_2-p}{\sqrt{\frac{p(1-p)}{n}}}\right) - \Phi\left(\frac{p_1-p}{\sqrt{\frac{p(1-p)}{n}}}\right)$$

$$S_n = \sum_{i=1}^{n} X_i$$

$$P(k_1 < S_n < k_2) = \Phi\left(\frac{k_2-np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k_1-np}{\sqrt{np(1-p)}}\right)$$

For smaller sample sizes a so-called continuity correction is often employed to improve the accuracy of the approximation. Thus we would compute the preceding probability as

$$P(k_1 < S_n < k_2) = \Phi\left(\frac{k_2 + 0{,}5 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k_1 - 0{,}5 - np}{\sqrt{np(1-p)}}\right)$$

The Central Limit Theorem applies broadly to most distributions. In particular, the normal distribution can be used to approximate the Poisson distribution when the interval of observation, t, and hence the expected number of events, λt, is large. Let

$$\hat{\lambda} = \frac{X}{t} = \frac{count\ \ per}{unit\ \ time} = \frac{rate\ \ of}{occurrence}$$

We know that the mean and variance of the number of events during an interval t is λt, and therefore the mean and variance of the rate at which events occur is

$$E(\frac{X}{t}) = \lambda; \qquad D(\frac{X}{t}) = \frac{\lambda}{t}$$

Probabilities concerning Poisson counts or rates can then be calculated as

$$P(k_1 < X < k_2) = \Phi\left(\frac{k_2 - \lambda t}{\sqrt{\lambda t}}\right) - \Phi\left(\frac{k_1 - \lambda t}{\sqrt{\lambda t}}\right)$$

$$P(g_1 < \frac{X}{t} < g_2) = \Phi\left(\frac{g_2 - \lambda}{\sqrt{\frac{\lambda}{t}}}\right) - \Phi\left(\frac{g_1 - \lambda}{\sqrt{\frac{\lambda}{t}}}\right)$$

where,

$$g_1 = \frac{k_1}{t}; \qquad g_2 = \frac{k_2}{t}$$

Applying the continuity correction, we would calculate the probability as,

$$P(k_1 < X < k_2) = \Phi\left(\frac{k_2 + 0{,}5 - \lambda t}{\sqrt{\lambda t}}\right) - \Phi\left(\frac{k_1 - 0{,}5 - \lambda t}{\sqrt{\lambda t}}\right)$$

6.3. Special sampling distribution

Chi-square distribution

The chi-squared random variable arises as the sum of squared standard normal random variables. The distribution has a single parameter n, the number of squared normal random variables in the sum. This parameter is called the degrees of freedom of the chi-squared distribution.

$$\chi_n^2 = \sum_{i=1}^{n} Z_i^2$$

A χ-squared random variable with one degree of freedom χ_1^2 is simply a squared standard normal random variable. The distribution function of this random variable is

$$\begin{aligned} F_{\chi_1^2}(y) &= P(Z^2 < y) = P(-\sqrt{y} < Z < \sqrt{y}) \\ &= \Phi\left(\sqrt{y}\right) - \Phi\left(-\sqrt{y}\right) \qquad y > 0 \end{aligned}$$

The probability density function of a χ_1^2 random variable can be found from the derivative of its distribution function.

$$\begin{aligned} f_{\chi_1^2}(y) &= \frac{\partial F_{\chi_1^2}(y)}{\partial y} = \left(\frac{\partial \Phi(\sqrt{y})}{\partial \sqrt{y}} + \frac{\partial \Phi(-\sqrt{y})}{\partial \sqrt{y}} \right) \frac{\partial \sqrt{y}}{\partial y} \\ &= \frac{e^{-\frac{y}{2}}}{\sqrt{2\pi y}} \qquad y > 0 \end{aligned}$$

