10.2. Young's inequality. *Let $p, q \in (1, \infty)$, $\frac{1}{p}+\frac{1}{q}=1$. If a, b are nonnegative numbers, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. We can assume $ab > 0$. Making use of concavity of the function ln,

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) = \ln a + \ln b = \ln(ab)$$

holds, and we easily establish the required inequality. ■

10.3. Hölder's inequality. *Suppose that $f \in \mathscr{L}^p$ and $g \in \mathscr{L}^q$, where $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in \mathscr{L}^1$ and*

$$\left|\int_X fg \,\mathrm{d}\mu\right| \leq \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p} \left(\int_X |g|^q \,\mathrm{d}\mu\right)^{1/q}.$$

Proof. Denote

$$s = \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p}, \qquad t = \left(\int_X |g|^q \,\mathrm{d}\mu\right)^{1/q}.$$

We may assume that $st > 0$. Thanks to Young's inequality ($a = |f(x)|/s$, $b = |g(x)|/t$) we have

$$\frac{f(x)g(x)}{st} \leq \frac{|f(x)|\,|g(x)|}{st} \leq \frac{|f(x)|^p}{ps^p} + \frac{|g(x)|^q}{qt^q}$$

for each $x \in X$. Thus

$$\frac{1}{st}\int_X fg \,\mathrm{d}\mu \leq \frac{\int_X |f|^p \,\mathrm{d}\mu}{ps^p} + \frac{\int_X |g|^q \,\mathrm{d}\mu}{qt^q} \leq \frac{1}{p} + \frac{1}{q} = 1$$

which is what we wanted to prove. ■

10.4. Minkowski's inequality. *Let $p \in [1, \infty]$ and $f, g \in \mathscr{L}^p$. Then $f + g \in \mathscr{L}^p$ and*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. It is not hard to verify that $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$.
If $p = \infty$, then $|f| \leq s$ μ-almost everywhere and $|g| \leq t$ μ-almost everywhere, which implies $|f+g| \leq s + t$ μ-almost everywhere. Therefore $\|f+g\|_\infty \leq s+t$, and consequently

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

With these trivial cases out of the way, there remains the case $1 < p < \infty$. Hölder's inequality yields

$$\int_X |f|\,|f+g|^{p-1}\,\mathrm{d}\mu \le \left(\int_X |f|^p\,\mathrm{d}\mu\right)^{1/p}\left(\int_X |f+g|^{(p-1)q}\,\mathrm{d}\mu\right)^{1/q} = \left(\int_X |f|^p\,\mathrm{d}\mu\right)^{1/p}\left(\int_X |f+g|^p\,\mathrm{d}\mu\right)^{1-1/p}.$$

Analogously

$$\int_X |g|\,|f+g|^{p-1}\,\mathrm{d}\mu \le \left(\int_X |g|^p\,\mathrm{d}\mu\right)^{1/p}\left(\int_X |f+g|^p\,\mathrm{d}\mu\right)^{1-1/p}.$$

This entails

$$\int_X |f+g|^p\,\mathrm{d}\mu \le \int_X |f|\,|f+g|^{p-1}\,\mathrm{d}\mu + \int_X |g|\,|f+g|^{p-1}\,\mathrm{d}\mu \le \left(\left(\int_X |f|^p\right)^{1/p} + \left(\int_X |g|^p\right)^{1/p}\right)\left(\int_X |f+g|^p\,\mathrm{d}\mu\right)^{1-1/p}.$$

■

10.5. The L^p Spaces. The behavior of the function $f \mapsto \|f\|_p$ on $\mathscr{L}^p$, where $p \in [1, \infty]$, resembles axioms of a norm. However, in general, $\mathscr{L}^p$ is not a linear space and a nonzero function may have zero norm. To apply the theory of normed linear spaces, we identify functions which are equal almost everywhere. Formally, we assign to every function $f \in \mathscr{L}^p$ the class of functions

$$[f] = \{g \in \mathscr{L}^p : g = f \ \mu\text{-almost everywhere on } X\}$$

and define

$$L^p = L^p(X, \mathscr{S}, \mu) = \{[f] : f \in \mathscr{L}^p\}.$$

Then L^p is a linear space equipped with operations

$$[f] + [g] := [f+g], \quad \alpha[f] := [\alpha f] \quad (\alpha \in \mathbf{R}),$$

and with the (true) norm

$$\|\,[f]\,\|_p := \|f\|_p\,.$$

It can easily be seen that these definitions do not depend on the choice of representatives.

It is customary not to distinguish between functions and classes, often even between the spaces $\mathscr{L}^p$ and L^p. For instance, we say that $\{f_j\}$ is a Cauchy sequence in L^p while the meaning is that f_j are functions and $\{[f_j]\}$ is a Cauchy sequence in L^p.