The general density for a χ_n^2 random variable is

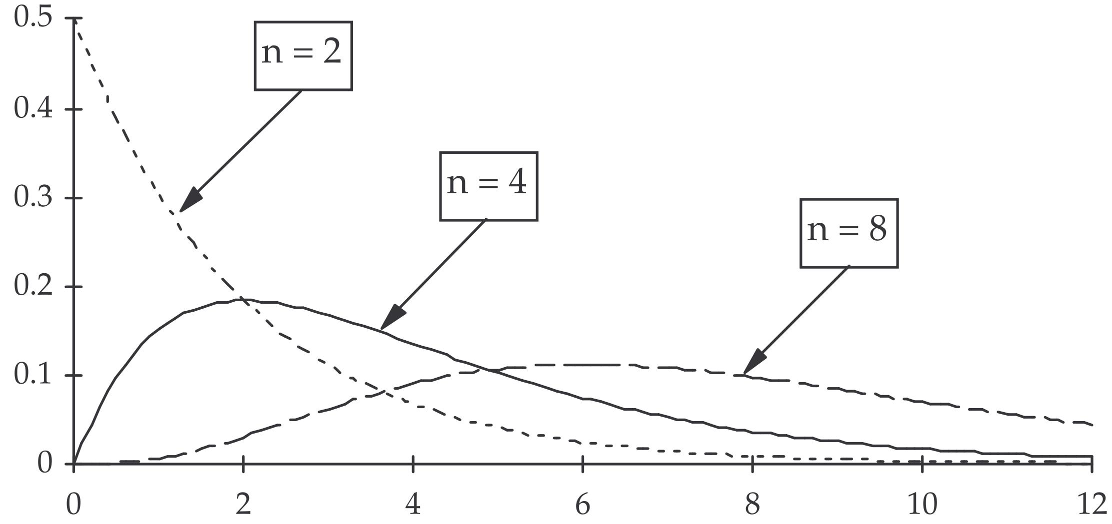
$$f_\chi(x) = \frac{x^{\left(\frac{n}{2}-1\right)} e^{-\frac{x}{2}}}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}$$

where $\Gamma(t)$ is a gamma function.

The mean and variance of the chi-squared distribution are

$$\begin{aligned} E(\chi_n^2) &= n \\ D(\chi_n^2) &= 2n \end{aligned}$$

The density function for various values of the parameter *n*:



The χ^2_n arises as the sampling distribution of the sample variance

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

If the X_i are a sample from a normal population with mean μ and standard deviation σ, then the sample variance has distribution

$$s^2 \rightarrow \chi^2_{n-1}\frac{\sigma^2}{n-1}$$

To see this, consider the sum of squared standardized observations from a normal population with mean μ and standard deviation σ.

$$\sum_{i=1}^{n}\left(\frac{x-\mu}{\sigma}\right)^2 \rightarrow \chi^2_n$$

This expression clearly has a χ^2_n distribution. Now re-express the numerator by added and subtracting the sample mean from the squared terms.

$$\frac{\sum_{i=1}^{n}(x - \bar{x} + \bar{x} - \mu)^2}{\sigma^2} \rightarrow \chi^2_n$$

Expanding and simplifying, we obtain

$$\frac{\sum_{i=1}^{n}(x-\bar{x})^2}{\sigma^2}+\frac{(\bar{x}-\mu)^2}{\sigma^2/n}\to\chi_n^2$$

The second term is simply the squared standardized sample mean and therefore has a χ_1^2 distribution. Since the χ_n^2 is the sum of n squared normals, the first term must be the remaining *n-1* squared normals. Therefore,

$$\frac{(n-1)\,s^2}{\sigma^2}=\frac{\sum_{i=1}^{n}(x-\bar{x})^2}{\sigma^2}\to\chi_{n-1}^2$$

Note that this argument is only heuristic and not a formal proof. The independence of the sample mean from the deviations about the sample mean has not been established.

This fact is important at the statistical hypothesis testing.

1. We use this distribution for verification of the random variables independence.
2. We can use chi-square distribution when we test that the random variables follow from certain distribution. This test is known as "Goodness-of-Fit Test".

Student's distribution (*t* distribution)

The Student's t distribution is the sampling distribution of the standardized sample mean when the sample variance is used to estimate the true population variance. The origin of the distribution's name, Student's t has an interesting history. An Irish statistician, W. S. Gosset first published this distribution anonymously under the pseudonym "Student" because his employer, Guiness Breweries of Dublin, prohibited its employees from publishing under their own names for fear that its competitors would discover the secret of their excellent beer. In his original paper, Gosset used the designation "t" for his statistic. Hence the name is Student's.

The Student's *t* distribution with *n* degrees of freedom is the ratio of a standard normal random variable over the square root of a chi-squared random variable divided by its degrees of freedom. The *t* distribution has a single parameter, *n* the degrees of freedom of the chi-squared random variable in the denominator.

$$t_n=\frac{Z}{\sqrt{\frac{\chi_n^2}{n}}}$$

The probability density function of this random variable is

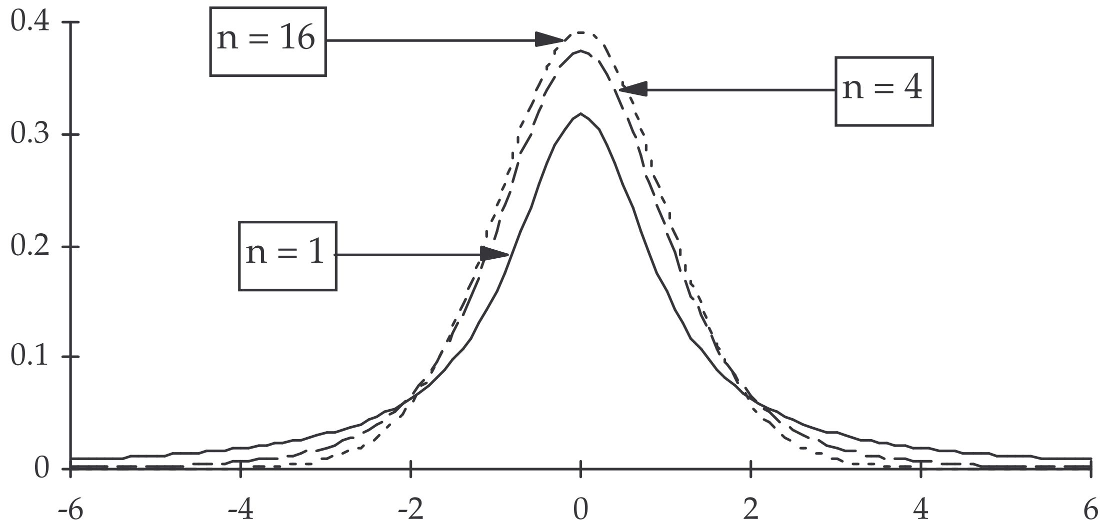
$$f(t)=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\;\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$

The mean and variance of the t-distribution are

$$E(t_n) = 0$$

$$D(t_n) = \frac{n}{n-2}$$

The following figure shows the density function for different values of number of the degrees of freedom:



If random variables $X_1, X_2, ..., X_n$ have the normal distribution $N(\mu, \sigma^2)$ and they are **independent** then we can show that

$$\overline{X} \to N(\mu, \sigma^2/n)$$

and

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \to N(0,1)$$

then

$$\frac{\overline{X} - \mu}{S}\sqrt{n} = \frac{\left(\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right)}{\sqrt{\dfrac{\dfrac{S^2}{\sigma^2}(n-1)}{n-1}}} \to t_{n\text{-}1}$$

Student's t-distribution has a wide exercise.

Fisher-Snedecor‘s distribution - F distribution

Snedecor's F distribution arises as the ratio of two chi-squared distributions divided by their respective degrees of freedom. The F distribution has two parameters, *n* the degrees of freedom of the chi-squared random variable in the numerator and *m* the degrees of freedom of the chi-squared random variable in the denominator.

$$F_{n,m} = \frac{\dfrac{\chi_n^2}{n}}{\dfrac{\chi_m^2}{m}}$$

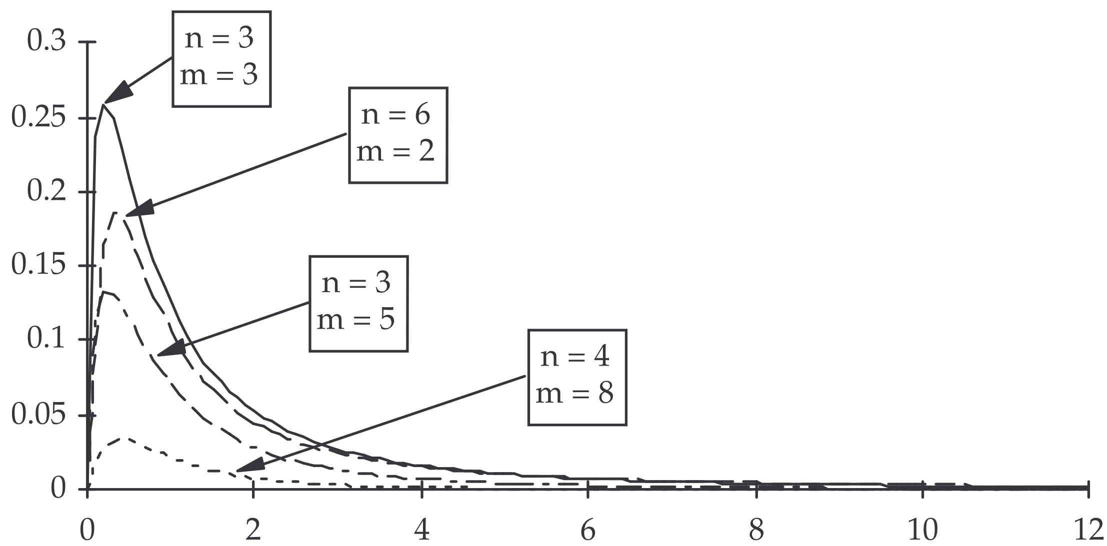
The probability density function of this random variable is

$$f(x) = \frac{\Gamma\left(\dfrac{n+m}{2}\right) n^{\frac{n}{2}} m^{\frac{m}{2}} x^{\frac{n}{2}-1}}{\Gamma\left(\dfrac{n}{2}\right)\Gamma\left(\dfrac{m}{2}\right)(m+nx)^{\frac{n+m}{2}}}$$

The mean and variance of the t distribution are

$$E(F_{n,m}) = \frac{m}{m-2} \qquad D(F_{n,m}) = \frac{2m^2\left(1+\dfrac{m-2}{n}\right)}{(m-2)^2\,(m-4)}$$

The following figure shows the density function for different values *m* a *n*:



Clearly such a distribution would arise as the sampling distribution of the ratio of the sample variances from two independent populations with the same standard deviation σ. The degrees of freedom represent one less than the samples sizes of the numerator and denominator sample variances respectively.

$$\overline{X_1};\ \overline{X_2}$$

$$X_{1j} \rightarrow N(\mu_1, \sigma); \quad \mathrm{j} = 1, \mathrm{n}_1$$

$$X_{2j} \rightarrow N(\mu_2, \sigma); \quad \mathrm{j} = 1, \mathrm{n}_2$$

$$S_i^2 = \frac{\sum_{j=1}^{n_i} \left(X_{ij} - \overline{X}_i\right)^2}{n_i - 1};\ i = 1, 2$$

$$\frac{S_1^2}{S_2^2} \rightarrow F_{n_1, n_2}$$

We use this distribution for evaluation of statistical analysis results.

$\sum$ Summary of notions

One of the most important continuous distributions is a **normal distribution**. It is distribution with two parameters, when the first parameter is a mean and the second one is a variance. We get standard normal distribution for special choice of parameters (the mean is equal 0 and the variance is equal 1).
Chebyshev's inequality puts an upper bound on the probability that an observation should be far from its mean.
Chí-square distribution is a distribution derived from sum of squared standard normal random variables.
Central limit theorem describes asymptotic statistic behavior of mean. We can use it for substitution of binomial (Poisson) distribution by normal distribution.
Student's t-distribution with *n* degrees of freedom is the ratio of a standard normal random variable over the square root of a chi-squared random variable divided by its degrees of freedom.
F distribution is the ratio of two chi-squared distributions divided by their respective degrees of freedom.

? Questions

1. Define relation between normal and standard normal distributions.
2. What is Chebyshev's inequality?
3. Explain law of large numbers.
4. Describe chi-square distribution.

Problems

Example 1: If the mean (μ) height of a group of students is equal to 170cm with a standard deviation (σ) of 10 cm, calculate the probability that a student is between 160cm and 180cm.

{*Answer:* 0,6828}

Example 2: Let X = "height of a randomly chosen male", and suppose that X is normally distributed with μ = 176cm and σ^2 = 25 cm^2,i.e. X is N(176,25):

(i) calculate the probability that the height of a randomly chosen male is less than or equal to 182

(ii) calculate the probability that the height of a randomly chosen male is less than or equal to 170

(iii) calculate the probability that the height of a randomly chosen male is not greater than 176

(iv) calculate the probability that the height of a randomly chosen male is between 170 and 182 cm

(v) calculate the probability that the height of a randomly chosen male is not less than 160

{*Answer:* (i) 0.885, (ii) 0.115, (iii) 0.5, (iv) 0.7698. (v) 0.9993}

Example 3: We take the heights of 9 males and we assume that the heights are i.i.d. N(176,25) as before. What is the probability that the sample mean height is between 174 and 178 cm?

{*Answer:* 0.7698}

Example 4: Premature babies are those born more than 3 weeks early. A local newspaper reports that 10% of the live births in a country are premature. Suppose that 250 live births are randomly selected and the number Y of "preemies" is determined.

(i) What is the probability that X lies in between 15 and 30 (both included)?

(ii) Find the proportion of the event fewer than 20 births are premature?

{*Answer:* (i) 0.86, (ii) 0.12}